# Differential and Integral Calculus

## by R.Courant

### Volume 1



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## Differential and Integral Calculus

## Chapter I: Introduction

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# Chapter I.

# Introduction

The differential and integral calculus is based on two concepts of outstanding importance, apart from the concept of number, namely, the concepts of function and limit. While these concepts can be recognized here and there even in the mathematics of the ancients, it is only in modern mathematics that their essential character and significance have been fully clarified. We shall attempt here to explain these concepts as simply and clearly as possible.

## 1.1 The Continuum of Numbers

We shall consider the numbers and start with the natural numbers 1, 2, 3, ··· as given as well as the rules

$$(a + b) + c = a + (b + c)$$ - **associative law of addition**,
$$a + b = b + a$$ - **commutative law of addition,**
$$(ab)c = a(bc)$$ - **associative law of multiplication**,
$$ab = ba$$ - **commutative law of multiplication**,
$$a(b + c) = ab + ac$$ - **distributive law of multiplication**.

by which we calculate with them; we shall only briefly recall the way in which the concept of the positive integers (the **natural numbers**) has had to be extended.

1.1.1 **The System of Rational Numbers and the Need for its Extension:** In the domain of the natural numbers, the fundamental operations of addition and multiplication can always be performed without restriction, i.e., the sum and the product of two natural numbers are themselves always natural numbers. But the inverses of these operations, **subtraction** and **division**, cannot invariably be performed within the domain of natural numbers, whence mathematicians were long ago obliged to invent the **number 0**, the **negative integers**, and **positive and negative fractions**. The totality of all these numbers is usually called the **class of rational numbers**, since all of them are obtained from unity by means of the **rational operations of calculation:** Addition, multiplication, subtraction and division.

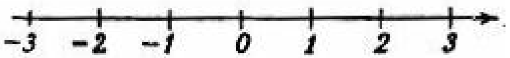



Fig. 1. The number axis

Numbers are usually represented graphically by means of the points on a straight line - the **number axis** - by taking an arbitrary point of the line as the **origin** or **zero point** and another arbitrary point as the **point 1**; the distance between these two points (the length of the **unit interval**) then serves as a **scale** by which we can assign a point on the line to every **rational number**, positive or negative. It is customary to mark off the **positive numbers** to the right and the **negative numbers** to the left of the origin (Fig. 1). If, as is usually done, we define the **absolute value** (also called the **numerical value** or **modulus**) $|a|$ of a number $a$ to be $a$ itself when $a \geq 0$, and $-a$ when $a < 0$, then $|a|$ simply denotes the distance of the corresponding point on the number axis from the origin.

The symbol $\geq$ means that either the sign $>$ or the sign $=$ shall hold. A corresponding statement holds for the signs $\pm$ and $\mp$ which will be used later on.

The geometrical representation of the rational numbers by points on the number axis suggests an important property which can be stated as follows: **The set of rational numbers is everywhere dense.** This means that in every interval of the number axis, no matter how small, there are always rational numbers; in geometrical terms, in the segment of the number axis between any two rational points, however close together, there are points corresponding to rational numbers. This density of the rational numbers at once becomes clear if we start from the fact that the numbers $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots, \frac{1}{2^n}, \cdots$ become steadily smaller and approach nearer and nearer to zero as $n$ increases. If we now divide the number axis into equal parts of length $1/2^n$, beginning at the origin, the end-points $1/2^n, 2/2^n, 3/2^n, \cdots$ of these intervals represent rational numbers of the form $m/2^n$, where we still have the number $n$ at our disposal. Now, if we are given a fixed interval of the number axis, no matter how small, we need only choose $n$ so large that $1/2^n$ is less than the length of the interval; the intervals of the above subdivision are then small enough for us to be sure that at least one of the points of the sub-division $m/2^n$ lies in the interval.

Yet, in spite of this property of density, the rational numbers are not sufficient to represent every point on the number axis. Even the Greek mathematicians recognized that, if a given line segment of unit length has been chosen, there are intervals, the lengths of which cannot be represented by rational numbers; these are the so-called **segments incommensurable** with the unit. For example, the hypotenuse $l$ of a right-angled, isosceles triangle with sides of unit length is not commensurable with the length unit, because, by Pythagoras' Theorem, the square of this length must equal 2. Therefore, if $l$ were a rational number, and consequently equal to $p/q$, where $p$ and $q$ are non-zero integers, we should have $p^2 = 2q^2$. We can assume that $p$ and $q$ have no common factors, for such common factors could be cancelled out to begin with. Since, according to the above equation, $p^2$ is an even number, $p$ itself must be even, say $p = 2p'$. Substituting this expression for $p$ yields $4p'^2 = 2q^2$ or $q^2 == 2p'^2$, whence $q^2$ is even, and so is $q$. Hence $p$ and $q$ have the common factor 2, which contradicts our hypothesis that $p$ and $q$ do not have a common factor. Thus, the assumption that the hypotenuse can be represented by a fraction $p/q$ leads to contradiction and is therefore false.

The above reasoning - a characteristic example of an **indirect proof** - shows that the symbol $\sqrt{2}$ cannot correspond to any rational number. Thus, if we insist that, after choice of a unit interval, every point of the number axis shall have a number corresponding to it, we are forced to extend the domain of rational numbers by the introduction of the new **irrational numbers**. This system of rational and irrational numbers, such that each point on the axis corresponds to just one number and each number corresponds to just one point on the axis, is called the **system of real numbers**.

Thus named to distinguish it from the system of **complex numbers**, obtained by yet another extension.

**1.1.2 Real Numbers and Infinite Decimals:** Our requirement that there shall correspond to each point of the axis one real number states nothing *a priori* about the possibility of calculating with these numbers in the same manner as with rational numbers. We establish our right to do this by showing that our requirement is equivalent to the following fact: The totality of all real numbers is represented by the totality of all finite and infinite decimals.

We first recall the fact, familiar from elementary mathematics, that every rational number can be represented by a terminating or by a recurring decimal; and conversely, that every such decimal represents a rational number. We shall now show that we can assign to every point of the number axis a uniquely determined decimal (usually an infinite one), so that we can represent the irrational points or irrational numbers by infinite decimals. (In accordance with the above remark, the irrational numbers must be represented by **infinite non-recurring decimals**, for example, 0.101101110···).

Let the points which correspond to the integers be marked on the number axis. By means of these points, the axis is subdivided into intervals or segments of length 1. In what follows, we shall say that a point of the line belongs to an interval, if it is an **interior point** or an **end-point** of the interval. Now let $P$ be an arbitrary point of the number axis. Then this point belongs to one or, if it is a point of division, to two of the above intervals. If we agree that, in the second case, the right-hand point of the two intervals meeting at P is to be chosen, we have in all cases an interval with end-points $g$ and $g + 1$ to which $P$ belongs, where $g$ is an integer. We subdivide this interval into ten equal sub-intervals by means of the points corresponding to the numbers

$$g+\frac{1}{10}, g+\frac{2}{10}, \ldots, g+\frac{9}{10},$$

and we number these sub-intervals 0, 1, ··· , 9 in their natural order from the left to the right. The sub-interval with the number $a$ then has the end-points $g+a/10$ and $g+a/10 + 1/10$. The point $P$ must be contained in one of these sub-intervals. (If $P$ is one of the new points of division, it belongs to two consecutive intervals; as before, we choose the one on the right hand side.) Let the interval thus determined be associated with the number $a_1$. The end-points of this interval then correspond to the numbers $g+a_1/10$ and $g+a_1/10+1/10$. We again sub-divide this sub-interval into ten equal parts and determine that one to which $P$ belongs; as before, if $P$ belongs to two sub-intervals, we choose the one on the right hand side. Thus, we obtain an interval with the end-points $g+a_1/10+a_2/10^2$ and $g+a_1/10+a_2/10^2+1/10^3$, where $a_2$ is one of the digits 0, 1, ··· , 9. We subdivide this sub-interval again and continue to repeat this process. After $n$ steps, we arrive at a sub-interval, which contains $P$, has the length $1/10^n$ and end-points corresponding to the numbers

$$g+\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots+\frac{a_n}{10^n} \text{ and } g+\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots+\frac{a_n}{10^n}+\frac{1}{10^n},$$

where each a is one of the numbers 0, 1, ⋯ , 9, but

$$\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots+\frac{a_n}{10^n}$$

is simply the decimal fraction $0.a_1a_2\cdots a_n$. Hence, the end-points of the interval may also be written in the form

$$g+0{\cdot}a_1a_2\ldots a_n \text{ and } g+0{\cdot}a_1a_2\ldots a_n+\frac{1}{10^n}.$$

If we consider the above process repeated indefinitely, we obtain an infinite decimal $0.a_1a_2\cdots$, which has the following meaning: If we break off this decimal at any place, say, the $n$-th, the point **P** will lie in the interval of length $1/10^n$ the end-points (**approximating points**) of which are

$$g+0{\cdot}a_1a_2\ldots a_n \text{ and } g+0{\cdot}a_1a_2\ldots a_n+\frac{1}{10^n}.$$

In particular, the point corresponding to the rational number $g + 0.a_1a_2\cdots a_n$ will lie arbitrarily near to the point P if only $n$ is large enough; for this reason, the points $g + 0.a_1a_2\cdots a_n$ are called **approximating points**. We say that the infinite decimal $g + 0.a_1a_2\cdots$ is the real number corresponding to the point P.

Thus, we emphasize the fundamental assumption that we can calculate in the usual way with real numbers, and hence with decimals. It is possible to prove this using only the properties of the integers as a starting-point. But this is no light task and, rather than allowing it to bar our progress at this early stage,we regard the fact that the ordinary rules of calculation apply to the real numbers to be an **axiom**, on which we shall base all of the differential and integral calculus.

We insert here a remark concerning the possibility arising in certain cases of choosing in the above scheme of expansion the interval in **two** ways. It follows from our construction that the points of division, arising in our repeated process of sub-division, and such points only can be represented by finite decimals $g + 0.a_1a_2\cdots a_n$. Assume that such a point $P$ first appears as a point of sub-division at the $n$-th stage of the sub-division. Then, according to the above process, we have chosen at the $n$-th stage the interval to the right of $P$. In the following stages, we must choose a sub-interval of this interval. But such an interval must have $P$

as its left end-point. Therefore, in all further stages of the sub-division, we must choose the first sub-interval, which has the number 0. Thus, the infinite decimal corresponding to $P$ is $g + 0.a_1a_2\cdots a_n000\cdots$. If, on the other hand, we had at the $n$-th stage chosen the left-hand interval containing P, then, at all later stages of sub-division, we should have had to choose the sub-interval furthest to the right, which has $P$ as its right end-point. Such a sub-interval has the number 9. Thus, we should have obtained for $P$ a decimal expansion in which all the digits from the $(n + 1)$-th onwards are nines. The double possibility of choice in our construction therefore corresponds to the fact that, for example, the number ¼ has the two decimal expansions 0.25000··· and 0.24999···.

**1.1.3 Expression of Numbers in Scales other than that of 10:** In our representation of the real numbers, we gave the number 10 a special role, because each interval was subdivided into ten equal parts. The only reason for this is the widely spread use of the decimal system. We could just as well have taken $p$ equal sub-intervals, where $p$ is an arbitrary integer greater than 1. We should then have obtained an expression of the form

$$g + \frac{b_1}{p} + \frac{b_2}{p^2} + \cdots,$$

where each $b$ is one of the numbers $0, 1, \cdots, p - 1$. Here we find again that the rational numbers, and only the rational numbers, have recurring or terminating expansions of this kind. For theoretical purposes, it is often convenient to choose $p = 2$. We then obtain the **binary expansion of the real numbers**

$$g + \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots,$$

where each $b$ is either* 0 or 1.

Even for numerical calculations, the decimal system is not the best. The **sexagesimal system**, with which the Babylonians calculated, has the advantage that a comparatively large proportion of the rational. numbers, the decimal expansions of which do not terminate, possess terminating sexagesimal expansions.

For numerical calculations, it is customary to express the whole number $g$, which, for the sake of for simplicity, we assume here to be positive, in the decimal system, that is, in the form

$$a_m 10^m + a_{m-1}10^{m-1} + \ldots + a_1 10 + a_0,$$

where each $a_\nu$ is one of the digits 0, 1, ···, 9. Then, for $g + 0.a_1a_2\cdots$, we write simply

$$a_m a_{m-1} \ldots a_1 a_0 \cdot a_1 a_2 \ldots$$

Similarly, the positive whole number *g* can be written in one and only one way in the form

$$\beta_k p^k + \beta_{k-1} p^{k-1} + \ldots + \beta_1 p + \beta_0,$$

where each of the numbers $\beta_\nu$ is one of the numbers 0, 1, ⋯ , *p* - 1. Together with our previous expression, this yields: Every positive real number can be represented in the form

$$\beta_k p^k + \beta_{k-1} p^{k-1} + \ldots + \beta_1 p + \beta_0 + \frac{b_1}{p} + \frac{b_2}{p^2} + \ldots,$$

where $\beta_\nu$ and $b_\nu$ are whole numbers between 0 and *p* — 1. Thus, for example, the binary expansion of the fraction 21/4 is

$$\frac{21}{4} = 1 \times 2^2 + 0 \times 2 + 1 + \frac{0}{2} + \frac{1}{2^2}.$$

**1.1.4 Inequalities:** Calculation with inequalities has a far larger role in higher than in elementary mathematics. We shall therefore briefly recall some of the simplest rules concerning them.

If $a > b$ and $c > d$, then $a + c > b + d$, but not $a - c > b - d$. Moreover, if $a>b$, it follows that $ac > bc$, provided *c* is positive. On multiplication by a negative number, the sense of the inequality is reversed. If $a>b>0$ and $c>d>0$, it follows that $ac>bd$.

For the absolute values of numbers, the following inequalities hold:

$$|a \pm b| \leqq |a| + |b|, \quad |a \pm b| \geqq |a| - |b|.$$

The square of any real number is larger than or equal to zero, whence, if *x* and *y* are arbitrary real numbers

$$(x - y)^2 = x^2 + y^2 - 2xy \geqq 0,$$

or

$$2xy \leqq x^2 + y^2.$$

**1.1.5 Schwarz's Inequality:** Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be any real numbers. Substitute in the preceding inequality*

$$x = \frac{|a_i|}{\sqrt{(a_1^2 + a_2^2 + \ldots + a_n^2)}}, \quad y = \frac{|b_i|}{\sqrt{(b_1^2 + b_2^2 + \ldots + b_n^2)}}$$

for $i = 1$, $i = 2$, $\cdots$, $i = n$ successively and add the resulting inequalities. We obtain on the right hand side the sum 2, because

$$\left(\frac{|a_1|}{\sqrt{(a_1^2 + \ldots + a_n^2)}}\right)^2 + \ldots + \left(\frac{|a_n|}{\sqrt{(a_1^2 + \ldots + a_n^2)}}\right)^2 = 1,$$

$$\left(\frac{|b_1|}{\sqrt{(b_1^2 + \ldots + b_n^2)}}\right)^2 + \ldots + \left(\frac{|b_n|}{\sqrt{(b_1^2 + \ldots + b_n^2)}}\right)^2 = 1.$$

If we divide both sides of the inequality by 2, we obtain

$$\frac{|a_1 b_1| + |a_2 b_2| + \ldots + |a_n b_n|}{\sqrt{(a_1^2 + \ldots + a_n^2)}\sqrt{(b_1^2 + \ldots + b_n^2)}} \leqq 1,$$

or, finally,

$$|a_1 b_1| + |a_2 b_2| + \ldots + |a_n b_n| \leqq \sqrt{(a_1^2 + \ldots + a_n^2)}\sqrt{(b_1^2 + \ldots + b_n^2)}.$$

* Here and hereafter, the **symbol $\sqrt{x}$**, where $x > 0$, denotes that positive number the square of which is $x$.

Since the expressions on both sides of this inequality are positive, we may take the square and then omit the modulus signs:

$$(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2 \leqq (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2).$$

This is the **Cauchy-Schwarz inequality**.

**Exercises 1.1:** (more difficult examples are indicated by an asterisk)

1. Prove that the following numbers are irrational: (*a*) $\sqrt{3}$, (*b*) $\sqrt{n}$, where *n* is not a perfect square, (*c*) $\sqrt[3]{3}$, (*d*)* $x = \sqrt{2} + \sqrt[3]{3}$, $x = \sqrt{3} + \sqrt[3]{2}$.
2*. In an ordinary system of rectangular co-ordinates, the points for which both co-ordinates are integers axe called **lattice points**. Prove that a triangle the vertices of which are lattice points cannot be equilateral.
3. Prove the inequalities:

$$(a)\ x + \frac{1}{x} \geqq 2,\ x > 0. \qquad (b)\ x + \frac{1}{x} \leqq -2,\ x < 0.$$
$$(c)\ \left| x + \frac{1}{x} \right| \geqq 2,\ x \neq 0.$$

4. Show that, if a > 0, $ax^3+2bx+c\geq 0$ for all values of *x*, if and only if $b^2-ac\leq 0$. 5. Prove the inequalities:

$$(a)\ x^2 + xy + y^2 \geqq 0.$$
$$(b)^*\ x^{2n} + x^{2n-1}y + x^{2n-2}y^2 + \ldots + y^{2n} \geqq 0.$$
$$(c)^*\ x^4 - 3x^3 + 4x^2 - 3x + 1 \geqq 0.$$

6. Prove Schwarz's ineqality by considering the expression

$$(a_1x + b_1)^2 + (a_2x + b_2)^2 + \ldots + (a_nx + b_n)^2,$$

collecting terms and applying Ex. 4.
7. Show that the equality sign in Schwarz's inequality holds if, and only if, the *a*'s and b's are proportional, that is, $ca_\nu + db_\nu = 0$ for all $\nu$'s, where *c*, *d* are independent of $\nu$ and not both zero.
8. For *n* = 2, 3, state the geometrical interpretation of Schwarz's inequality.
9. The numbers $\gamma_1$, $\gamma_2$ are direction cosines of a line; that is, $\gamma_1^2 + \gamma_2^2 = 1$. Similarly, $\eta_1^2 + \eta_2^2 = 1$. Prove that the equation $\gamma_1\eta_1 + \gamma_2\eta_2 = 1$ implies the equations $\gamma_1 = \eta_1$, $\gamma_2 = \eta_2$.
10. Prove the inequality

$$\sqrt{(a_1 - b_1)^2 + \ldots + (a_n - b_n)^2} \leqq \sqrt{(a_1^2 + \ldots + a_n^2)} + \sqrt{(b_1^2 + \ldots + b_n^2)}$$

and state its geometrical interpretation.

## 1.2. The Concept of Function

**1.2.1 Examples:** (*a*) If an **ideal gas** is compressed in a vessel by means of a piston, the temperature being kept constant, the pressure *p* and the volume *v* are connected by the relation

$$pv = C,$$

where *C* is a constant. This formula - **Boyle's Law** - states nothing about the quantities *v* and *p* themselves; its meaning is: If *p* has a definite value, arbitrarily chosen in a certain range (the range being determined physically and not mathematically), then *v* can be determined, and conversely:

$$v = \frac{C}{p}, \quad p = \frac{C}{v}.$$

We then say that *v* is a function of *p* or, in the converse case, that *p* is a function of *v*.

(*b*) If we heat a metal rod, which at temperature 0º has the length $l_0$, to the temperature $\theta°$, then its length *l* will be given, under the simplest physical assumptions, by the law

$$l = l_0 (1 + \beta\theta),$$

where $\beta$ – the **coefficient of thermal expansion** - is a constant. Again, we say that *l* is a function of $\theta$.

(*c*) Let there be given in a triangle the lengths of two sides, say *a* and *b*. If we choose for the angle $\gamma$ between these two sides any arbitrary value less than 180°, the triangle is completely determined; in particular, the third side *c* is determined. In this case, we say that if *a* and *b* are given, *c* is a function of the angle $\gamma$. As we know from trigonometry, this function is represented by the formula

$$c = \sqrt{(a^2 + b^2 - 2ab \cos \gamma)}.$$

**1.2.2 Formulation of the Concept of Function:** In order to give a general definition of the mathematical concept of function, we fix upon a definite interval of our number scale, say, the interval between the numbers *a* and b, and consider the totality of numbers *x* which belong to this interval, that is, which satisfy the relation

$$a \leqq x \leqq b.$$

If we consider the symbol *x* as denoting any of the numbers in this interval, we call it a **continuous variable** in the interval.

If now there corresponds to each value of *x* in this interval a single definite value *y*, where *x* and *y* are connected by any law whatsoever, we say that *y* is a function of *x* and write symbolically

$$y = f(x), \quad y = F(x), \quad y = g(x),$$

or some similar expression. We then call *x* the **independent variable** and *y* the **dependent variable**, or we call *x* the **argument of the function** *y*.

It should be noted that, for certain purposes, it makes a difference whether we include in the interval from *a* to *b* the end-points, as we have done above, or exclude them; in the latter case, the variable *x* is restricted by the inequalities

$$a < x < b.$$

In order to avoid a misunderstanding, we may call the first kind of interval - including its end-points - a **closed interval**, the second kind an **open interval**. If only one end-point and not the other is included, as, for example, in $a<x\leq b$, we speak of an interval which is **open at one end** (in this case the end *a*). Finally,we may also consider open intervals which extend without bound in one direction or both. We then say that the variable *x* ranges over an **infinite open interval** and write symbolically

$$a < x < \infty \quad \text{or} \quad -\infty < x < b \quad \text{or} \quad -\infty < x < \infty .$$

In the general definition of a function, which is defined in an interval, nothing is said about the nature of the relation, by which the dependent variable is determined when the independent variable is given. This relation may be as complicated as we please and in theoretical investigations this wide generality is an advantage. But in applications and, in particular, in the differential and integral calculus, the functions with which we have to deal are

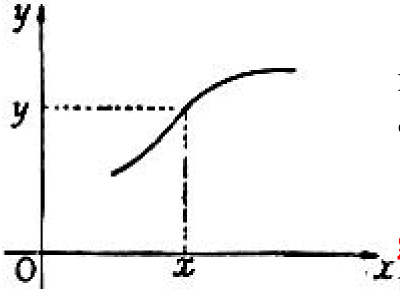


Fig. 2.—Rectangular axes

not of the **widest** generality; on the contrary, the laws of correspondence by which a value of *y* is assigned to each *x* are subject to certain simplifying restrictions.

**1.2.3 Graphical Representation. Continuity. Monotonic Function:** Natural restrictions of the **general function concept** are suggested when we consider the connection with geometry. In fact, the fundamental idea of **analytical geometry** is one of giving a curve, defined by some geometrical property, a characteristic analytical representation by regarding one of the rectangular co-ordinates, say *y*, as a function $y = f(x)$ of the other co-ordinate *x*; for example, a parabola is represented by the function $y = x^2$, the circle with radius 1 about the origin by the two functions $y = \sqrt{(1 - x^2)}$ and $y = -\sqrt{(1 - x^2)}$. In the first example, we may think of the function as defined in the infinite interval $-\infty<x<\infty$; in the second example, we must restrict ourselves to the interval $-1 \leq x \leq 1$, since outside this interval the function has no meaning (when *x* and *y* are real).

Conversely, if instead of starting with a curve which is determined geometrically, we consider a given function $y = f(x)$, we can represent the functional dependence of *y* on *x* graphically by making use of a rectangular co-ordinate system in the usual way (fig.2). If, for each **abscissa** *x*, we mark off the corresponding ordinate $y = f(x)$, we obtain the geometrical representation of the function. The restriction which we now wish to impose on the function concept is: The geometrical representation of the function shall take the form of a reasonable geometrical curve. It is true that this implies a vague general idea rather than a strict mathematical condition. But we shall soon formulate conditions, such as **continuity**, **differentiability**, etc., which will ensure that the graph of a function has the character of a curve capable of being visualized geometrically. At any rate, we shall exclude a function such as the following one: For every rational value of *x*, the function *y* has the value 1, for every irrational value of *x*, the value 0. This assigns a definite value of *y* to each *x*, but in every interval of *x*, no matter how small, the value of *y* jumps from 0 to 1 and back an infinite number of times.

Unless the contrary is expressly stated, it will always be assumed that the law, which assigns a value of the function to each value of *x*, assigns just one value of *y* to each value of *x*, as, for example, $y = x^2$ or $y = \sin x$. If we begin with a geometrically given curve, it may happen, as in the case of the circle $x^2+y^2=1$, that the whole course of the curve is not given by one single (**single-valued**) function, but requires several functions - in the case of the circle, the two functions $y = \sqrt{(1 - x^2)}$ and $y = -\sqrt{(1 - x^2)}$. The same is true for the hyperbola $y^2-x^2=1$, which is represented by the two functions $y = \sqrt{(1+x^2)}$ and $y= -\sqrt{(1+x^2)}$. Hence such curves do not determine the corresponding functions uniquely.

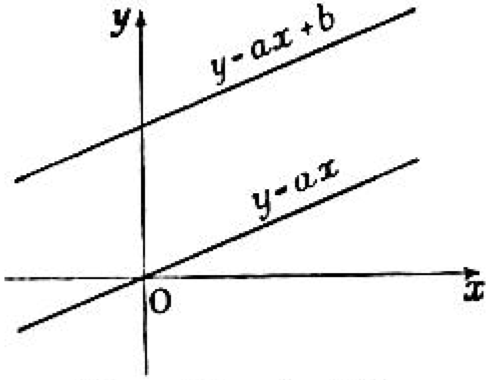



Fig. 5.—Linear functions

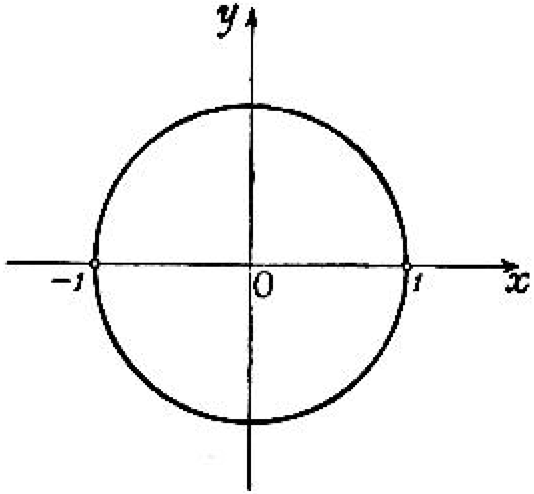



Fig. 3

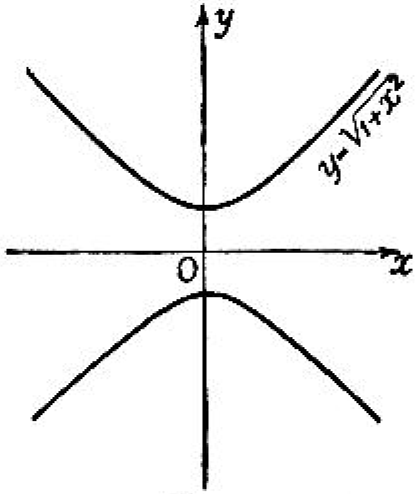



Fig. 4

Multiple-valued functions

Consequently, it is sometimes said that the function corresponding to a curve is **multi-valued**. The separate functions representing a curve are then called the **single-valued branches** belonging to the curve. For the sake of clearness, we shall henceforth use the word **function** to mean a single-valued function. In conformity with this, the symbol $\sqrt{x}$ (for $x \geq 0$) will always denote the **non-negative** number, the square of which is $x$.

If a curve is the geometrical representation of **one** function, it will be cut by any parallel to the $y$-axis in at most **one** point, since there corresponds to each point $x$ in the interval of definition just one value of $y$. Otherwise, as, for example, in the case of the circle, represented by the two functions

$$y = \sqrt{(1 - x^2)} \text{ and } y = -\sqrt{(1 - x^2)},$$

such parallels to the $y$-axis may intersect the curve in more than one point. The portions of a curve corresponding to different single-valued branches are sometimes so interlinked that the complete curve is a single figure which can be drawn with one stroke of the pen, for example, the circle ( Fig. 3), or, on the other hand, the branches may be completely separated, for example, the hyperbola (Fig. 4).

Here follow some more examples of the graphical representation of functions.

(a) $$y = ax.$$

$y$ is proportional to $x$. The graph (Fig. 5) is a straight line through the origin of the co-ordinate system.

$$(b) \qquad y = ax + b.$$

$y$ is a **linear function** of $x$. The graph is a straight line through the point $x = 0$, $y = b$, which, if $a \neq 0$, also passes through the point $x=-b/a$, $y=0$, and, if $a=0$, runs horizontally.

$$(c) \qquad y = \frac{a}{x}.$$

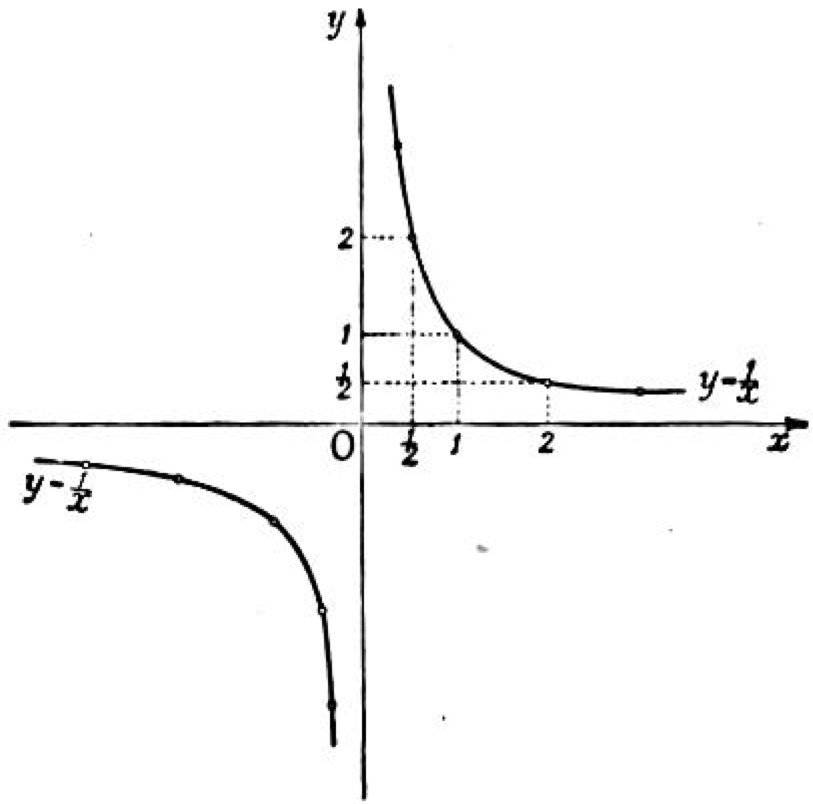



**Fig. 6.—Infinite discontinuities**

$y$ is inversely proportional to $x$. In particular, if $a = 1$, so that

$$y = \frac{1}{x},$$

we find, for example, that

$$y = 1 \text{ for } x = 1,\ y = 2 \text{ for } x = \tfrac{1}{2},\ y = \tfrac{1}{2} \text{ for } x = 2.$$

The graph (Fig. 6) is a curve - a **rectangular hyperbola**, symmetrical with respect to the bi-sectors of the angles between the co-ordinate axes.

This last function is obviously not defined for the value $x = 0$, since division by zero has no meaning. The exceptional point $x = 0$, in the neighbourhood of which there occur arbitrarily large values of the function, both positive and negative, is the simplest example of an **infinite discontinuity**, a subject to which we shall return later.

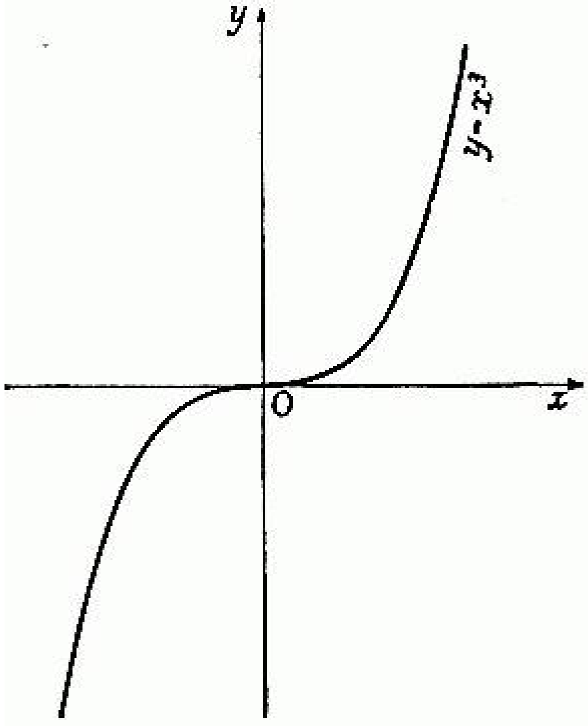


Fig. 8 Cubical parabola

$$(d) \qquad y = x^2.$$

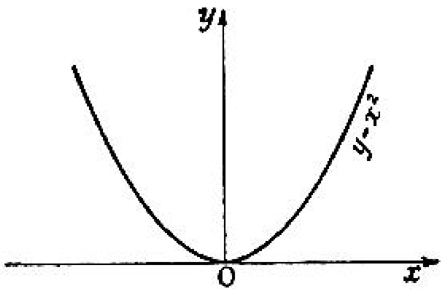


Fig. 7.—Parabola

As is well known, this function is represented by a **parabola** (Fig. 7).

Similarly, the function $y = x^3$ is represented by the so-called **cubical parabola** (Fig. 8).

The Curves just considered and their graphs exhibit a property which is of the greatest importance in the discussion of functions, namely, the **property of continuity**. We shall later analyze this concept in more detail; intuitively speaking, it amounts to: A small change in $x$ causes only a small change in $y$ and not a sudden jump in its value, that is, the graph is not broken off or else the change in $y$ remains less than any arbitrarily chosen positive bound, provided that the change in $x$ is correspondingly small.

A function which for all values of $x$ in an interval has the same value $y = a$ is called a **constant**; it is graphically represented by a horizontal straight line. A function $y = f(x)$ such that throughout the interval, in which it is defined, an increase in the value of $x$ always causes an increase in the value of $y$ is said to be **monotonic increasing**; if, on the other hand, an increase in the value of $x$ always causes a decrease in the value of $y$, the function is said to be

**monotonic decreasing**. Such functions are represented graphically by curves, which in the corresponding interval always rise or always fall (from the left to the right.)( Fig. 9).

If the curve, represented by $y = f(x)$, is symmetrical with respect to the $y$-axis, that is, if $x = -a$ and $x = a$ give the same value for the function, or

$$f(-x) = f(x),$$

we say that the function is **even**. For example, the function $y = x^2$

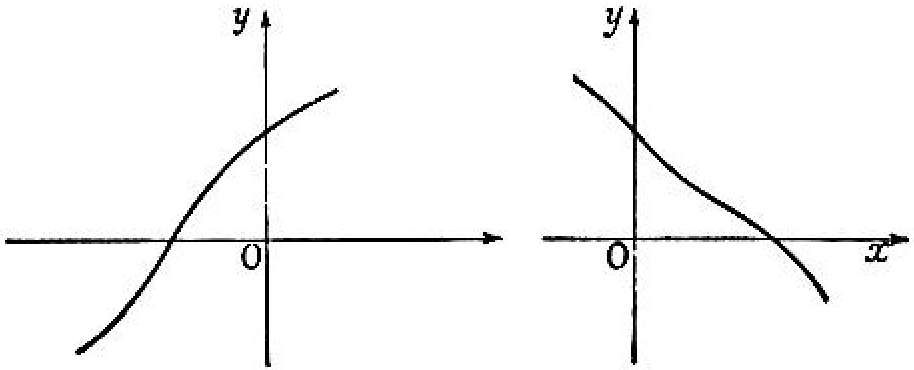



Fig. 9.—Monotonic functions

is even (Fig. 7). On the other hand, if the curve is symmetrical with respect to the origin, that is, if

$$f(-x) = -f(x),$$

it is an **odd function**; for example, the functions $y = x$ and $y = 1/x^3$ (Fig. 8) and $y = 1/x$ are odd.

**1.2.4 Inverse Functions:** Even in our first example, it was made evident that a formal relationship between two quantities may be regarded in two different ways, since it is possible either to consider the first variable to be a function of the second or the second one a function of the first vatiable.For example, if $y=ax+b$, where we assume that $a \neq 0$, $x$ is represented as a function of $y$ by the equation $x = (y - b)/a$. Again, the functional relationship, represented by the equation $y = x^2$, can also be represented by the equation $x = \pm \sqrt{y}$, so that the function $y=x^2$ amounts to the same thing as the two functions $x=\sqrt{y}$ and $x=-\sqrt{y}$. Thus, when an arbitrary function $y = f(x)$ is given, we can attempt to determine $x$ as a function of $y$, or, as we shall say, to replace the function $y=f(x)$ by the **inverse function** $x = \phi(y)$.

Geometrically speaking, this has the meaning: We consider the curve obtained by reflecting the graph of $y = f(x)$ in the line bisecting the angle between the positive $x$-axis and the positive $y$-axis*(Fig. 10). This gives us at once a graphical representation of $x$ as a function of $y$ and thus represents the inverse function $x = \phi(y)$.

* Instead of reflecting the graph in this way, we could first rotate the co-ordinate axes and the curve $y = f(x)$ by a right angle and then reflect the graph in the $x$-axis.

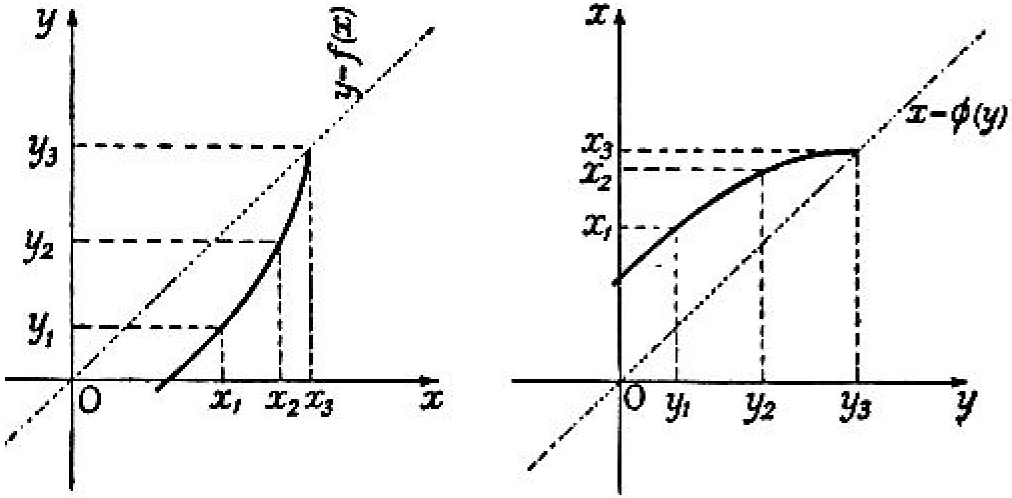



Fig. 10.—Inversion of a function

However, these geometrical ideas show at once that a function $y = f(x)$, defined in an interval, has not a single-valued inverse unless certain conditions are satisfied. If the graph of the function is cut in more than one point by a line $y=c$, parallel to the $x$-axis, the value $y = c$ will correspond to more than one value of x, so that the function cannot have a single-valued inverse. This case cannot occur if $y = f(x)$ is **continuous** and **monotonic**, because then Fig. 10 shows us that there corresponds to each value of $y$ in the interval $y_1\, y\, y_3$ just one value of $x$ in the interval $x_1\, x\, x_3$, and we infer from the figure that a function which is continuous and monotonic in an interval always has a single-valued inverse, and this inverse function is also continuous and monotonic.

## 1.3 More Detailed Study of the Elementary Functions

**1.3.1 The Rational Functions:** We now continue with a brief review of the elementary functions which the reader has already encountered in his previous studies. The simplest types of function are obtained by repeated application of

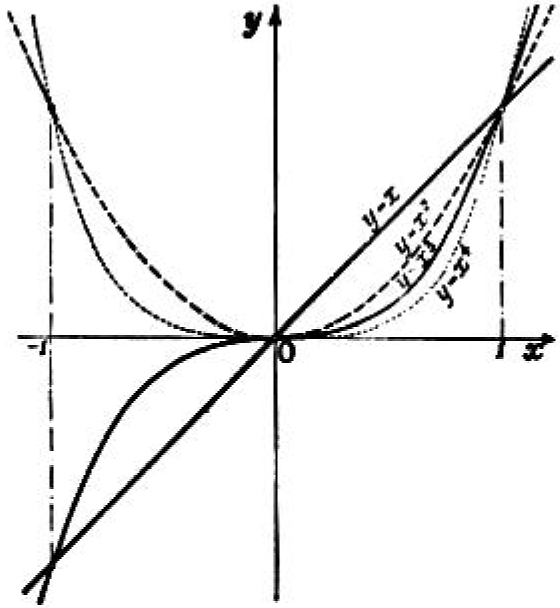


Fig. 11.—Powers of x

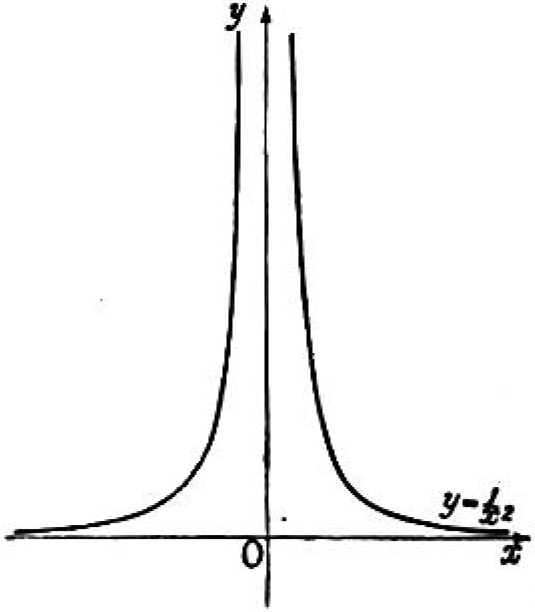


Fig. 12

the elementary operations: **Addition**, **multiplication**, **subtraction**. If we apply these operations to an independent variable *x* and any real numbers, we obtain the **rational, integral functions** or **polynomials**:

$$y = a_0 + a_1 x + \ldots + a_n x^n.$$

The polynomials are the simplest and, in a sense, the basic functions of analysis.

If we now form the quotients of such functions, i.e., expressions of the form

$$y = \frac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_m x^m},$$

we obtain the **general** or **fractional rational functions**, which are defined at all points where the denominator differs from zero.

The simplest rational, integral function is the **linear function**

$$y = ax + b.$$

It is represented graphically by a straight line. Every quadratic function of the form

$$y = ax^2 + bx + c$$

is represented by a parabola. The curves which represent rational integral functions of the third degree

$$y = ax^3 + bx^2 + cx + d,$$

are occasionally called parabolas of the third order or cubical parabolas, and so on.

As examples, we have in Fig. 11 above the graphs of the function $y = x^n$ for the exponents 1, 2, 3, 4 . We see that for even values of $n$, the function $y= x^n$ satisfies the equation $f(- x) = f(x)$, whence it is an **even** function, while for odd values of $n$ it satisfies the condition $f(- x) = -f(x)$ and is an **odd** function.

The simplest example of a rational function which is not a polynomial is $y = 1/x$ (Fig.6); its graph is a rectangular hyperbola. Another example is the function $y =1/x^2$ (Fig. 12 above).

**1.3.2 The Algebraic Functions:** We are at once led away from the domain of the rational functions by the problem of forming their inverses. The most important example of this is the introduction of the function $\sqrt[n]{x}$. We start with the function $y = x^n$, which is monotonic for $x \geq 0$. Hence it has a single-valued inverse, which we denote by the symbol $x = \sqrt[n]{y}$ or, interchanging the letters used for the dependent and independent variables,

$$y = \sqrt[n]{x} = x^{1/n}.$$

In accordance with its definition, this root is always non-negative. In the case of odd $n$, the function $x^n$ is monotonic for all values of x, including negative values, whence, for odd values of $n$, we can also define $\sqrt[n]{x}$ uniquely for all values of $x$; in this case, $\sqrt[n]{x}$ is negative for negative values of $x$.

More generally, we may consider

$$y = \sqrt[n]{R(x)},$$

where $R(x)$ is a **rational function**. We arrive at further functions of a similar type by applying rational operations to one or more of these special functions. Thus, for example, we may form the functions

$$y = \sqrt[m]{x} + \sqrt[m]{(x^2+1)}, \quad y = x + \sqrt{(x^2+1)}.$$

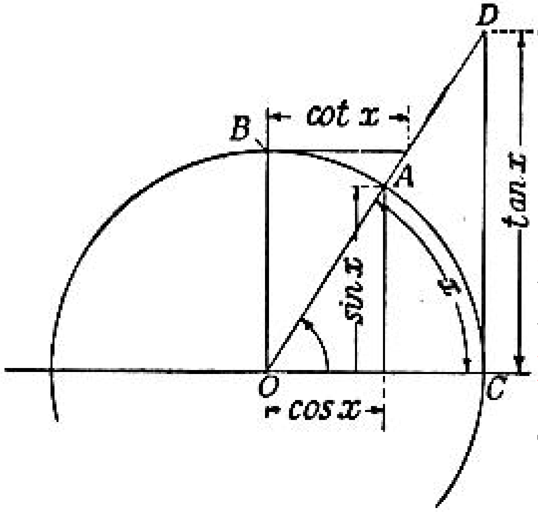


Fig. 13.—The trigonometric functions

These functions are special cases of **algebraic functions**. (The general concept of an algebraic function cannot be defined here; cf. **Chapter X.**)

**1.3.3 The Trigonometric Functions:** While the rational and algebraic functions just considered are defined directly in terms of the elementary, computational operations, **geometry** is the source, from which we first draw our knowledge of the other functions, the so-called **transcendental functions**. We shall here consider the elementary transcendental functions - the trigonometric functions, the **exponential function** and the **logarithm**.

The word **transcendental** does not mean anything particularly deep or mysterious; it merely suggests the fact that the definition of these functions by means of the elementary operations of calculation is not possible, "quod algebrae vires transcendit" (Latin for what exceeds the forces of algebra).

In all higher analytical investigations, where there occur angles, it is customary to measure these angles not in **degrees**, **minutes** and **seconds**, but in **radians**. We place the angle to be measured with its **vertex** at the centre of a circle of radius 1 and measure the size of the angle by the length of the arc of the circumference which the angle cuts out. Thus, an angle of 180° is the same as an angle of $\pi$ radians (has **radian measure** $\pi$), an angle of 90° has radian measure $\pi/2$, an angle of 45° a radian measure $\pi/4$), an angle of 360° a radian measure $2\pi$. Conversely, an angle of 1 radian, expressed in degrees, is

$$\frac{180^\circ}{\pi}, \text{ or approximately } 57^\circ\ 17'\ 45''.$$

From here on, whenever we speak of an angle *x*, we shall mean an angle the radian measure of which is *x*.

After these preliminary remarks, we may briefly remind the reader of the meanings of the **trigonometric functions** sin *x*, cos *x*, tan *x*, cot *x**. These are shown in Fig. 13, in which the angle *x* is measured from the arm *OC* (of length 1), angles being positive in the counter-clockwise direction.

At times, it is convenient to introduce the functions sec *x* = 1/cos *x*, cosec *x* = 1/sin *x*.

The rectangular co-ordinates of the point *A* yields at once the functions cos *x* and sin *x*. The graphs of the functions sin *x*, cos *x*, tan *x*, cot *x* are given in Figs. 14 and 15.

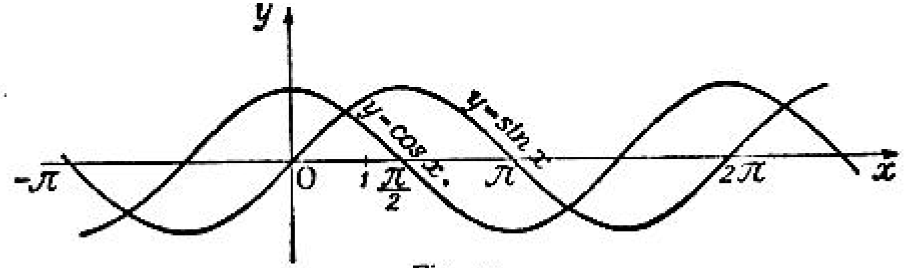



Fig. 14

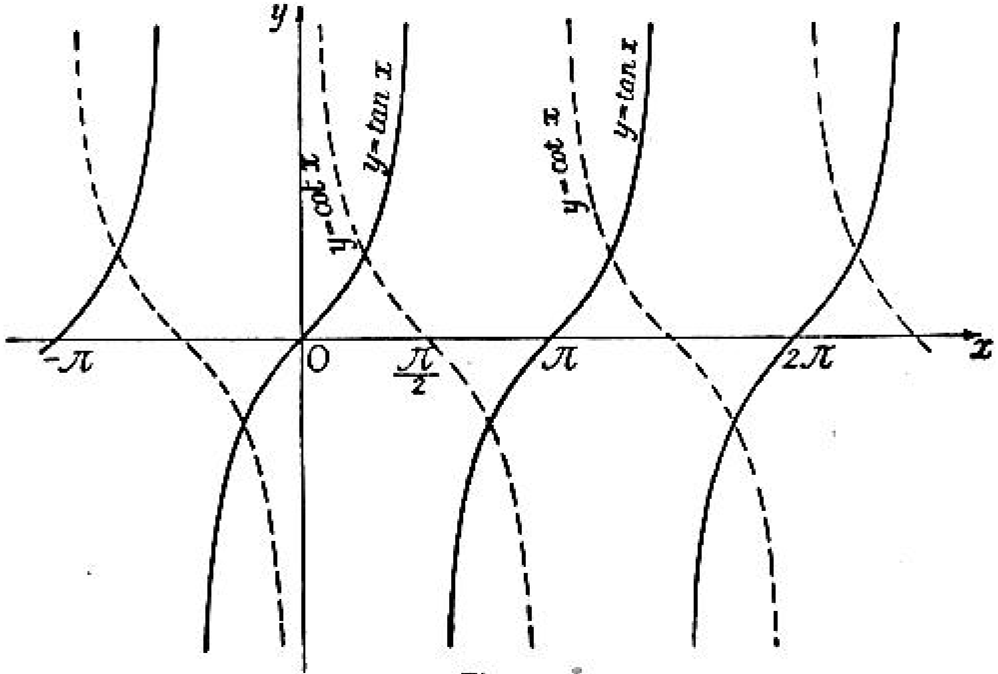



Fig. 15

**1.3.4 The Exponential Function and the Logarithm:** In addition to the trigonometric functions, the exponential function with the positive base *a*,

$$y = a^x,$$

and its inverse, the logarithm to the base *a*,

$$x = \log_a y,$$

are also referred to as **elementary transcendental functions**. In elementary mathematics, it is customary to disregard certain inherent difficulties in the definition of these functions; we too shall postpone the exact discussion of these functions until we have better methods at our disposal (3.6 *et sequ.*). However, we will at least state here the basis of the definitions. If x = $p/q$ is a rational number (where $p$ and $q$ are positive integers), then **-** assuming the number $a$ to be positive **-** we define $a^x$ as $\sqrt[q]{a^p} = a^{p/q}$ , where the root, by convention, is to be taken as positive. Since the rational values of $x$ are everywhere dense, it is natural to extend this function $a^x$ so as to make it into a continuous function, defined also for irrational values of $x$, giving $a^x$ values when $x$ is irrational, which are continuous with the values already defined when $x$ is rational. This yields a continuous function $y = a^x$ **-** the **exponential function** **-** which for all **rational** values of $x$ gives the value of $a^x$ found above. Meanwhile, we will take for granted the fact that this extension is actually possible and can be carried out in only one way; but it must be kept in mind that we still have to prove that this is so (A1.2.5 and 3.6.5).

The function

$$x = \log_a y$$

can then be defined for y > 0 as the **inverse of the exponential function**.

**Exercises 1.2:**

1. Plot the graph of $y = x^3$. Without further calculation, find from this the graph of $y = \sqrt[3]{x}$.
2. Sketch the these graphs and state whether the functions are even or odd:

(a) $y = \sin 2x$.
(b) $y = 5\cos x$.
(c) $y = \sin x + \cos x$.
(d) $y = 2\sin x + \sin 2x$.
(e) $y = \sin(x + \pi)$.
(f) $y = 2\cos\left(x + \frac{\pi}{3}\right)$.
(g) $y = \tan x - x$.

3. Sketch the graphs of the following functions and state whether they are (1) monotonic or not, (2) even or odd.

(a) $y = x^2 \ (-\infty < x < \infty)$.
(b) $y = x^2 \ (0 \leqq x \leqq 1)$.
(c) $y = x \ (-1 \leqq x \leqq 1)$.
(d) $y = |x| \ (-1 \leqq x \leqq 1)$.
(e) $y = \sqrt{x^2} \ (-1 \leqq x \leqq 1)$.
(f) $y = |x - 1| \ (-\infty < x < \infty)$.
(g) $y = |x^2 + 4x + 2| \ (-4 \leqq x \leqq 3)$.
(h) $y = [x] \ (-\infty < x < \infty)$, where $[x]$ means the greatest integer which does not exceed $x$; that is, $[x] \leqq x < [x] + 1$.
(i) $y = x - [x] \ (-\infty < x < \infty)$.
(j) $y = \sqrt{x - [x]} \ (-\infty < x < \infty)$.
(k) $y = x + \sqrt{x - [x]} \ (-\infty < x < \infty)$.
(l) $y = |x - 1| + |x + 1| - 2 \ (-5 \leqq x \leqq 5)$.
(m) $y = |x - 1| - 2|x| + |x + 1| \ (-\infty < x < \infty)$.

Which two of these functions are identical?

4. A body dropped from rest falls approximately 16 $t^2$ ft in $t$ sec. If a ball falls from a window 25 ft. above ground, plot its height above the ground as a function of $t$ for the first 4 sec. after it starts to fall.

Answers and Hints

### 1.4. Functions of an Integral Variable. Sequences of Numbers

Hitherto, we have considered the independent variable as a continuous variable, that is, as varying over a complete interval. However, there occur numerous cases in mathematics in which a quantity depends only on an integer, a number *n* which can take the values 1, 2, 3, ···; it is called a **function of an integral variable**. This idea will most easily be grasped by means of examples:

**Example 1:** The sum of the first *n* integers

$$S_1(n) = 1 + 2 + 3 + 4 + \ldots + n = \tfrac{1}{2}n(n+1)$$

is a function of *n*. Similarly, the sum of the first *n* squares

$$S_2(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2,$$

is a function * of the integer *n*.

* This last sum may easily be represented as a simple rational expression in *n* as follows: We start with the formula

$$(\nu + 1)^3 - \nu^3 = 3\nu^2 + 3\nu + 1,$$

write down this equation for the values ν = 0, 1, 2, ··· , *n* and add them. We thus obtain

$$(n+1)^3 = 3S_2 + 3S_1 + n + 1;$$

on substituting the formula for $S_1$ just given, this becomes

$$3S_2 = (n+1)\left\{(n+1)^2 - 1 - \frac{3}{2}n\right\} = (n+1)\left\{n^2 + \frac{1}{2}n\right\},$$

so that

$$S_2 = \tfrac{1}{6}n(n+1)(2n+1).$$

By a similar process, the functions

$$S_3(n) = 1^3 + 2^3 + \ldots + n^3,$$
$$S_4(n) = 1^4 + 2^4 + \ldots + n^4,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

can be represented as rational functions of *n*.

**Example 2:** Other simple functions of integers are the **factorials**

$$n! = 1.2.3 \ldots n$$

and the **binomial coefficients**

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

for a fixed value of *k*.

**Example 3:** Every whole number $n > 1$, which is not a **prime number**, is divisible by more than two positive integers, while the prime numbers are only divisible by themselves and by 1. Obviously, we can consider the number $T(n)$ of divisors of *n* as a function of the number *n* itself. For the first few numbers, this function is given by

$$\begin{array}{rcccccccccccc} n = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ T(n) = & 1 & 2 & 2 & 3 & 2 & 4 & 2 & 4 & 3 & 4 & 2 & 6 \end{array}$$

**Example 4:** A function of this type, which is of great importance in the theory of numbers, is $\pi(n)$, the number of **primes** which are less than the number *n*. Its detailed investigation is one of the most interesting and attractive problems in the **theory of numbers**. We mention here merely the principal result of these investigations: For large values of *n*, the **number $\pi(n)$** is given approximately by the function * $n/\log n$, where we mean by $\log n$ the logarithm to the natural base *e*, to be defined later on.

* That is, the quotient of the number $\pi(n)$ by the number $n/\log n$ differs arbitrarily little from 1, provided only that *n* is large enough.

As a rule, functions of an integral variable occur in the form of so-called **sequences of numbers**. By a sequence of numbers, we understand an ordered array of infinitely many numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$ (not necessarily all different), determined by any law whatsoever. In other words, we are dealing simply with a function *a* of the integral variable *n*, the only difference being that we are using the **subscript notation** $a_n$ instead of the symbol $a(n)$.

## Exercises 1.3:

1. Prove that

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2.$$

2. From the formula for

$$1^2 + 2^2 + \ldots + n^2,$$

find a formula for

$$1^2 + 3^2 + 5^2 + \ldots + (2n + 1)^2.$$

3. Prove the following **properties of the binomial coefficients**:

$$(a)\ \binom{n}{k} = \binom{n}{n-k} (k \leq n). \quad (b)\ \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \text{ (for } k > 0).$$

$$(c)\ 1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

4. Evaluate the sums:

$$(a)\ 1.2 + 2.3 + \ldots + n(n + 1).$$

$$(b)\ \frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)}.$$

$$(c)\ \frac{3}{1^2.2^2} + \frac{5}{2^2.3^2} + \ldots + \frac{2n+1}{n^2(n+1)^2}.$$

5. A sequence is called an **arithmetic progression of the first order**, if the differences of successive terms are constant, an **arithmetic progression of the second order**, if the differences of successive terms form an arithmetic progression of the first order and, in general, an arithmetic progression of order $k$, if the differences of successive terms form an arithmetic progression of order ($k$ - 1).

The numbers 4, 6, 13, 27, 50, 84 are the first six terms of an arithmetic progression. What is its order? What is the eighth term?

6. Prove that the $n$-th term of an arithmetic progression of the second order can be written in the form $an^2 + bn + c$, where $a, b, c$ are independent of $n$.

7*. Prove that the $n$-th term of an arithmetic progression of order $k$ can be written in the form

$$an^k + bn^{k-1} + \ldots + pn + q,$$

where $a$, $b$, $\cdots$, $p$, $q$ are independent of $n$.
Find the $n$-th term of the progression in 6.

Answers and Hints

## 1.5 The Concept of the Limit of a Sequence

Th concept on which the whole of **analysis** ultimately rests is that of the **limit of a sequence**. We shall first make the position clear by considering several examples.

### 1.5.1 $a_n = 1/n$: We consider the sequence

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \ldots, \quad a_n = \frac{1}{n}, \ldots .$$

No number of this sequence is zero; but we see that the larger the number $n$, the closer to zero is the number $a_n$. Hence, if we mark off around the point 0 an interval as small as we please, then starting with a definite value of the subscript all the numbers $a_n$ will fall into this interval. We express this state of affairs by saying that, as $n$ increases, the numbers $a_n$ tend to 0, or that they possess the limit 0, or that the sequence $a_1$, $a_2$, $a_3$, $\cdots$ converges to 0.

If numbers are represented as points on a line, this means that the points $1/n$ crowd closer and closer to the point 0 as $n$ increases.

The situation is similar in the case of the sequence

$$a_1 = 1, \quad a_2 = -\frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = -\frac{1}{4}, \ldots, \quad a_n = \frac{(-1)^{n-1}}{n}, \ldots .$$

Here too, the numbers $a_n$ tend to zero as $n$ increases; the only difference is that the numbers $a_n$ are sometimes larger and sometimes smaller than the limit 0; as we say, they **oscillate about the limit**.

The **convergence** of the sequence to 0 is usually expressed symbolically by the equation

$$\lim_{n \to \infty} a_n = 0,$$

or occasionally by the abbreviation

$$a_n \to 0.$$

**1.5.2 $a_{2m}$ = 1/m; $a_{2m-1}$ = 1/2m:** In the preceding examples, the absolute value of the difference between $a_n$ and the limit steadily becomes smaller as $n$ increases. This is not necessarily always the case, as is shown by the sequence

$$a_1 = \frac{1}{2}, \quad a_2 = 1, \quad a_3 = \frac{1}{4}, \quad a_4 = \frac{1}{2}, \quad a_5 = \frac{1}{6}, \quad a_6 = \frac{1}{3}, \ldots;$$

that is, in general, for even values $n = 2m$, $a_n = a_{2m}$, $= 1/m$, for odd values $n = 2m - 1$, $a_n = a_{2m-1} = l$. This sequence also has a limit, namely zero, since every interval about the origin, no matter how small, will contain all the numbers $a_n$ from a certain value of $n$ onwards; but it is not true that every number lies nearer to the limit zero than the preceding one.

**1.5.3 $a_n = n/(n + 1)$:** We consider the sequence

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{2}{3}, \ldots, \quad a_n = \frac{n}{n+1}, \ldots.$$

where the integral subscript $n$ takes all the values 1, 2, 3, ··· . If we write $a_n = 1 - 1/(n + 1)$, we see at once that, as $n$ increases, the numbers $a_n$ will approach closer and closer to the number 1, in the sense that, if we mark off any interval about the point 1, all the numbers $a_n$ following a certain $a_N$ must fall into that interval. We write

$$\lim_{n \to \infty} a_n = 1.$$

The sequence

$$a_n = \frac{n^2 - 1}{n^2 + n + 1}$$

behaves in a similar manner. This sequence also tends to a limit as $n$ increases to the limit 1, in fact, in symbols, $\lim_{n \to \infty} a_n = 1.$ We see this most readily, if we write

$$a_n = 1 - \frac{n+2}{n^2+n+1} = 1 - r_n;$$

now we have only to show that the numbers $r_n$ tend to 0 as $n$ increases. For all values of $n$ greater than 2, we have $n + 3 < 2n$ and $n^2 + n + 1 > n^2$. Hence we have for the remainder $r_n$

$$0 < r_n < \frac{2n}{n^2} = \frac{2}{n} \ (n > 2),$$

which shows at once that $r_n$ tends to 0 as $n$ increases. Our discussion yields at the same time an estimate of the amount by which the number $a_n$ (for $n > 2$) can at most differ from the limit 1; this difference certainly cannot exceed 2/n.

This example illustrates the fact, which we should naturally expect, that for large values of $n$ the terms with the highest indices in the numerator and denominator of the fraction for $a_n$ predominate and that they determine the limit.

**1.5.4** $a_n = \sqrt[n]{p}$ **:** Let $p$ be any fixed positive number. We consider the sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$, where

$$a_n = \sqrt[n]{p}.$$

We assert that

$$\lim_{n\to\infty} a_n \equiv \lim_{n\to\infty} \sqrt[n]{p} = 1.$$

We can prove this very easily by using a lemma which we shall find also useful for other purposes.

If $1 + h$ is a positive number (that is, if $h > -1$) and $n$ is an integer greater than 1, then

$$(1+h)^n > 1 + nh. \quad \ldots\ldots \quad (1)$$

Assume that Inequality (1) already has been proved for a certain $m > 1$; multiply both sides by $(1+h)$ and obtain

$$(1+h)^{m+1} > (1+mh)(1+h) = 1 + (m+1)h + mh^2.$$

If we omit on the right hand side the positive term $mh^2$, the inequality remains valid. We thus obtain

$$(1 + h)^{m+1} > 1 + (m + 1)h.$$

However, this is our inequality for the index $m + 1$. Hence, if the inequality holds for the index $m$, is also holds for $m + 1$. Since it holds for m= 2, it also holds for $m = 3$, whence for $m = 4$, and so on, whence it holds for every index. This is a simple example of a proof by **mathematical induction**, a type of proof which is often useful.

Returning to our sequence, we distinguish between the case $p > 1$ and the case $p < 1$ ( if $p = 1$, then $\sqrt[n]{p}$ is also equal to 1 for every $n$ and our statement becomes trivial).

If $p > 1$, then $\sqrt[n]{p}$ will also be greater than 1; let $\sqrt[n]{p} = 1 + h_n$, where $h_n$ is a positive quantity depending on $n$ and we find by Inequality (1)

$$p = (1 + h_n)^n > 1 + nh_n,$$

whence follows at once that

$$0 < h_n < \frac{p - 1}{n}.$$

Thus, as n increases, the number $h_n$ must tend to 0, which proves that the numbers $a_n$ converge to the limit 1, as stated. At the same time, we have a means for estimating how close any $a_n$ is to the limit 1; the difference between $a_n$ and 1 is certainly not greater than $(p — 1)/n$.

If $p < 1$, then $\sqrt[n]{p}$ will likewise be less than 1 and therefore may be taken equal to $1/(1 + h_n)$, where $h_n$ is a positive number. It follows from this, using Inequality (1), that

$$p = \frac{1}{(1 + h_n)^n} < \frac{1}{1 + nh_n}.$$

(By decreasing the denominator, we increase the fraction. It follows that

$$1 + nh_n < \frac{1}{p},$$

whence

$$h_n < \frac{1/p - 1}{n}.$$

This shows that $h_n$ tends to 0 as $n$ increases. As the reciprocal of a quantity tending to 1, $\sqrt[n]{p}$ itself tends to 1.

### 1.5.5 $a_n = \alpha^n$:

We consider the sequence $a_n = \alpha^n$, where $\alpha$ is fixed and $n$ runs through the sequence of positive integers.

First, let $\alpha$ be a positive number less than 1. We may then put $\alpha = 1/(1 + h)$, where $h$ is positive and Inequality (1) yields

$$a_n = \frac{1}{(1+h)^n} < \frac{1}{1+nh} < \frac{1}{nh}.$$

Since the number $h$ and, consequently, $1/h$ depends only on $n$ and does not change as $n$ increases, we see that as $n$ increases $\alpha^n$ tends to 0:

$$\lim_{n \to \infty} \alpha^n = 0 \qquad (0 < \alpha < 1).$$

The same relationship holds when $\alpha$ is zero or negative, but greater than - 1. This is immediately obvious, since in any case

$$\lim_{n \to \infty} |\alpha|^n = 0.$$

If $\alpha = 1$, then $\alpha^n$ will obviously be always equal to 1 and we shall have to regard the number 1 as the limit of $\alpha^n$.

If $\alpha > 1$, we set $\alpha = 1 + h$, where $h$ is positive and see at once from our inequality that, as $n$ increases, $\alpha^n$ does not tend to any definite limit, but increases beyond all bounds. We express this state of affairs by saying that $\alpha^n$ tends to infinity as $n$ increases, or that $\alpha^n$ becomes infinite; in symbols

$$\lim_{n \to \infty} \alpha^n = \infty \qquad (\alpha > 1).$$

Nevertheless, as **we must explicitly emphasize**, the symbol $\infty$ does not denote a number with which we can calculate as with any other number; equations or statements which express that a quantity is or becomes infinite never have the same sense as an equation between definite quantities. In spite of this, such modes of expression and the use of the **symbol $\infty$** are extremely convenient, as we shall often see in the following pages.

*If* $\alpha = -1$, the values of $\alpha^n$ will not tend to any limit, but, as $n$ runs through the sequence of positive integers, it will take the values +1 and - 1 alternately. Similarly, if $\alpha < -1$, the value of $\alpha^n$ will increase numerically beyond all bounds, but its sign will be in sequence positive and. negative.

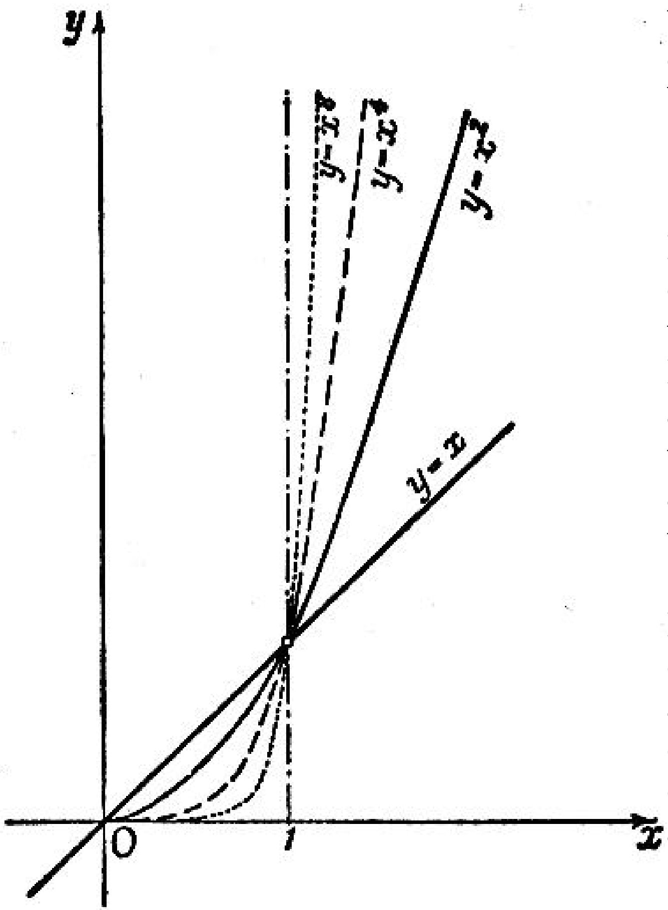



Fig. 16.—$x^n$ as $n$ increases

### 1.5.6. Geometrical Illustration of the Limits of $\alpha^n$ and $\sqrt[n]{p}$:

If we consider the curves $\mathbf{y = x^n}$ and $y = x^{1/n} = \sqrt[n]{x}$ and restrict ourselves, for the sake of convenience, to non-negative values of $x$, the preceding limits are illustrated by Figs. 16 and. 17, respectively. In the case of the curves y = $x^n$, we see that in the interval 0 to 1 they approach closer and closer to the $x$-axis as $n$ increases, while outside that interval they climb more and more steeply and draw in closer and closer to a line parallel to the $y$-axis. All the curves pass through the point with co-ordinates $x$=1, $y$=1 and through the origin.

In the case of the functions $y = x^{1/n} = +\sqrt[n]{x}$, the curves approach closer and closer to the line parallel to the $x$-axis and at a distance 1 above it. On the other hand, all the curves must pass through the origin. Hence, in the limit, the curves approach the broken line consisting of the part of the $y$-axis between the points $y$ = 0 and $y$ = 1 and of the parallel to the $x$-axis y = 1. Moreover, it is clear that the

two figures are closely related, as one would expect from the fact that the functions $y = \sqrt[n]{x}$ are actually the inverse functions of the *n*-th powers, from which we infer that each figure is transformed into the other on reflection in the line $y = x$.

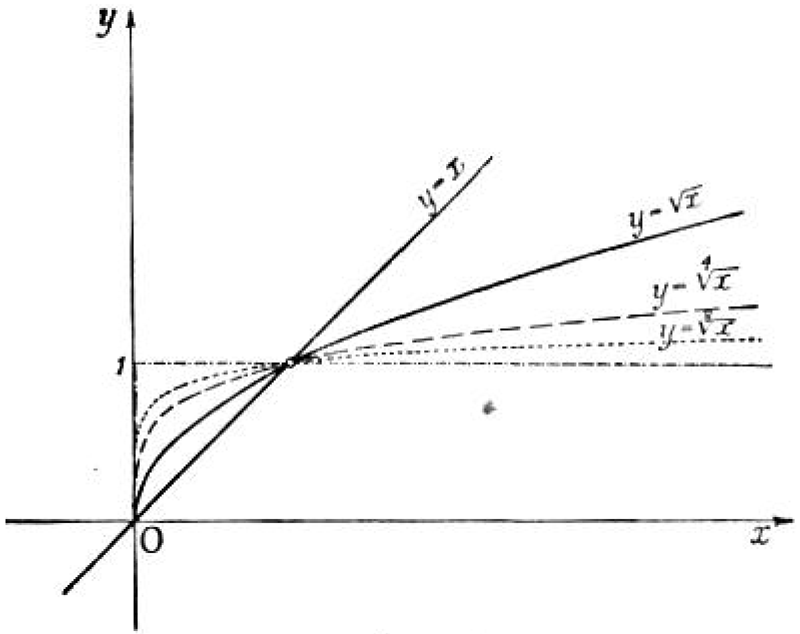



Fig. 17.—$x^{1/n}$ as $n$ increases

**1.5.7 The Geometric Series:** An example of a limit which is more or less familiar from elementary mathematics is the geometric series

$$1 + q + q^2 + \ldots + q^{n-1} = S_n;$$

the number $q$ is called the **common ratio** of the series. The value of this sum may, as is well known, be expressed in. the form

$$S_n = \frac{1 - q^n}{1 - q},$$

provided that $q \neq 1$; we can derive this expression by multiplying the sum $S_n$ by $q$ and subtracting the equation thus obtained from the original equation, or we may verify the formula by division.

There arises now the question what happens to the sum $S_n$ when $n$. increases indefinitely? The answer is: The sum $S_n$ has a definite limit $S$ if $q$ lies between -1 and +1, these end values being excluded, and it is then true that

$$S = \lim_{n \to \infty} S_n = \frac{1}{1-q}.$$

In order to verify this statement, we write the numbers $S_n$ in the form

$$S_n = \frac{1-q^n}{1-q} = \frac{1}{1-q} - \frac{q^n}{1-q}.$$

We have already shown that, provided $|q| < 1$, the quantity $q^n$ and with it $q^n/(1 - q)$ tends to 0 as $n$ increases; hence, with the above assumption, the number $S_n$ tends, as was stated, to the limit $1/(1 - q)$ as $n$ increases.

The passage to the limit

$$\lim_{n \to \infty} (1 + q + q^2 + \ldots + q^{n-1}) = \frac{1}{1-q}$$

is usually expressed by saying that when $|q| < 1$, the geometric series can be extended to infinity and that the sum of the infinite geometric series is the expression $1/(1 - q)$.

The sums $S_n$ of the finite geometric series are also called the **partial sums** of the infinite geometric series $1 + q + q^2 + \cdots$. (We must draw a sharp distinction between the sequence of numbers $S_1, S_2, \cdots$, and the geometric series.)

The fact that the partial sums $S_n$ of a geometric series tend to the limit $S = 1/(1 - q)$ as $n$ increases may also be expressed by saying that the infinite geometric series $1 + q + q^2 + \cdots$ converges to the sum $S = 1/(1 - q)$ when $|q| < 1$.

## 1.5.8 $a_n = \sqrt[n]{n}$ :

We shall show that the sequence of numbers

$$a_1 = 1, \quad a_2 = \sqrt{2}, \quad a_3 = \sqrt[3]{3}, \quad \ldots, \quad a_n = \sqrt[n]{n}, \quad \ldots$$

tends to 1 as $n$ increases, i.e., that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Here we make use of a slight artifice. Instead of the sequence $a_n = \sqrt[n]{n}$, we first consider the sequence $b_n = \sqrt{a_n} = \sqrt{\sqrt[n]{n}} = \sqrt[n]{\sqrt{n}}$. When $n > 1$, the term $b_n$ is also greater than 1. We can therefore set $b_n = 1 + h_n$, where $h_n$ is positive and depends on $n$. By Inequality (l), we therefore have

$$\sqrt{n} = (b_n)^n = (1 + h_n)^n \geqq 1 + nh_n,$$

so that

$$h_n \leqq \frac{\sqrt{n} - 1}{n} \leqq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}.$$

We now have

$$1 \leqq a_n = b_n^2 = 1 + 2h_n + h_n^2 \leqq 1 + \frac{2}{\sqrt{n}} + \frac{1}{n}.$$

Obviously, the right hand side of this inequality tends to 1, and so does $a_n$.

## 1.5.9 $a_n = \sqrt{n+1} - \sqrt{n}$ :

We assert that

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0.$$

In order to prove this formula, we need only write this expression in the form

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}};$$

we see at once that this expression tends to 0 as $n$ increases.

**1.5.10** $a_n = \frac{n}{\alpha^n}$ **:** Let $\alpha$ be a number greater than **1.** We assert that as $n$ increases the sequence of the numbers $a_n = n/a_n$ tends to the limit 0**.**

As in the case of $\sqrt[n]{n}$ above, we consider the sequence

$$\sqrt{a_n} = \frac{\sqrt{n}}{(\sqrt{\alpha})^n}.$$

We set $\sqrt{\alpha} = 1 + h$. Here $h > 0$, since $\alpha$ and hence $\sqrt{\alpha}$ is greater than 1. By Inequality (1), we have

$$\sqrt{\alpha^n} = (1 + h)^n > 1 + nh,$$

so that

$$\sqrt{a_n} = \frac{\sqrt{n}}{(1+h)^n} \leqq \frac{\sqrt{n}}{1+nh} \leqq \frac{\sqrt{n}}{nh} = \frac{1}{h\sqrt{n}}.$$

Hence

$$a_n \leqq \frac{1}{nh^2}.$$

## Exercises 1.4:

1.Prove that

$$\lim_{n\to\infty} \frac{n^2+n-1}{3n^2+1} = \frac{1}{3}.$$

Find an $N$ such that for $n > N$ the difference between $\frac{n^2+n-1}{3n^2+1}$ and $\frac{1}{3}$ is ($a$) less than 1/10, ($b$) less than 1/1,000, ($c$) lees than 1/1,000,000.

2. Find the limits of the following expressions as $n \to \infty$:

$$(a)\ \frac{n^5+3n+1}{n^6+7n^2+2}. \qquad (b)\ \frac{n^6+3n+1}{n^5+7n^2+2}.$$

$$(c)\ \frac{6n^2+2n+1}{n^3+n^2}. \qquad (d)\ \frac{a_0n^k+a_1n^{k-1}+\dots+a_k}{b_0n^k+b_1n^{k-1}+\dots+b_k}.$$

$$(e)\ \frac{\sum_{k=1}^{n} k^2}{n^3}.$$

3. Prove that $\lim_{n\to\infty} \sqrt[n]{n^2} = 1.$

4. Prove that $\lim_{n\to\infty} \frac{n^2}{2^n} = 0.$ Find an $n$ such that $n^2/2n < 1/1{,}000$ whenever $n > N$.

5. Find numbers $N_1$, $N_2$, $N_3$ such that

$$(a)\ \frac{n}{2^n} < \frac{1}{10} \text{ for every } n > N_1;$$

$$(b)\ \frac{n}{2^n} < \frac{1}{100} \text{ for every } n > N_2;$$

$$(c)\ \frac{n}{2^n} < \frac{1}{1000} \text{ for every } n > N_3.$$

6. Do the same thing for the sequence $a_n = \sqrt{n+1} - \sqrt{n}.$

7. Prove that $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})(\sqrt{n} + \tfrac{1}{2}) = \tfrac{1}{2}.$

8. Prove that $\lim_{n\to\infty} (\sqrt[3]{n+1} - \sqrt[3]{n}) = 0.$

9. Let $a_n = 10^n/n!$.
(*a*) To what limit does $a_n$ converge?
(*b*) Is the sequence monotonic?
(*c*) Is it monotonic from a certain $n$ onwards?

(d) Give an estimate of the difference between $a_n$ and the limit.
(e) From what value of $n$ onwards is this difference less than 1/100?

10. Prove that $\lim_{n\to\infty} \frac{n!}{n^n} = 0.$

11. Prove that $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n}{n^2}\right) = \frac{1}{2}.$

12. Prove that $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2}\right) = 0.$

13. Prove that $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \dots + \frac{1}{\sqrt{2n}}\right) = \infty.$

14.* Prove that $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$

15. Prove that if $a$ and b $\le a$ are positive, the sequence $\sqrt[n]{a^n + b^n}$ converges to $a$. Similarly, for any $k$ fixed positive numbers $a_1$, $a_2$, ⋯ , $a_k$, prove that $\sqrt[n]{a_1^n + a_2^n + \dots + a_k^n}$ converges and find its limit.

16. Prove that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$ converges. Find its limit.

17.* If $\nu(n)$ is the number of distinct prime factors of $n$, prove that $\lim_{n\to\infty} \frac{\nu(n)}{n} = 0.$

Answers and Hints

## 1.6 Further Discussion of the Concept of Limit

**1.6.1 First Definition of Convergence:** The examples discussed in the last section guide us to the general concept of limit:

If an infinite sequence of numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$ is given and if there is a number $\boldsymbol{l}$ such that every interval, however small, marked off about the point $\boldsymbol{l}$, contains all the points $a_n$, except for at most a finite number, we say that the number $\boldsymbol{l}$ is the limit of the sequence $a_1, a_2, \cdots$ or that the sequence $a_1, a_2, \cdots$ converges to $\boldsymbol{l}$; in symbols, $\lim_{n \to \infty} a_n = l$. Here we expressly remark that this includes the trivial case in which all the numbers $a_n$ are equal to one another, and hence also coincide with the limit.

Instead of the above, we may use the following equivalent statement: If any positive number $\varepsilon$ be assigned - however small - one can find a whole number $N=N(\varepsilon)$ such that from $N$ onward (i.e., for $n > N(\varepsilon)$), it is always true that $|a_n - l| < \varepsilon$. Of course, it is as a rule true that the bound $N(\varepsilon)$ will have to be chosen larger and larger with smaller and smaller values of $\varepsilon$; in other words, $N(\varepsilon)$ will increase beyond all bounds as $\varepsilon$ tends to 0.

It is important to remember that every convergent sequence is **bounded**, that is, there corresponds to every sequence $a_1, a_2, a_3, \cdots$, for which a limit $l$ exists, a positive number $M$, independent of $n$, such that for all the terms $a_n$ of the sequence the inequality $|a_n| < M$ is valid.

This theorem readily follows from our definition. We choose $\varepsilon$ equal to 1; then there is an index $N$ such that $|a_n - l| < 1$ for $n > N$. Let $A$ be the largest among the numbers

$$|a_1 - l|, \ |a_2 - l|, \ \ldots, \ |a_N - l|.$$

We can then put $M = [l] + A + 1$. Since, by the definition of $A$, the inequality $|a_n - l| < A + 1$ certainly holds for n= 1, 2, $\cdots$, $N$, while for $n > N$

$$|a_n - l| < 1 \leqq A + 1.$$

A sequence which does not converge is said to be **divergent**. If, as $n$ increases, the numbers $a_n$ increase beyond all bounds, we say that the sequence diverges to $+\infty$ and, as we have already done occasionally, we write $\lim_{n \to \infty} a_n = \infty$. Similarly, we write $\lim_{n \to \infty} a_n = -\infty$, if, as n increases, the numbers $-a_n$ increase beyond all bounds in the positive direction. But divergence may manifest itself in other ways, as, for example, in the case of the sequence $a_1 = -1, \ a_2 = +1, \ a_3 = -1, \ a_4 = +1, \ \ldots$, the terms of which oscillate between two different values.*

*Another useful remark: As regards convergence, the behaviour of a sequence is unaltered, if we omit a. finite number of the terms $a_n$. In what follows, we shall frequently use this, speaking of the convergence or divergence of series in which the term $a_n$ is undefined for a finite number of values of $n$.

In all the examples given above, the limit of the sequence considered is a known number. If the concept of limit were to yield nothing more than the recognition that certain known numbers can be approximated as closely as we like by certain sequences of other known numbers, we should have gained very little. The fruitfulness of the **concept of limit** in analysis rests essentially on the fact that limits of sequences of known numbers provide a means of dealing with other numbers, which are not directly known or expressible. The whole of higher analysis consists of a succession of examples of this fact, which will become steadily clearer to us in the following chapters. The representation of the irrational numbers as limits of rational numbers may be regarded as a first example. In this section, we shall become acquainted with further examples. However, before we take up this subject, we shall make a few general, preliminary remarks.

**1.6.2 Second (Intrinsic) Definition of Convergence:** How can we tell that a given sequence of numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$ converges to a limit, even when we do not know beforehand what that limit is? This important question is answered by **Cauchy's convergence test**.*

It is sometimes referred to as the **general principle of convergence**.

We say that a sequence of numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$ is convergent if there corresponds to every arbitrarily selected small positive number $\varepsilon$ a number $N=N(\varepsilon)$, which usually depends on $\varepsilon$, such that $|a_n - a_m| < \varepsilon$, provided that $n$ and $m$ are both at least equal to $N(\varepsilon)$. Cauchy's convergence test can then be expressed as follows:

Every intrinsically convergent sequence of numbers possesses a limit.

The importance of Cauchy's test lies in the fact that it allows to speak of the limit of a sequence after considering the sequence itself without any further information about the limit itself. The converse of Cauchy's test is very easy to prove. For, if the sequence $a_1, a_2, \cdots$ tends to the limit $l$, then, by the definition of convergence, we have

$$|l - a_n| < \frac{\epsilon}{2} \quad \text{and} \quad |l - a_m| < \frac{\epsilon}{2},$$

where $\varepsilon$ is a positive quantity as small as we please, provided only that both $m$ and $n$ are large enough, whence

$$|a_n - a_m| = |(l - a_m) - (l - a_n)| \leq |l - a_m| + |l - a_n| < \epsilon.$$

Since $\varepsilon$ can be chosen as small as we please, this inequality expresses our statement.

Cauchy's test itself becomes intuitively obvious, if we think of the numbers as they are represented on the number axis. It then states that a sequence certainly has a limit, if after a certain point $N$ all the terms of the sequence are restricted to an interval which can be made arbitrarily small by choosing $N$ large enough.

In **Appendix 1**, we shall show how Cauchy's test can be proved by purely analytical methods. For the time being, we will accept it as a postulate.

**1.6.3 Monotonic Sequence:** The question whether a given sequence converges to a limit is particularly easy to answer when the sequence is a so-called monotonic sequence, that is, if either every number of the sequence is larger than the preceding number (**monotonic increasing**) or else every number is smaller than the preceding number (**monotonic decreasing**). We have the **theorem:**

Every monotonic increasing sequence the terms of which are bounded above (that is, lie below a fixed number) possesses a limit; similarly, every monotonic decreasing sequence the terms of which never drop below a certain fixed bound possesses a limit.

For the present, we shall regard these results as obvious, merely referring the student to the rigorous proof in **Appendix 1**. Naturally, a convergent, monotonic, increasing sequence must tend to a limit which is greater than any term of the sequence, while for a convergent monotonic decreasing sequence the numbers tend to a limit which is smaller than any number of the sequence. Thus, for example, the numbers $1/n$ form a monotonic decreasing sequence with the limit 0, while the numbers $1 - 1/n$ form a monotonic increasing sequence with the limit 1.

In many cases, it is convenient to replace the condition that a sequence shall increase monotonically by the weaker condition that its terms shall never decrease, in other words, to allow its successive terms to be equal to one another. We then speak of a **monotonic non-decreasing sequence**, or of a **monotonic increasing sequence in the wider sense**. Our theorem on limits remains true for such sequences as well as for sequences which are monotonic non-increasing or monotonic decreasing in the wider sense.

**1.6.4 Operations with Limits:** We conclude with a remark concerning calculations with limits. It follows almost at once from the definition of limit that we can perform the elementary operations of addition, multiplication, subtraction, and division according to the rules:

If $a_1, a_2, \cdots$ is a sequence with the limit $a$ and $b_1, b_2, \cdots$ a sequence with the limit $b$, then the sequence of numbers $c_n = a_n + b_n$ also has a limit and

$$\lim_{n \to \infty} c_n = a + b.$$

The sequence of numbers $c_n = a_n \cdot b_n$ likewise converges, and

$$\lim_{n \to \infty} c_n = ab.$$

Similarly, the sequence $c_n = a_n - b_n$ converges, and

$$\lim_{n \to \infty} c_n = a - b.$$

Provided the limit $b$ differs from 0, the numbers $c_n = a_n/b_n$ also converge, and have the limit

$$\lim_{n \to \infty} c_n = \frac{a}{b}.$$

In words: we can interchange the rational operations of calculation with the process of forming a limit, that is, we obtain the same result whether we first perform a passage to the limit and then a rational operation or *vice versa*.

For the proof of these simple rules, it is sufficient to give one example; using this as a model, the reader can establish the other statements by himself. As this example, we consider **multiplication of limits**. The relations $a_n \to a$ and $b_n \to b$ amount to the following: If we choose any positive number $\varepsilon$, we need only take $n$ greater than $N$, where $N = N(\varepsilon)$ is a sufficiently large number depending on $\varepsilon$, in order to have both

$$|a - a_n| < \epsilon \quad \text{and} \quad |b - b_n| < \epsilon.$$

If we write $ab - a_n b_n = b(a - a_n) + a_n(b - b_n)$ and recall that there is a positive bound $M$, independent of $n$, such that $|a_n| < M$, we obtain

$$|ab - a_n b_n| \leq |b||a - a_n| + |a_n||b - b_n| < (|b| + M)\epsilon.$$

Since the quantity $(|b| + M)\varepsilon$ can be made arbitrarily small by choosing $\varepsilon$ small enough, we see that the difference between $ab$ and $a_n b_n$ actually becomes as small as we please for all sufficiently large values of $n$, which is precisely the statement made in the equation

$$ab = \lim_{n \to \infty} a_n b_n.$$

By means of these rules, many limits can be evaluated very easily; for example, we have

$$\lim_{n \to \infty} \frac{n^2 - 1}{n^2 + n + 1} = \lim_{n \to \infty} \frac{1 - \frac{1}{n^2}}{1 + \frac{1}{n} + \frac{1}{n^2}} = 1,$$

since in the second expression the passage to the limit in the numerator and denominator can be made directly.

Another simple and obvious rule is worth stating. If $\lim a_n = a$ and $\lim b_n = b$, and if, in addition, $a_n > b_n$ for every $n$, then $a \geq b$. However, we are by no means entitled to expect that, in general, $a$ will be greater than $b$, as is shown by the case of the sequences $a_n = 1/n$, $b_n = 1/2n$, for which $a = 0 = b$.

**1.6.5 The Number *e*:** As a first example of the generation of a number, which cannot be stated in advance as the limit of a sequence of known numbers, we consider the sums

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}.$$

We assert that, as $n$ increases, $S_n$ tends to a definite limit.

In order to prove the existence of the limit, we observe that, as $n$ increases, the numbers $S_n$ increase monotonically. For all values of $n$, we also have

$$S_n \leqq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} = 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} < 3.$$

Hence the numbers $S_n$ have the upper bound 3 and, being a monotonic increasing sequence, they possess a limit, which we denote by $e$:

$$e = \lim_{n \to \infty} S_n.$$

Moreover, we assert that the number $e$, defined as the above limit, is also the limit of the sequence

$$T_n = \left(1 + \frac{1}{n}\right)^n.$$

The proof is simple and at the same time an instructive example of operations with limits. According to the **binomial theorem**, which we shall here assume,

$$\begin{aligned} T_n &= \left(1 + \frac{1}{n}\right)^n \\ &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \dots + \frac{n(n-1)(n-2)\dots 1}{n!}\frac{1}{n^n} = \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \dots = \\ &\quad + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$

Hence we see at once: (1) that $T_n \leq S_n$, and (2) that the $T_n$ also form a monotonic increasing sequence*, whence there follows the existence of the limit $\lim_{n \to \infty} T_n = T$. In order to prove that $T = e$, we observe that

$$T_m > 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{m}\right) + \dots + \frac{1}{n!}\left(1 - \frac{1}{m}\right)\dots\left(1 - \frac{n-1}{m}\right),$$

provided that $m > n$. If we now keep $n$ fixed and let $m$ increase beyond all bounds, we obtain on the left hand side the number $T$ and on the right hand side the expression $S_n$, so that $T \geq S_n$. We have thus established the relationship

$T \geq S_n \geq T_n$ for every value of $n$. We can now let $n$ increase, so that $T_n$ tends to $T$; it follows from the double inequality that $T = \lim_{n\to\infty} S_n = e$, as was to be proved.

* We obtain $T_{n+1}$ from $T_n$ by replacing the factors 1-1/$n$, 1-2/$n$,··· by the larger factors 1-1/($n$+1), 1-1/($n$+1), ··· and finally adding a positive term.

We shall later meet this number $e$ again from still another point of view.

**1.6.6 The number $\pi$ as a limit:** A limiting process, which in essence goes back to classical antiquity (**Archimedes**), is that by which the number $\pi$ is defined. Geometrically speaking, $\pi$ means the area of the circle of radius 1. We therefore accept the existence of this number as intuitive, regarding it as obvious that this area can be expressed by a (rational or irrational) number, which we then simply denote by $\pi$. However, this definition is not of much help to us, if we wish to calculate the number with any accuracy. We have then no choice but to represent it by means of a limiting process, namely, as the limit of a sequence of known and easily calculated numbers. Archimedes himself used this process in his **method of exhaustions**, where he steadily approximated the circle by means of regular polygons with an increasing number of sides fitting it more and more closely. If we denote by $f$ the area of the regular polygon with $m$ sides, inscribed in the circle, then the area of the inscribed 2$m$ polygon is given by the formula (proved by elementary geometry)

$$f_{2m} = \frac{m}{2}\sqrt{2 - 2\sqrt{1 - \left(\frac{2f_m}{m}\right)^2}}.$$

We now let $m$ run, not through the sequence of all positive integers, but through the sequence of powers of 2, that is, $m = 2^n$; in other words, we form those regular polygons the vertices of which are obtained by repeated bisection of the circumference. The area of the circle is then given by the limit

$$\pi = \lim_{n\to\infty} f_{2^n}.$$

In fact, this representation of $\pi$ as a limit serves as a base for numerical computations; starting with the value $f_4 = 2$, we can calculate in order the terms of our sequence tending to $\pi$. An estimate of the accuracy with which any term $f_{2^n}$ represents $\pi$ can be obtained by constructing the lines touching the circle and parallel to the sides of the

inscribed $2^n$ polygon. These lines form a circumscribed polygon, similar to the inscribed $2^n$ polygon, and have greater dimensions in the ratio 1: $\cos\pi/2^{n-1}$, whence the area $F_{2^n}$ of the circumscribed polygon is given by

$$\frac{f_{2^n}}{F_{2^n}} = \left(\cos\frac{\pi}{2^{n-1}}\right)^2.$$

Since the area of the circumscribed polygon is evidently larger than that of the circle, we have

$$f_{2^n} < \pi < F_{2^n} = \frac{f_{2^n}}{\left(\cos\frac{\pi}{2^{n-1}}\right)^2}.$$

These are matters with which the reader will be more or less familiar. What we wish to point out here is that the calculation of areas by means of exhaustion by rectilinear figures, the areas of which are readily calculated, forms the basis for the concept of integral to be introduced in the next chapter.

## Exercises 1.5:

1.* (*a*) Replace the statement the sequence $a_n$ is not absolutely bounded by an equivalent statement not involving any form of the words bounded or unbounded.
(*b*) Replace the statement the sequence $a_n$ is divergent by an equivalent statement not involving any form of the words convergent or divergent.

2.* Let $a_1$ and $b_1$ be two positive numbers and $a_1 < b_1$. Let $a_2$ and $b_2$ be defined by the equations

$$a_2 = \sqrt{a_1 b_1}, \quad b_2 = \frac{a_1 + b_1}{2}.$$

Similarly, let

$$a_3 = \sqrt{a_2 b_2}, \quad b_3 = \frac{a_2 + b_2}{2},$$

and, in general,

$$a_n = \sqrt{a_{n-1}b_{n-1}}, \quad b_n = \frac{a_{n-1} + b_{n-1}}{2}.$$

Prove (*a*) that the sequence an $a_1, a_2, \cdots$ converges, (*b*) that the sequence $b_1, b_2, \cdots$ converges, (*c*) that the two sequences have the same limit. (This limit is called the arithmetic-geometric mean of $a_1$ and $b_1$.)

3.* Prove that if $\lim_{n\to\infty} a_n = \xi,$ then $\lim_{n\to\infty} \sigma_n = \xi,$ , where $\sigma_n$ is the arithmetic mean $(a_1 + a_2 + \ldots + a_n)/n.$

4. If $\lim_{n\to\infty} a_n = \xi$ , shows that the arithmetic means of the arithmetic means $\sigma_n$ tend towards $\xi$

5. Find the error involved in using $S_n = 1 + \frac{1}{1!} + \ldots + \frac{1}{n!}$ as an approximation to $e$. Calculate $e$ accurately to 6 decimal places.

Answers and Hints

## 1.7 The Concept of Limit where the Variable is Continuous

Hitherto, we have considered limits of sequences, that is, of functions of an integral variable $n$. However, the notion of limit frequently occurs in connection with the concepts of a continuous variable $x$ and a function.

We say that the value of the function $f(x)$ tends to a limit $l$ as $x$ tends to $\xi$, or in symbols

$$\lim_{x\to\xi} f(x) = l,$$

if all the values of the function $f(x)$, for which $x$ lies near enough to $\xi$, differ arbitrarily little from $l$. Expressed more precisely, the condition is:

If an arbitrarily small positive quantity $\varepsilon$ is assigned, we can mark off about $\xi$ an interval $|x - \xi| < \delta$ so small that there applies for every point $x$ in this interval different from $\xi$ itself the inequality $|f(x) - l| < \varepsilon$.

We expressly exclude here the equality of $x$ and $\xi$. This is done purely for reasons of expediency, so as to have the definition in a form more convenient for applications, e.g., in the case where the function $f(x)$ is not defined at the point $\xi$, although it is defined for all other points in a neighbourhood of $\xi$.

If our function is defined or considered in a given interval only, for example,

$$\sqrt{1-x^2} \text{ in } -1 \leqq x \leqq 1,$$

we shall restrict the values of $x$ to this interval. Thus, if $\xi$ denotes an end-point of an interval, $x$ is made to approach $\xi$ by values on one side of $\xi$ only (**limit from the interior of the interval or one-sided limit**).

As an immediate consequence of this definition, we have the fact:

If $\lim_{x \to \xi} f(x) = l$ and $x_1, x_2, \cdots, x_n, \cdots$ is a sequence of numbers all different from $\xi$, but approaching $\xi$ as a limit, then $\lim_{n \to \infty} f(x_n) = l.$ .

In fact, let $\varepsilon$ be any positive number; we wish to show that for all values of $n$ greater than a certain $n_0$ there applies the inequality

$$|f(x_n) - l| < \epsilon.$$

By definition, there exists a $\delta > 0$ such that, whenever $|x - \xi| < \delta$, one has the inequality

$$|f(x_n) - l| < \epsilon.$$

Since $x_n \to \xi$, the relation $|x - \xi| < \delta$ is satisfied for all sufficiently large values of $n$, and it follows for such values that $|f(x_n) - l| < \varepsilon$, as was to be proved.

We shall now attempt to clarify this abstract definition by means of simple examples. Consider first the function

$$f(x) = \frac{\sin x}{x},$$

defined for $x \neq 0$. We state that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

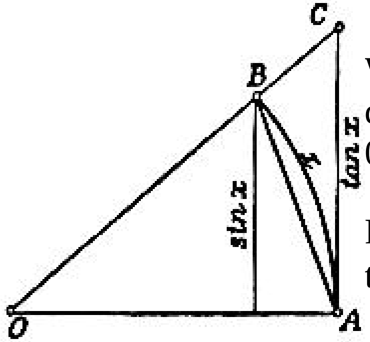



Fig. 18

We cannot prove this statement simply by carrying out the passage to the limit in the numerator and denominator separately, because the numerator and denominator vanish when $x = 0$, and the symbol 0/0 has no meaning. We arrive at the proof as follows:

From Fig. 18, we find by comparing the areas of the triangles *OAB* and *OAC* and the sector *OAB* that, if $0<x<\pi/2$,

$$\sin x < x < \tan x.$$

From this follows that, if $0<|x|<\pi/2$,

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

Hence the quotient $\sin x/x$ lies between the numbers 1 and $\cos x$ . We know that $\cos x \to 1$ as $x \to 0$, whence the quotient $\sin x/x$ can only differ arbitrarily little from 1, provided that $x$ is near enough to 0. This is exactly what is meant by the equation which was to be proved.

By the result just proved

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{1}{\cos x} = 1,$$

and also

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

This last result follows from the formula, valid for $0 < |x| < \pi/2$,

$$\frac{1-\cos x}{x} = \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \frac{1-\cos^2 x}{x(1+\cos x)} =$$
$$= \frac{\sin x}{x} \cdot \frac{1}{1+\cos x} \cdot \sin x.$$

As $x \to 0$, the first factor on the right hand side tends to 1, the second one to 1/2 and the third one to 0, whence the product tends to 0, as has been stated.

The same formula, on division by $x$, yields

$$\frac{1-\cos x}{x^2} = \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x},$$

whence

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

Finally, consider the function $\sqrt{x^2}$, defined for all values of $x$. This function is never negative; it is equal to $x$ for $x \geq 0$ and to $-x$ for $x < 0$. In other words, $\sqrt{x^2} = |x|$. Hence the function $\sqrt{x^2}/x$, defined for all non-zero values of $x$, has the value +1 when $x > 0$ and —1 when $x < 0$, whence the limit cannot exist, since arbitrarily near to 0 we can find values of $x$ for which the quotient is +1 and other values for which it is -1.

In concluding this discussion on limits in connection with a continuous variable, we remark that it is, of course, possible to consider limiting processes in which the continuous variable $x$ increases beyond all bounds. We state the example

$$\lim_{x \to \infty} \frac{x^2+1}{x^2-1} = \lim_{x \to \infty} \frac{1+\dfrac{1}{x^2}}{1-\dfrac{1}{x^2}} = 1$$

without further discussion. It signifies that the function on the left hand side differs arbitrarily little from 1, provided only that $x$ is sufficiently large.

In these examples, we have proceeded as if operations with limits obeyed the same laws in the case of continuous variables as in the case of sequences. That this is actually true, the reader himself can verify; the proofs are essentially the same as for limits of sequences.

**Exercises 1.6**: 1. Find the following limits, giving at each step the theorem on limits which justifies it:

(a) $\lim\limits_{x \to 2} 3x.$ (c) $\lim\limits_{x \to 1} \dfrac{x^2 + 2x - 1}{2x + 2}.$

(b) $\lim\limits_{x \to 2} 4x + 3.$ (d) $\lim\limits_{x \to 2} \sqrt{5 + \sqrt[3]{2x^5}}.$

2. Prove that

$$(a)\ \lim_{x \to 0} \frac{\sqrt{1-x}}{x};\quad (b)\ \lim_{x \to 0} \frac{\sqrt{1+x}}{x};\quad (c)\ \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}.$$

3. Determine whether or not the following limits exist and, if they do exist, find their values:

$$(a)\ \lim_{x \to 0} \frac{\sqrt{1-x}}{x};\quad (b)\ \lim_{x \to 0} \frac{\sqrt{1+x}}{x};\quad (c)\ \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}.$$

Answers and Hints

## 1.8. The Concept of Continuity

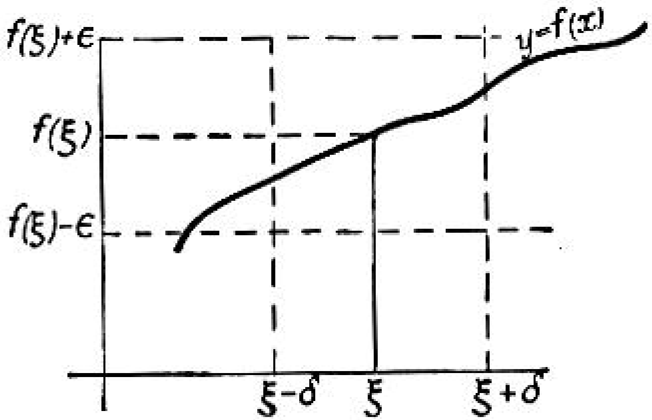


Fig. 19

**1.8.1 Definitions:** We have already illustrated in 1.2.3 by means of examples the notion of continuity. Now, using the idea of limit, we can make the concept of continuity precise.

We thought of the graph of a function, which is continuous in an interval, as a curve consisting of one unbroken piece; we also stated that the change in the function $y$ must remain arbitrarily small, provided only that the change of the independent variable $x$ is restricted to a sufficiently small interval. This state of affairs is usually formulated as follows, with a greater range but increased precision. A function $f(x)$ is said to be continuous at the point $\xi$, if it possesses the property: At the point $\xi$, the value of the function $f(\xi)$ is approximated to within an arbitrary pre-assigned degree of accuracy $\varepsilon$ by all functional values $f(x)$ for which $x$ is near enough to $\xi$. In other words, $f(x)$ is continuous at $\xi$, if for every positive number $\varepsilon$, no matter how small, there can be determined

another positive number $\delta = \delta(\varepsilon)$ such that $|f(x) - f(\xi)| < \varepsilon$ (Fig. 19) for all points $x$ for which $|x - \xi| < \delta$. Or again: The condition of continuity requires that for the point $\xi$

$$\lim_{x \to \xi} f(x) = f(\xi).$$

The value of the function at the point $\xi$ is the same as the limit of the functional values $f(x_n)$ for any arbitrary sequence $x_n$ of numbers converging to $\xi$.

It is important to observe that our condition involves two different matters: (1) the **existence** of $\lim_{x \to \xi} f(x)$ and (2) the **coincidence** of this limit with $f(\xi)$, the value of the function at the point $\xi$.

Having now defined continuity of a function $f(x)$ at a point $\xi$, we proceed to state what we mean by the continuity of a function $f(x)$ in an interval. This may be defined simply as follows: The function $f(x)$ is continuous in an interval, if it is continuous at each point of that interval. Stated fully, this requires that, if a positive number $\varepsilon$ be assigned, then there exists for each point $x$ of the interval a number $\delta$, depending as a rule on $\varepsilon$ and on $x$, such that

$$|f(\bar{x}) - f(x)| < \epsilon \quad \text{if} \quad |\bar{x} - x| < \delta,$$

and $\bar{x}$ lies in the interval $a \leq \bar{x} \leq b$.

Closely related to this is another concept, namely that of **uniform continuity**. A function $f(x)$ is uniformly continuous in the interval $a \leq x \leq b$, if there exists for every positive number $\varepsilon$ a corresponding positive number $\delta$ such that, for every pair of points $x_1$, $x_2$ in the interval, the distance $|x_1 - x_2|$ a part of which is less than $\delta$, one has the inequality $|f(x_1) - f(x_2)| < \varepsilon$. This differs from the definition stated above in that $\delta$ in the definition of uniform continuity does not depend on $x$, but is equally effective for all values of $x$, whence follows the term **uniform continuity**.

It is quite obvious that a uniformly continuous function is necessarily continuous. Conversely, it can be shown that every function $f(x)$, which is continuous in a closed interval $a \leq x \leq b$, is also uniformly continuous. The proof of this is given in Appendix I. Even though the reader may not desire to read the proof at present, he will find it helpful to study the examples at the beginning of A1.2.2. But until the student has worked through this proof, he may assume, whenever a function is said to be continuous in a closed interval, uniform continuity is implied.

**1.8.2 Concept of Discontinuity:** We can understand the concept of continuity better if we study its opposite, the concept of **discontinuity**. The simplest discontinuity occurs at those points at which a function has a **jump**, that is, at which the function has a definite limit as $x$ tends to the point from the right and a definite limit as $x$ tends to the point from the left, while these two limits are different. It does not matter whether or how the function is defined at the point of discontinuity itself.

For example, the function $f(x)$ defined by the equations

$$f(x) = 0 \text{ for } x^2 > 1, \quad f(x) = 1 \text{ for } x^2 < 1, \quad f(x) = \tfrac{1}{2} \text{ for } x^2 = 1$$

has discontinuities at the points $\xi = 1$ and $\xi = $ -l. The limits on approaching these points from the right and from the left hand side differ by 1, and the values of the function at these points agree with neither limit, but are equal to the arithmetic mean of the two limits.

It may be noted in passing that our function can be represented, using the idea of a limit, by the expression

$$f(x) = \lim_{n \to \infty} \frac{1}{1 + x^{2n}}.$$

For, if $x^2 < 1$, that is, if $x$ lies in the interval -1 < » < 1, the numbers $x^{2n}$ will have the limit 0, and the function will have the value 1. However, if $x^2 > 1$, as $n$ increases, $x^{2n}$ will increase beyond all bounds; our function will then have the value 0. Finally, obviously, for $x^2 = 1$, that is, for $x = +1$ and $x = -1$, the value of the function is ½ (Fig. 20).

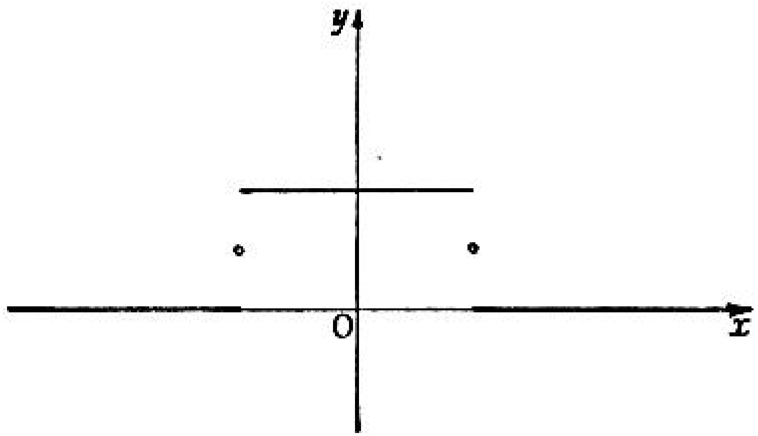



Fig. 20

;

Other curves with jumps are shown in Figs. 21*a* and 21*b*; they represent functions with obvious discontinuities.

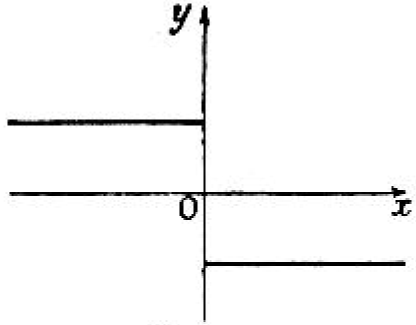



Fig. 21a

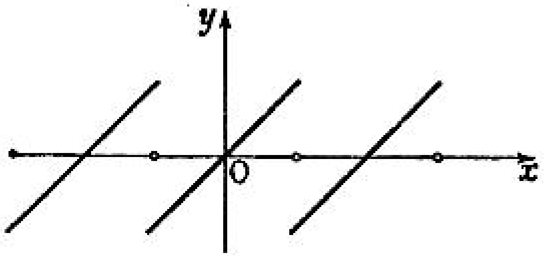



Fig. 21b

In the case of discontinuities of this kind, both the limits from the right and from the left exist. We now proceed to the consideration of discontinuities in which this is not the case. These most important of such discontinuities are the **infinite discontinuities** or **infinities**. These are discontinuities which are exhibited by the functions $1/x$ or $1/x^2$ at the point $\xi = 0$; as $x \to \xi$, the absolute value $|f(x)|$ of the function increases beyond all bounds. In the case of $1/x$, the function increases numerically beyond all bounds through positive and through negative values, respectively, as $x$ approaches the origin from the right and from the left. On the other hand, the function $1/x^2$ has for $x = 0$ an infinite discontinuity at which its value becomes positive from both sides (Figs. 6 and 12). The function $y = 1/(x^2 - 1)$, shown in Fig. 22, has infinite discontinuities both at $x = 1$ and at $x = -1$.

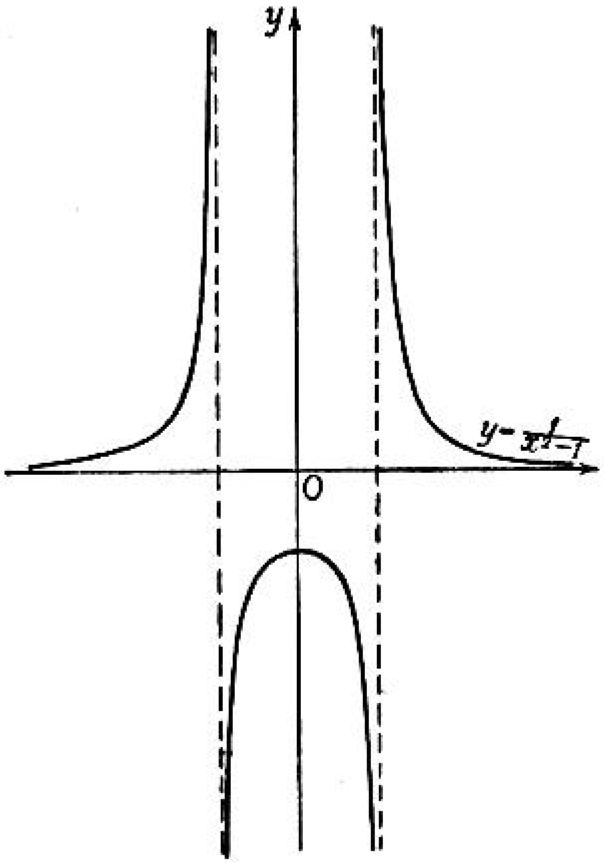



Fig. 22.—Function with infinite discontinuities

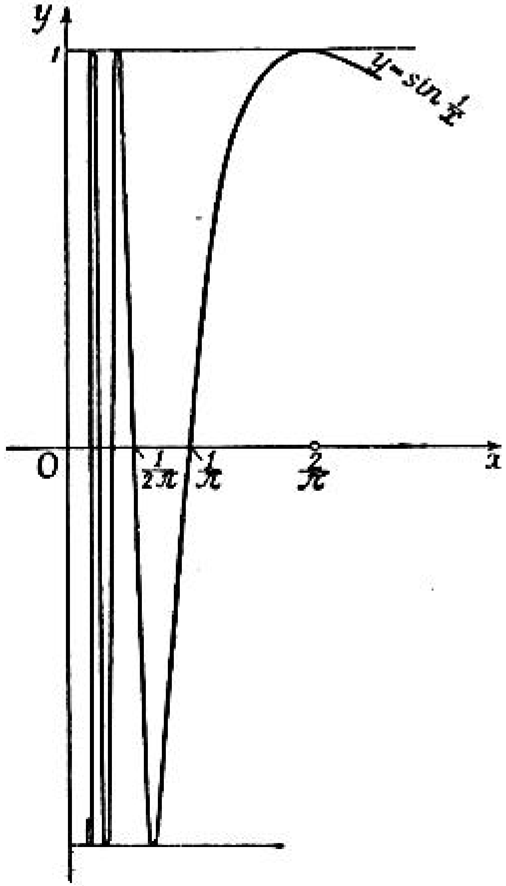



Fig. 23.—Oscillating function with discontinuity

Finally, we shall illustrate by an example another type of discontinuity in which there does not exist a limit from the right or from the left. Consider the function

$$y = \sin\frac{1}{x},$$

defined for all non-zero values of $x$. This function takes all values between -1 and + 1 as the number $1/x$ ranges through the values from $(2n - 1/2)\pi$ to $(2n+1/2)\pi$, no matter what value has $n$. At the points $x=2/(4n-1)\pi$, the function will have the value -1, at the points $x=2/(4n+1)\pi$ the value +1. We see from this that the function swings backwards and forwards more rapidly between the values +1 and -1 as $x$ approaches closer and closer to the point 0

and that there occur in the immediate neighbourhood of the point $x = 0$ an infinite number of oscillations (Fig. 23 above).

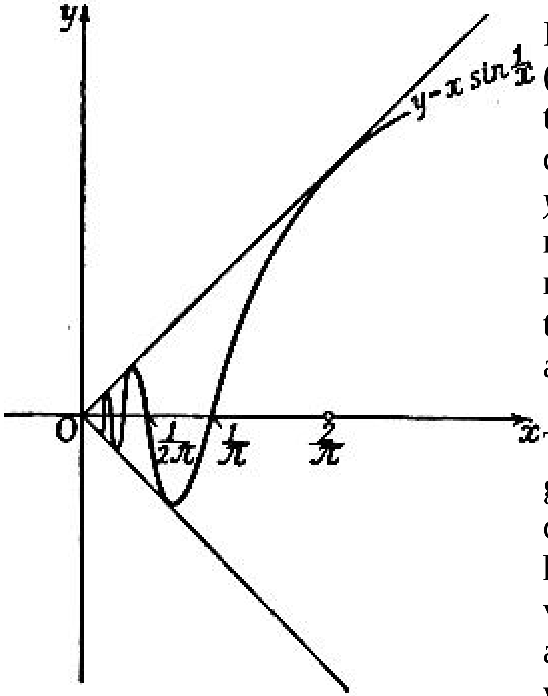



Fig. 24.—Continuous oscillating function

It is interesting to observe that, in contrast to the above example, the function $y=x\sin x/x$ (Fig. 24 below) remains continuous at the point $x = 0$, if we assign to it the value 0 at that point. This continuity is due to the fact that, as the origin is approached, the factor $x$ damps the oscillations of the sine. Yet, in the neighbourhood of the origin, the function $y = x \sin 1/x$ does not change a **finite** number of times from monotonic increasing to monotonic decreasing. On the contrary, it oscillates backwards and forwards an infinite number of times, the magnitude of these oscillations becoming as small as we please as the origin is approached. This example shows us that even the simple idea of continuity admits all sorts of remarkable possibilities foreign to our naive intuition.

There is one important fact which must be taken into account, if we are to give our ideas greater precision. It may happen that at a certain point a function is not defined by the original law, as for example at the point $x = 0$ in the last two examples discussed. We have then the right to extend the definition of the function by assigning to it any desired value at such a point. However, in the last example, we can extend the definition in such a way that the function also remains continuous at that point, namely, by setting $y = 0$ when $x = 0$. This can be done whenever both the limits from the left and from the right exist and are equal to each other; we need then only make the value of the function at the point in question equal to these limits, in order to make the function continuous there. In the case of the function $y=\sin 1/a$, this cannot be done.

**1.8.3 Theorems on Continuous Functions:** In conclusion, we quote the following important, general theorems, the proofs of which follow immediately from the remarks on operations with limits (cf. 1.6.4):

The sum, difference, and product of two continuous functions are themselves continuous. The quotient of two continuous functions is continuous at every point at which the denominator does not vanish.

In particular, it follows that all polynomials and all rational functions are continuous except at the points where the denominator vanishes. The fact that the other elementary functions, such as the trigonometric functions, are continuous will follow naturally from later considerations (2.3.5).

**Exercises 1.7:**

1. Prove that

$$\lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = 0.$$

2.Prove that

$$(a)\ \lim_{x \to \alpha} \frac{\sin (x - \alpha)}{x^2 - \alpha^2} = \frac{1}{2\alpha}; \qquad (b)\ \lim_{x \to \infty} \frac{x + \cos x}{x + 1} = 1;$$

$$(c)\ \lim_{x \to \infty} \cos 1/x = 1.$$

3. (*a*) Let $f(x)$ be defined by the equation $y = 6x$. Find a $\delta$, depending on $\xi$, so small that $|f(x)-f(\xi)| < \varepsilon$ whenever $|x - \xi| < \delta$, where (1) $\varepsilon = 1/10$; (2) $\varepsilon = 1/100$ (3) $\varepsilon = 1/1{,}000$.
Do the same for

$$(b)\ f(x) = x^2 - 2x;$$
$$(c)\ f(x) = 3x^4 + x^2 - 7;$$
$$(d)\ f(x) = \sqrt{x},\ x \geqq 0;$$
$$(e)\ f(x) = \sqrt{x^2}.$$

4. (a) Let f($x$) = 6$x$ in the interval $0 \leq x \leq 10$. Find a $\delta$ so small that $|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$, where (1) $\varepsilon = 1/100$, (2) $\varepsilon$ is arbitrary, but > 0.

Do the same for

$$(b)\ f(x) = x^2 - 2x,\ -1 \leqq x \leqq 1;$$
$$(c)\ f(x) = 3x^4 + x^3 - 7,\ 2 \leqq x \leqq 4;$$
$$(d)\ f(x) = \sqrt{x},\ 0 \leqq x \leqq 4;$$
$$(e)\ f(x) = \sqrt{x^2},\ -2 \leqq x \leqq 2.$$

5. Determine which of the following functions are continuous. For those which are discontinuous, find the points of discontinuity:

(a) $x^3 \sin x$.

(b) $x \sin^3(x^2)$.

(c) $\frac{1}{x} \sin x$.

(d) $\frac{\sin x}{\sqrt{x}}$.

(e) $\frac{x^3 + 3x + 7}{x^2 - 6x + 8}$.

(f) $\frac{x^3 + 3x + 7}{x^2 - 6x + 9}$.

(g) $\frac{x^3 + 3x + 7}{x^2 - 6x + 10}$.

(h) $\tan x$.

(i) $\frac{1}{\sin x}$.

(j) $\cot x$.

(k) $\frac{1}{\cos x}$.

(l) $x \cot x$.

(m) $(\pi - x) \tan x$.

Answers and Hints

# Appendix I to Chapter I

**Preliminary Remarks:** In **Greek mathematics**, we find an extensive development of the principle that all theorems are to be proved in a logically coherent way by reducing them to a system of axioms, as few in number as possible and not themselves to be proved. At the beginning of the modem era, this axiomatic method of presentation, which at the same time served as a test of the accuracy of the investigation, was regarded as a model for other branches of knowledge. For example, in philosophy, men like **Descartes** and **Spinoza** believed that they had made their investigations more convincing by presenting them **axiomatically**, or, as they called it, **geometrically**.

But it was a different matter with modern mathematics, which began to develop at about the same time as the new philosophy. In mathematics, the principle of reduction of material to axioms was frequently abandoned. **Intuitive evidence** in each separate case became a favourite method of proof. Even in the case of scientists of the first rank, we find operations with the new concepts based chiefly on a feeling for the right result and not always free from mystical associations—particularly in the case of the ominous **infinitely small quantities** or **infinitesimals**. Blind faith in the omnipotence of the new methods carried the investigator away along paths which he could never have travelled if subjected to the limitations of complete rigour. It is no wonder that only the sure instinct of a great master could guard against gross errors.

It is fortunate that this was so and that the critical counter-currents, which arose in the Eighteenth Century and reached their full strength in the Nineteenth Century, did not come in time to check the development of modem mathematics, but only in time to establish and extend its results. But the need for critical investigation and

consolidation of the advances made gradually increased to such an extent that its satisfaction is rightly regarded as one of the most important mathematical achievements of the Nineteenth Century.

In the differential and integral calculus, the critical work of **Cauchy** is particularly important. By formulating the fundamental concepts in a clear and satisfactory way, Cauchy rounded off in many directions the work, which began in the Eighteenth Century, of presenting higher analysis in an intelligible manner free from the vagueness due to the use of infinitesimals.

The principal thing, which remained to be done, was to replace intuitive considerations in proofs and discussions by considerations of pure analysis, depending only on numbers and on the operations, which can be performed with numbers **-** as we say, to **arithmetize analysis**. As a matter of fact, the critically trained mind feels there is something unsatisfactory about appeals to intuition in proofs in analysis. We need not go into the question of the accuracy or inaccuracy of intuition or of the existence of a **pure *a priori* intuition** in **Kant**'s sense, in order to recognize that naive intuitive thinking includes much vagueness which hinders the approach to completely rigorous proofs in analysis. In the following chapters, this will strike us more and more clearly. Even here, we may mention, for example, that the concept of a **continuous curve** is very difficult to grasp intuitively. A continuous curve need not by any means possess a definite direction at every point. In fact, there actually exist continuous curves, which at no point posses a direction, and continuous curves, to which no length can be assigned. In the face of such facts, even the beginner will admit the need for arithmetizing analysis.

Rigorous mathematical concepts are always very highly idealized forms of the ideas which arise intuitively. Hence it is absolutely impossible to dispose of problems relating to the ultimate foundations of mathematics by appealing to naive intuition.

Yet, we must not allow ourselves to forget that a century of brilliant and fruitful development of mathematics was possible before these requirements were fulfilled. In spite of all its defects, intuition still remains the most important driving force for mathematical discovery, and intuition alone can bridge the gap between theory and application.

We shall now follow **Bolzano** and **Weierstrass** in developing those lines of thought which yield the rigorous and complete proofs of the theorems which we have formulated by intuitive means in the first chapter.

## A1.1 The Principle of the Point of Accumulation and its Applications

**A1.1.1 The Principle of the Point of Accumulation:** In the rigorous discussion of the fundamentals of analysis, the leading part is played by Weierstrass' principle of the point of accumulation. From an intuitive point of view,

this principle is merely the statement of a triviality; however, just because it summarizes a state of affairs which occurs frequently, it is as useful as small change of money is in daily life. The principle follows:

If infinitely many numbers are given in a finite interval, these numbers possess at least one point of accumulation, that is, there is at least one point $\xi$ such that in every interval about the point $\xi$, however small, there lie infinitely many of the given numbers.

In order to prove this principle arithmetically, we assume to begin with that the given interval is the interval from 0 to 1. We now subdivide this interval into ten equal parts by means of the points 0.1, 0.2, ⋯ , 0.9. At least one of these subintervals must contain infinitely many points. Let the interval beginning with the number $0.a_1$ be that interval (or one of those intervals if there are several). We now subdivide this interval into ten parts by means of the points of subdivision $0.a_1 1$, $0.a_1 2$, ⋯ , $0.a_1 9$. Again, it is true that at least one of these subintervals must contain infinitely many points; let it be the subinterval beginning with the number $0.a_1 a_2$. We again subdivide into ten parts, note that one of these parts must contain infinitely many points and continue the process. We thus arrive at a sequence of digits $a_1, a_2, a_3, \cdots$ , each having one of the values 0, 1, 2, ⋯ , 9. We now consider the decimal

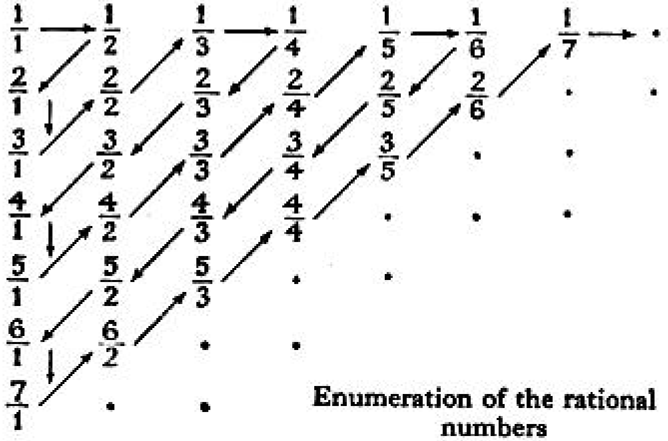
$$\xi = 0.a_1 a_2 a_3 \cdots .$$

It is clear that this is a point of accumulation of our set of numbers. In fact, every interval, no matter how small, in the interior of which the point $\xi$ lies, contains the subintervals of our system of subdivision from a certain degree of fineness onward, and these subintervals contain infinitely many numbers of the set.

If the interval under consideration instead of being the interval from 0 to 1 is, say, the interval from $a$ to $a + A$, nothing essential in the above argument is changed. The point of accumulation is then represented simply by a number of the form

$$a + \mathrm{h} \times 0.a_1 a_2 a_3 \cdots .$$

**A1.1.2 Limits of Sequences:**. These considerations throw new light upon the concept of the limit of an infinite sequence of numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$ . We first consider the exceptional case in which infinitely many numbers of the sequence are equal to one another and extend our definition by applying the name **point of accumulation also** to this point (or these points). If there are infinitely many different numbers in the sequence and if we assume that the numbers $a_n$ of this sequence are bounded, i.e., that there is a number $M$ such that $|a_n| < M$ for all values of $n$, the numbers of the sequence form an infinite set of numbers in a finite interval, since they all lie between **-** $M$ and $M$. Hence they must possess at least one point of accumulation ($\xi$). If there is only one point of accumulation, it is easy to show that the sequence converges and that its limit is the number $\xi$. In fact, let us mark off any small



Enumeration of the rational numbers

interval about the point $\xi$. If infinitely many points of the sequence were outside this interval, they would have a limit point other than $\xi$, contrary to the hypothesis, whence only a finite number of the numbers of the sequence are exterior to the interval and, by definition, the sequence approaches $\xi$. On the other hand, if there are several points of accumulation, the sequence approaches no limit. The existence of a limit and the uniqueness of the point of accumulation of a bounded sequence of numbers are therefore equivalent ideas.

The case of the non-existence of a limit is to be regarded as the rule rather than the exception. For example, the sequence with the terms $a_{2n} = 1/n$, $a_{2n-1} = 1 - 1/n$, (n = 1, 2, ···.) has the two points of accumulation 0 and 1.

The aggregate of the positive rational numbers may be regarded as a sequence of numbers, in which the ordering by magnitude is, of course, completely destroyed. We arrive most easily at such an arrangement in a sequence by first writing down the rational numbers as shown and then running through this array as shown by the arrows, disregarding those numbers which have already been encountered (such as 2/4). The system of rational numbers obviously has all rational and irrational points as points of accumulation. It therefore forms a simple example of a sequence with an infinite number of points of accumulation.

By means of the concept of convergence, we can state the principle of the point of accumulation in a remarkable form which is often convenient for applications.

From every bounded infinite set of numbers, it is possible to choose an infinite sequence $a_1, a_2, a_3, \cdots$ which converges to a definite limit $\xi$. For this purpose, we have only to take a point of accumulation $\xi$ of the given set of numbers, then select a number $a_1$ of the set the distance of which from $\xi$ is less than 1/10, then a second number $a_2$ of the set the distance of which from $\xi$ is less than 1/100, then a third number $a_3$, the distance of which from $\xi$ is less than 1/1,000, and so on. We see at once that this sequence actually converges to the limit $\xi$.

**A1.1.3 Proof of Cauchy's Convergence Test:**. Let us now return to convergent sequences, i.e., to bounded sequences with only one point of accumulation. Cauchy's convergence test now reduces almost to a triviality. In fact, when $m$ and $n$ are sufficiently large, let $|a_m - a_n|$ be arbitrarily small . Then all the numbers $a_n$ lie in a finite interval and therefore have at least one point of accumulation $\xi$. If there were a second point of accumulation $\eta$, the distance of this point from $\xi$ would be $|\xi - \eta| = a$, a positive quantity. Within an arbitrarily small distance from $\xi$, say, within a distance less than $a/3$ from $\xi$, there must be infinitely many numbers $a_n$ and hence, in particular, infinitely many numbers $a_n$ for which $n > N$, however large $N$ is chosen. Similarly, within an arbitrarily small

distance from the point $\eta$, say, within a distance less than $a/3$ from $\eta$, there are infinitely many numbers $a_m$ of the sequence; in particular, infinitely many numbers $a_m$, for which $m > N$. For these values $a_n$ and $a_m$, it is true that $|a_m - a_n| > a/3$, and this relation is incompatible with the hypothesis that for sufficiently large values of $N$ the difference $|a_n - a_m|$ is arbitrarily small provided that $n$ and $m$ are both greater than $N$. Hence there are not two distinct points of accumulation and Cauchy's test has been proved.

**A1.1.4 The Existence of Limits of Bounded Monotonic Sequences:** It is equally easy to see that a bounded monotonic increasing or monotonic decreasing sequence of numbers must possess a limit. In fact, let the sequence be monotonic increasing and $\xi$ be a point of accumulation of the sequence; such a point of accumulation must certainly exist. Then $\xi$ must be greater than any number of the sequence. For if a number $a_1$ of the sequence were equal to or greater than $\xi$, every number $a_n$ for which n > $l$ + 1 would satisfy the inequality $a_n > a_{l+1} > a_l \geq \xi$. Hence all numbers of the sequence, except at most the first ($l$ + 1), would lie outside the interval of length $2(a_{l+1}-\xi)$ the mid-point of which is at the point $\xi$. However, this contradicts the assumption that $\xi$ is a point of accumulation. Hence no numbers of the sequence, and, *a fortiori*, no points of accumulation lie above $\xi$. Thus, if another point of accumulation $\eta$ exists, we must have $\eta < \xi$. But if we repeat the above argument with $\eta$ in place of $\xi$, we obtain $\xi < \eta$, which is a contradiction. Hence only one point of accumulation can exist and the convergence is proved. Naturally, an argument exactly analogous to this one applies to monotonic decreasing sequences.

As in 1.6.4, we can extend our statements about monotonic sequences by including the limiting case in which successive numbers of the sequence are equal, to one another. It is in this case better to speak of **monotonic non-decreasing and monotonic non-increasing sequences**, respectively. The theorem about the existence of a limit remains valid for such sequences.

**A1.1.5 Upper and Lower Points of Accumulation; Upper and Lower Bounds of a Set of Numbers:** In the construction in A1.1.1, which led us to a point of accumulation $\xi$, we must at each step coose a subinterval containing infinitely many points of the set. Had we always chosen the last subinterval which contained an infinite number of points, we should have been led to a certain definite point of accumulation $\beta$. This point of accumulation $\beta$ is called the **upper point of accumulation** or the **upper limit of the set of numbers** and is denoted by $\overline{\lim}$. It is that point of accumulation of the sequence which lies furthest to the right, i.e., it is quite possible that an infinite number of points of the sequence lie above $\beta$, but no matter how small the positive number $\varepsilon$ may be, there are not an infinite number above $\beta + \varepsilon$.

If we had always chosen in the construction of A1.1.1 the first of the intervals containing an infinite number of points of the set, we should again have arrived at a certain definite point of accumulation $\alpha$. This point $\alpha$ is called

the **lower point of accumulation** or **lower limit** of the set and is denoted by $\underline{\lim}$. There may be infinitely many numbers of the set below $\alpha$, but, no matter how small is the positive number $\varepsilon$, there are only a finite number below $\alpha - \varepsilon$. The proofs of these facts can be left to the reader.

Neither the upper limit $\beta$ nor the lower limit $\alpha$ need belong to the set. For example, for the set of numbers $a_{2n} = 1/n$, $a_{2n-1} = 2 - 1/n$, these limits are $\alpha = 0$ and $\beta = 2$, respectively, but the numbers 0 and 2 do not themselves occur in the set.

There is in this example no number of the set above $\beta = 2$. In this case, we say that $\beta = 2$ is also the upper bound $M$ of the set, according to the following definition: $M$ is called the **least upper bound** or simply the **upper bound** of a set of numbers, if: (1) there is no number of the set greater than $M$, but (2) there is a number of the set greater than $M - \varepsilon$ for every positive number $\varepsilon$. The upper bound may coincide with the upper bound, as in the example above. But the set $a_n = 1 + 1/n$ ($n = 1, 2 \cdots$) shows that this is not necessarily the case. Here $M = 2$ and $\beta = 1$.

Every bounded set of numbers has a **least upper bound**. In fact, let $\beta$ be the upper limit of the set. Either there are no numbers of the set larger than $\beta$ or there are such numbers. In the first case, $\beta$ is the least upper bound, since no numbers are above $\beta$ and there are numbers arbitrarily close to $\beta$ below it. In the second case, let $a$ be a number of the set greater than $\beta$. There are only a finite number of numbers of the set equal to or greater than $a$, since otherwise there would be a point of accumulation above $\beta$, which is impossible. We therefore need only choose the greatest of these numbers; it will be the **upper bound** of the set.

We see that in any case $M \geq \beta$ and we recognize the fact: If the upper bound of a set does not coincide with the upper limit, it must belong to the set and is an **isolated point of the set**.

Corresponding statements hold for the **lower bound** $m$; it is always equal to or less than $a$, and if $m$ and $a$ do not coincide, $m$ belongs to the set and is an **isolated point** of the set.

## A1.2. Theorems on Continuous Functions

**A1.2.1. Greatest and Least Values of Continuous functions:** A bounded, infinite set of numbers must possess a **least upper bound** $M$ and a **largest lower bound** $m$. However, as we have seen, these numbers $M$ and $m$ do not necessarily belong to the set; as we say: The set does not necessarily have a greatest or a least value.

In view of this fact, the following theorem on continuous functions is by no means as obvious as it appears to be to simple intuition: Every function $f(x)$, which is continuous in a closed interval $a \leq x \leq b$, assumes at least once a greatest and a least value or, as we say, it possesses a greatest and a least value.

This may easily be proved as follows: The values assumed by the continuous function $f(x)$ in the interval $a \leq x \leq b$ form a bounded set of numbers and therefore possess a least upper bound $M$. Otherwise, a sequence of numbers $\xi_1$, , $\xi_2, \cdots, \xi_n, \cdots$ in our interval would exist for which $f(\xi)$ increases beyond all bounds. This sequence would have at least one point of accumulation $\bar{\xi}$ in the interval. In that case, arbitrarily near to $\bar{\xi}$, there would always be numbers $\xi_n$ of our sequence for which the expression $|f(\bar{\xi}) - f(\xi_n)|$ exceeds 1 (and, in fact, is arbitrarily large), that is, the function would be discontinuous at the point $\bar{\xi}$. Thus, a least upper bound $M$ exists and hence either there is a point $\xi$ such that $f(\xi) = M$, which would prove the statement, or there is a sequence of numbers $x_1, x_2, \cdots, x_n, \cdots$ in the interval for which

$$\lim_{n \to \infty} f(x_n) = M.$$

According to the formulated principle of the point of accumulation, we can select a subsequence of the numbers $x_n$,, which converges to a limit $\xi$. Let us call this subsequence $\xi_1, \xi_2, \cdots, \xi_n, \cdots$, so that

$$\lim_{n \to \infty} \xi_n = \xi$$

It is then certain that

$$\lim_{n \to \infty} f(\xi_n) = M.$$

Ott the other hand, the function has been assumed to be continuous in the interval and, in particular, at $\xi$, whence

$$\lim_{n \to \infty} f(\xi_n) = f(\xi).$$

Hence $f(\xi) = M$. The value $M$ is therefore assumed by the function at a definite point $\xi$ in the interior or on the boundary of the interval, as stated. An exactly similar discussion applies to the least value.

In general, the theorem about the greatest and least values of continuous functions does not remain true unless we expressly assume the interval to be closed, that is, unless we make the hypothesis of continuity also refer to the end-points. For example, the function $y = 1/x$ is continuous in the open interval $0 < x < \infty$. However, it has no greatest value, but has arbitrarily large values near $x = 0$. Similarly, it has no least value, but comes arbitrarily near 0 for sufficiently large values of $x$ without ever assuming the value 0.

**A1.2.2 The Uniformity of Continuity:** As we have already seen and as we shall further see, the continuity of a function $f(x)$ in a closed interval $a \leq x \leq b$ leaves room for a variety of possibilities, which do not suggest themselves intuitively. For this reason, we shall give logically rigorous proofs of certain consequences of the idea of continuity which, from a naïve point of view, seem to be quite obvious. The definition of continuity simply states that there follows from the relation

$$\lim_{n \to \infty} \xi_n = \xi$$

the relation

$$\lim_{n \to \infty} f(\xi_n) = f(\xi).$$

We can also express this as follows: There correspond to every $\varepsilon > 0$ for each point $\xi$ a number $\delta > 0$ such that $|f(x) - f(\xi)| < \varepsilon$ whenever $|x - \xi| < \delta$, provided that all the numbers $x$ considered lie in the interval $a \leq x \leq b$.

For example, in the case of the function $y = cx$ (where $c \neq 0$), such a number $\delta$ is given by the relation $\delta = \varepsilon |c|$. For the function $y = x^2$, we can find such a number as follows: We assume that $a=0$ and $b=1$, and ask ourselves how near to $\xi$ the number $x$ must lie in order that the expression $|x^2 - \xi^2|$ may be less than $\varepsilon$. For this purpose, we write

$$|x^2 - \xi^2| = |x - \xi|\,|x + \xi| \leqq |x - \xi|\,(1 + \xi).$$

Hence, if we choose $\delta \leq \varepsilon/(1 + \xi)$, we can be sure that $|x^2 - \xi^2| < \varepsilon$. We see in this example that the number $\delta$ found in this way depends not only on $\varepsilon$, but also on the point of the interval at which we are investigating the continuity of the function. However, if we give up the attempt to make the best possible choice of $\delta$ for each $\xi$, we can eliminate this dependence of $\delta$ on $\xi$. In fact, we need only replace $\xi$ on the right hand side by the number 1 and thus obtain for $\delta$ the expression $\varepsilon/2$, which is smaller than the previous expression for $\delta$, but serves equally well for all points $\xi$.

There arises now the question whether something similar does not hold for every function which is continuous in a closed interval, that is, we enquire whether it may not be possible to determine for each $\varepsilon$ a $\delta = \delta(\varepsilon)$ which depends only on $\varepsilon$ and not on $\xi$ such that the inequality

$$|f(x) - f(\xi)| < \epsilon$$

holds, provided $|x - \xi| < \delta$ for all values of $\xi$ at the same time (or, better expressed, uniformly with respect to $\xi$). As a matter of fact, this is possible merely as a consequence of the general definition of continuity without any additional assumptions. This fact, which first attracted attention late in the Nineteenth Century, is called the **Theorem of the Uniform Continuity of Continuous Functions**.

We shall prove this theorem indirectly, that is, we shall show that the assumption that a function $f(x)$ exists, which in a closed interval $a \leq x \leq b$ is continuous and yet not uniformly continuous, leads us to a contradiction. Uniform continuity means that, if we wish to make the difference $|f(u) - f(v)|$ less than an arbitrarily chosen positive number $\varepsilon$, the numbers $u$ and $v$ being chosen in the closed interval $a \leq x \leq b$, we need only choose $u$ and $v$ near enough to one another, namely, at a distance apart which is less than $\delta = \delta(\varepsilon)$; it is immaterial where in the interval the pair of numbers $u,v$ is chosen. Now, if $f(x)$ were not uniformly continuous, there would exist a positive (perhaps very small) number $\alpha$ with the property: There corresponds to every number $\delta_n$ of an arbitrary sequence $\delta_1, \delta_2, \cdots$ of positive numbers tending to zero a pair of values $u_n, v_n$ of the interval for which $|u_n - v_n| < \delta_n$ and $|f(u_n) - f(v_n)| > \alpha$. According to the principle of the point of accumulation, the numbers $u_n$ must have a point of accumulation $\xi$ and the numbers $v_n$ must have the same point of accumulation. If we select an arbitrarily small interval $|x - \xi| < \delta$ about this point $\xi$, an infinite number of the pairs $u_n, v_n$ will lie in this interval. But this contradicts the assumed continuity of $f(x)$ at the point $\xi$; in fact, by Cauchy's convergence test, this requires that the points $x_2$ and $x_1$ are near enough to $\xi$

$$|f(x_1) - f(x_2)| < a.$$

The **uniformity of the continuity** is thus proved.

In our proof, we have made essential use of the fact that the interval is closed*. In fact, the theorem of uniformity of continuity does not hold for intervals which are not closed.

* 0therwise the point of accumulation need not belong to the interval.

For example, the function $l/x$ is continuous in the half-open interval $0 < x \leq 1$, but it is not uniformly continuous, because no matter how small the length $\delta (< l)$ of an interval is chosen, the function will take values differing by any fixed number, say 1, in the interval, if only the interval lies near enough to the origin, say, $\delta/2 \leq x \leq 3\delta/2$. Of course, the non-uniformity of continuity is due to the fact that in the closed interval $0 \leq x \leq 1$the function possesses at the origin a discontinuity. If we had considered the example $y = x^2$ in the entire (open) interval $-\infty < x < \infty$ instead of in a closed interval, it would not have been continuous.

**A1.2.3 The Intermediate Value Theorem:** There is another theorem, which constantly recurs in analysis:

A function $f(x)$, continuous in a closed interval $a \leq x \leq b$, which is negative for $x = a$ and positive for $x = b$ (or conversely), assumes the value 0 at least once in the interval.

Geometrically speaking, this theorem is trivial, since it merely states that a curve, which begins below the $x$-axis and ends above it, must cut the axis somewhere in between. Analytically speaking, the theorem is very easily proved. There are in the interval an infinite number of points for which $f(x) < 0$; in fact, due to the continuity of the function, this is true for an entire interval beginning at the point $a$. The set consisting of those points $x$ for which $f(x) < 0$ has a least upper bound $\xi$ which is greater than $a$. Since there are points in every neighbourhood of $\xi$ for which $f(x) < 0$, we must have $f(\xi) \leq 0$ (whence, in particular, $\xi \neq b$). However, it is impossible that $f(\xi) \leq 0$, (whence, in particular, $\xi \neq \beta$). However, it is impossible that $f(\xi) < 0$, because then $f(x)$ would be negative in a sufficiently small neighbourhood of $\xi$, including values of $x > \xi$, in contradiction to the assumption that $\xi$ is the upper bound of the values of $x$ for which $f(x) < 0$. Hence $f(\xi) = 0$ and our assertion is proved.

A slight generalization of our theorem is: If we assume that $f(a)=\alpha$ and $f(b)=\beta$ and if $\mu$ is any value between $\alpha$ and $\beta$, the continuous function $f(x)$ assumes the value $f()\mu$ at least once in the interval, because the continuous function

$$\phi(x) = f(x) - \mu$$

will have different signs at the two ends of the interval and will therefore assume the value 0 somewhere inside it.

**A1.2.4 The Inverse of a Continuous Monotonic Function:** If the continuous function $y = f(x)$ is monotonic in the interval $a \leq x \leq b$, it will assume each value between $f(a)$ and $f(b)$ once and only once; hence, if $y$ describes the closed interval between the values $\alpha = f(a)$ and $\beta = f(b)$, there will correspond to each value of $y$ exactly one value of $x$. We can therefore think of $x$ as a single-valued function of $y$ in this interval, i.e., the function $y = f(x)$ has a unique inverse. We assert that this inverse function $x = \phi(y)$ is also a continuous, monotonic function of $y$, as $y$ varies within the interval between $\alpha$ and $\beta$.

The monotonic character of the inverse function $x = \phi(y)$ is obvious. In order to prove its continuity, we observe that it follows from the monotonic character of $f(x)$ that

$$|f(x_2) - f(x_1)| = |y_2 - y_1| > 0,$$

provided that $x_1$ and $x_2$ are distinct numbers of the interval. If $h$ is a positive number less than $b - a$, the function

$$|f(x + h) - f(x)|$$

is continuous in the closed interval $a \leq x \leq b - h$,whence it has a least value

$$|f(\xi + h) - f(\xi)| = \alpha(h),$$

which by our preceding remark is not zero.* We conclude from this that, if $x_1$ and $x_2$ are two points in the interval for which $|x_1 - x_2| \geq h$, then

$$|f(x_1) - f(x_2)| \geqq \alpha(h).$$

However, this implies the continuity of the inverse function, because, if $|y_1 - y_2|$ falls below the positive number $\alpha(h)$, then we must have $|x_1 - x_2| \geq h < h$, whence, if a positive number $\varepsilon$ is given, we need only choose $\delta$ equal to $\alpha(\varepsilon)$ in order to ensure that for all values $y$ for which $|y_1 - y_2| < \delta$ it is also true that $|\phi(y_1) - \phi(y_2)| < \varepsilon$.

* On account of the continuity of $f(x)$, naturally, $\alpha(h)$ itself tends to 0 with $h$.

Hence we have established the theorem: If the function $y = f(x)$ is continuous and monotonic in the interval $a \leq x \leq b$ and $f(a) = \alpha$, $f(b) = \beta$, then it has a single-valued inverse function $x = \phi(y)$, $\alpha \leq y \leq \beta$ and this inverse function is also continuous and monotonic.

**A1.2.5 Further Theorems on Continuous Functions:** We leave it to the reader to prove the following almost trivial fact: A continuous function of a continuous function is itself continuous, i.e., if $\phi(x)$ is a function, continuous in the interval $a \leq x \leq b$, its functional values lie in the interval $\alpha \leq \phi \leq \beta$ and, in addition, if $f(\phi)$ is a continuous function of $\phi$ in this last interval, then $f(\phi(x))$ is a continuous function of $x$ in the interval $a \leq x \leq b$. (**Theorem of the continuity of functions of continuous functions**.)

It has already been mentioned that the sum, difference and product of continuous functions are themselves continuous and that the quotient of continuous functions is continuous, provided that the denominator remains different from zero.

## A1.3 Some Remarks on the Elementary Functions:

In Chapter I, we have tacitly assumed that the elementary functions are continuous. The proof of this fact is very simple. Firstly, the function $f(x) = x$ is continuous, whence $y = x^2$ as the product of two continuous functions is continuous and every power of $x$ is likewise continuous. Thus, every polynomial is continuous, being a sum of continuous functions. Every rational, fractional function, as a quotient of continuous functions, is likewise continuous in every interval in which the denominator does not vanish.

The function $x^n$ is continuous and monotonic. Hence the $n$-th root, as the inverse function of the $n$-th power, is continuous. By the theorem of the continuity of functions of continuous functions, the $n$-th root of a rational function is continuous (except where the denominator vanishes).

The continuity of the trigonometric functions, with which the reader is familiar from elementary mathematics, could now readily be proved, using the concepts developed above. The discussion is not given here, since it will be seenin 2.3.5 that this continuity follows naturally as a consequence of their differentiability.

We shall merely comment here on the definition and continuity of the exponential function $a^x$, the general power function $x^\alpha$ and the logarithm. We assume, as in 1.3.4, that $a$ is a positive number, say, greater than 1, and, if $r=p/q$ is a positive rational number ($p$ and $q$ being integers), we take $a^r = a^{p/q}$ to mean the positive number the $q$-th power of which is $a^p$. If $\alpha$ is any irrational number and $r_1, r_2, \cdots, r_m, \cdots$ is a sequence of rational numbers approaching $\alpha$, we assert that $\lim\limits_{m\to\infty} a^{r_m}$ exists and then call this limit $a^\alpha$.

In order to prove the existence of this limit by Cauchy's test, we need only show that $|a^{r_n} - a^{r_m}|$ is arbitrarily small, provided that $n$ and $m$ are sufficiently large. For example, let $r_n > r_m$, i.e., that $r_n - r_m = \delta$, where $\delta > 0$. Then

$$a^{r_n} - a^{r_m} = a^{r_m}(a^\delta - 1).$$

Since $a^{r_m}$ remains bounded, we need only show that

$$|a^\delta - 1| = a^\delta - 1$$

is arbitrarily small when the values of $n$ and $m$ are sufficiently large. However, $\delta$ is a rational number and certainly may be made as small as we please, provided the values of $n$ and $m$ are sufficiently large. Hence, if $l$ is an arbitrarily large positive integer, $\delta < 1/l$, if $n$ and $m$ are large enough. Now, the relations $\delta < 1/l$ and $a > 1$ yield*

$$1 < a^\delta < a^{1/l},$$

and since $a^{1/l}$ tends to 1 as $l$ increases our assertion follows immediately.

This statement follows from the fact that, when a > 1, the power $a^{m/n}$ is greater than 1 provided $m/n$ is positive. This is clearly true. For if $a^{m/n}$ were less than 1, then $a^m = (a^{m/n})^n$ would be a product of $n$ factors all less than 1 and would be less than 1. On the contrary, $a^m$ is the product of $n$ factors all greater than 1, and so it is greater than 1.

It may be left to the reader to show that the function $a^x$, extended to irrational values in this manner, is also continuous everywhere and, moreover, that it is a monotonic function. For negative values of $x$, this function is naturally defined by the equation

$$a^x = \frac{1}{a^{-x}}.$$

As $x$ runs from $-\infty$ to $+\infty$, $a^x$ takes all values between 0 and $+\infty$. Consequently, it possesses a continuous and monotonic inverse function, which we call the **logarithm to the base a**. In a similar manner, we can prove that the general power $x^\alpha$ is a continuous function of $x$, where $\alpha$ is any fixed rational or irrational number and $x$ varies over the interval $0 < x < \infty$, as well as that $x^\alpha$ is monotonic if $\alpha \neq 0$.

The elementary discussion of the exponential function, the logarithm and the power $x^\alpha$, which has been sketched here will later on be replaced by another discussion which, in principle, is much simpler.

## Exercises 1.8:

1. Give the upper and lower bounds and upper and lower limits for the following sequences and state which belong to the sequence:

(a) $\frac{6^n}{n!}$, $n = 1, 2, \ldots$.  (b) $0, \frac{(-1)^n}{n!}$, $n = 1, 2, \ldots$.

(c) $\frac{(-1)^n}{n} + \frac{n}{2n+1}$, $n = 1, 2, \ldots$.  (d) $1 + \frac{(-1)^n}{n} + \frac{(-1)^n n}{2n+1}$, $n = 1, 2, \ldots$.

(e) $\frac{1}{m^2} + \frac{1}{n^2}$, $m, n = 1, 2, \ldots$.

2.* Prove that if $f(x)$ is continuous for $a \le x \le b$, then there exists for every $\varepsilon > 0$ a polygonal function $\phi(x)$ (i.e., a continuous function the graph of which consists of a finite number of rectilinear segments meeting at corners} such that $|f(x) - \phi(x)| < \varepsilon$ for every $x$ in the interval.

3. Prove that every polygonal function $\phi(x)$ can be represented by a sum

$$\varphi(x) = a + bx + \Sigma c_i \,|\, x - x_i \,|\,,$$

where the $x_i$ are the abscissae of the corners.
Find a formula of this kind for the function $f(x)$, defined *by* the equations:

$$\begin{aligned} f(x) &= 2x - 1 \; (0 \leqq x \leqq 2), \\ f(x) &= 5 - x \; (2 \leqq x \leqq 3), \\ f(x) &= x - 1 \; (3 \leqq x \leqq 5), \\ f(x) &= 4 \; (5 \leqq x \leqq 7). \end{aligned}$$

4. As in A1.2.2, find for the following functions $\delta(\varepsilon)$ such that $(f(x_1) - f(x_2)| < \delta(\varepsilon)$:

(a) $f(x) = 2x^3, -1 \leqq x \leqq 1.$
(b) $f(x) = x^n, -a \leqq x \leqq a.$
*(c) $f(x) = \sqrt[3]{1 - x^2}, -1 \leqq x \leqq 1.$

5.* The function $y = \sin(1/x)$ has a discontinuity in the interval $.0 < x < 1$. Prove that it is **not** uniformly continuous in that open interval.

6. A function $f(x)$ is defined for all values of $x$ as follows:

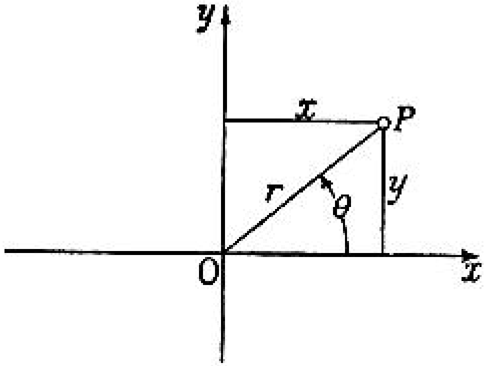



Fig. 25.—Polar co-ordinates

$f(x) = 0$ for all irrational values of $x$,
$f(x) = 1/q$ for all rational $x = p/q$,

where $p/q$ is a fraction in its lowest terms (thus, for $x = 16/29$, $f(x) = 1/29$).

Prove that $f(x)$ is continuous for all irrational and discontinuous for all rational values of $x$.

Answer and Hints

# Appendix II to Chapter I

**A2.1 Polar Co-ordinates:** In Chapter I, we have placed the concept of function in the foreground and represented functions geometrically by means of curves. However, it is useful to recall that analytical geometry follows the reverse procedure; it begins with a curve given by some geometrical property and represents this curve by a function, for example, by a function which expresses one of the coordinates of a point of the curve in terms of the other coordinate. This point of view naturally leads us to consider, apart from rectangular coordinates, to which we have restricted ourselves in Chapter I, other systems of coordinates which may be better suited for the representation of curves which are given geometrically, The most important example is that of the **polar coordinates** $r$, $\theta$, connected with the rectangular co-ordinates $x$, $y$ of a point $P$ by the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x},$$

and the geometrical interpretation of which follows from Fig. 25.

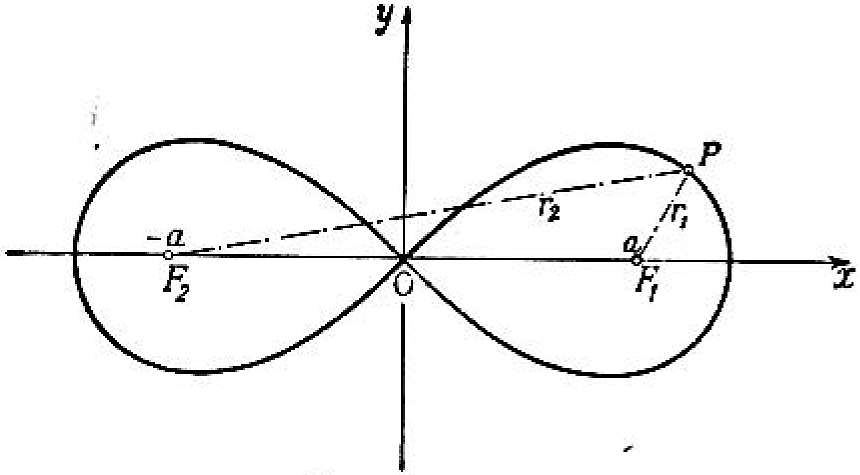



Fig. 26.—Lemniscate

Consider as an example the **lemniscate**. It is geometrically defined as the locus of all points for which the product of the distances $r_1$ and $r_2$ from the fixed points $P_1$ and $P_2$ with the rectangular co-ordinates $x = a$, $y = 0$ and $x = -a$, $y=0$, respectively, has the constant value $a^2$ (Fig. 26). Since

$$r_1^2 = (x - a)^2 + y^2, \quad r_2^2 = (x + a)^2 + y^2,$$

a simple calculation yields the equation of the lemniscate in the form

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0.$$

If we now introduce polar coordinates, we obtain

$$r^4 - 2a^2r^2(\cos^2\theta - \sin^2\theta) = 0;$$

now, division by $r^2$ and use of a simple trigonometric formula yields

$$r^2 = 2a^2 \cos 2\theta.$$

Thus, we see that the equation of the lemniscate in polar coordinates is simpler than in rectangular coordinates.

**A2.2. Remarks on Complex Numbers:** Our study will be based chiefly on the class of real numbers. Nevertheless, with a view to the discussions in Chapters VIII, IX and XI, we remind the reader that the problems of

algebra have led to a still wider extension of the number concept, namely, to the introduction of complex numbers. The advance from the natural numbers to the class of all real numbers arose from the desire to eliminate exceptional phenomena and to make always possible operations such as subtraction, division, and correspondence between points and numbers. Similarly, we are compelled, by the requirement that every quadratic equation and, in fact, every algebraic equation shall have a solution, to introduce complex numbers. For example, if we wish the equation

$$x^2 + 1 = 0$$

to have roots, we are obliged to introduce the new symbols $i$ and $-i$ as the roots of this equation. (It is shown in algebra that this is sufficient to ensure that **every** algebraic equation shall have a solution.*

* That every algebraic equation possesses real or complex roots is the statement of the **Fundamental Theorem of Algebra**.

If $a$ and $b$ are ordinary real numbers, the **complex number** $c = a + ib$ denotes a pair of numbers $(a, b)$, calculations with such pairs of numbers being performed according to the general rule: We add, multiply and divide complex numbers (which include the real numbers as the special case $b = 0$), treating the symbol $i$ as an undetermined quantity and then simplify all expressions by using the equation $i^2 = -1$ to remove all powers of $i$ higher than the first, thus leaving only an expression of the form $a + ib$.

We may assume that the reader has already a certain degree of familiarity with these complex numbers. We shall nevertheless emphasize a particularly important relationship which we shall explain in connection with the geometrical or trigonometrical representation of complex numbers. If $c = x + iy$ is such a number, we represent it in a **rectangular co-ordinate system** by the point $P$ with the co-ordinates $x$ and $y$. By means of the above equations $x= r\cos\theta$, $y=r\sin\theta$, we now introduce the **polar coordinates** $r$ and $\theta$ instead of the rectangular co-ordinates $x$ and $y$. Then $r^2 = \sqrt{x^2 + y^2}$ is the distance of the point $P$ from the origin and $\theta$ is the angle between the positive $x$-axis and the segment $OP$. The complex number $c$ is now represented in the form

$$c = r(\cos\theta + i\sin\theta).$$

The angle $\theta$ is called the **amplitude of the complex number** $c$, the quantity $r$ its **absolute value** or **modulus**) for which we also write $|c|$Id. Obviously, there corresponds to the **conjugate complex number** $c = x - iy$ the same absolute value, but (except in the case of negative real values of $c$) the angle $-\theta$. Obviously,

$$r^2 = |c|^2 = c\bar{c} = x^2 + y^2.$$

If we use this trigonometrical representation, the multiplication of complex numbers takes a particularly simple form, because then

$$\begin{aligned} c \,.\, c' &= r(\cos\theta + i\sin\theta)\,.\,r'(\cos\theta' + i\sin\theta') \\ &= rr'(\cos\theta\cos\theta' - \sin\theta\sin\theta') \\ &\qquad + i(\cos\theta\sin\theta' + \sin\theta\cos\theta'). \end{aligned}$$

If we recall the **addition theorems** for the trigonometric functions, this becomes

$$c\,.\,c' = rr'(\cos(\theta+\theta') + i\sin(\theta+\theta')).$$

Hence we multiply complex numbers by multiplying their absolute values and adding their amplitudes. The remarkable formula

$$(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') = \cos(\theta+\theta') + i\sin(\theta+\theta')$$

is usually called **De Moivre's Theorem**. It leads us at once to the relation

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

which, for example, enables us at once to solve the equation $x^n = 1$ for positive integers $n$, the roots (the so-called roots of unity) being

$$\epsilon_1 = \epsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},\quad \epsilon_2 = \epsilon^2 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n},\quad \ldots,$$

$$\epsilon_{n-1} = \epsilon^{n-1} = \cos\frac{(n-1)\pi}{n} + i\sin\frac{(n-1)\pi}{n},\quad \epsilon_n = \epsilon^n = 1.$$

Moreover, if we imagine the expression on the left-hand side of the equation

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

expanded by the **binomial theorem**, we need only separate the real and imaginary parts, in order to obtain expressions for cos $n\theta$ and sin $n\theta$ in terms of powers and products of powers of sin $\theta$ and cos $\theta$.

Exercises 1.9:

1. Plot the graphs of the functions:

$$r = \sin\varphi, \qquad r = \cos 5\varphi,$$
$$r = \varphi, \qquad r = \frac{1}{\cos(\varphi - \alpha)},\ \alpha \text{ constant.}$$
$$r = \sin 6\varphi,$$

2. Find the polar equation of

(*a*) the circle with radius *a* with centre at the origin,
(*b*) the circle with radius *a* with centre $(a, \phi_0)$;
(*c*) the general straight line.

3. Use De Moivre's theorem to express cos $2\theta$ and sin $2\theta$ in terms of sin $\theta$ and cos $\theta$; similarly, for cos $3\theta$, sin $3\theta$, cos $5\theta$, sin $5\theta$.
Prove that cos $n\theta$ is a polynomial in cos $\theta$, and also that, if $n$ is odd, sin $n\theta$ is a polynomial in sin $\theta$.
4. Work out the following expressions and state the modulus and amplitude of each of the numbers involved and of the answers:

(*a*) $-3, 2i$,
(*b*) $(4+4i)(\frac{1}{2} - \frac{1}{2}\sqrt{3}i)$,
(*c*) $(1+i)(1-i)$,
(*d*) $(\sqrt{3} - i)^2$,
(*e*) $1^{1/2}$,
(*f*) $i^{1/2}$,
(*g*) $(1+i)^{1/2}$,
(*h*) $(3-3i)^{2/3}$,
(*k*) $1^{1/3}$,
(*l*) $(16i)^{1/4}$.

6.* Prove that, if $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, where $n$ is an integer greater than 1,

$$\varepsilon^{\nu} + \varepsilon^{2\nu} + \varepsilon^{3\nu} + \dots + \varepsilon^{n\nu} = \begin{cases} 0 \text{ if } n \text{ is not a factor of } \nu, \\ n \text{ if } n \text{ is a factor of } \nu. \end{cases}$$

Answers and Hints

## Chapter II

# The Fundamental Ideas of the Integral and Differential Calculus

Among the limiting processes of analysis, there are two processes with an especially important role, not only because they arise in many different connections, but chiefly due to their very close reciprocal relationship. Isolated examples of these two limiting processes, **differentiation** and **integration**, have even been considered in classical times; but it is the recognition of their complementary nature and the resulting development of a new and methodical mathematical procedure which marks the beginning of the **real systematic differential and integral calculus**. The credit of initiating this development belongs equally to the two great geniuses of the Seventeenth Century, **Newton** and **Leibnitz**, who, as we know to-day, made their discoveries independently of each other. While Newton, in his investigations, may have succeeded in stating his concepts more clearly, Leibnitz's notation and methods of calculation were more highly developed; even to-day, these formal portions of Leibnitz's work form an indispensable element in the theory.

## 2.1 The Definite integral

We first encounter the integral in the problem of measuring the area of a plane region bounded by curved lines. Then, more refined considerations permit us to separate the notion of integral from the naïve intuitive idea of area and to express it analytically in terms of the notion of number only. We shall find this analytical definition of the integral to be of great significance not only because it alone enables us to attain complete clarity in our concepts, but also because its applications extend far beyond the calculation of areas. We shall begin by considering the matter intuitively.

**2.1.1 The integral as an Area:** Let there be are given a function $f(x)$, which is continuous and positive in an interval, and two values $a$ and $b$ $(a < b)$ in that interval. We think of the function as being represented by a curve and consider the area of the region which is bounded above by the curve, at the sides by the straight lines $x = a$ and $x = b$ and below by the portion of the $x$-axis between the points $a$ and $b$ (Fig. 1). That there is a definite meaning to speaking of the area of this region is an assumption inspired by intuition, which we state here

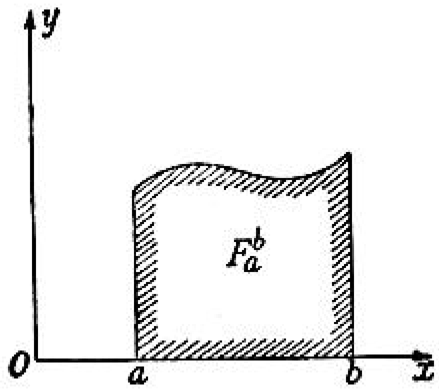



Fig. 1

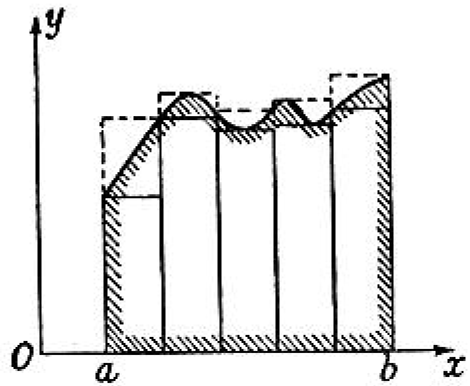



Fig. 2.—Upper sum and lower sum

expressly as a hypothesis. We call this area $F_a^b$ the **definite integral** of the function $f(x)$ between the limits $a$ and $b$. When we actually seek to assign a numerical value to this area, we find that we are, in general, unable to measure areas with curved boundaries, but we can measure polygons with straight sides by dividing them into rectangles and triangles. Such a sub-division of our area is usually impossible. It is, however, only a short step to conceive in the following manner the area as the limiting value of a sum of areas of rectangles. We subdivide the part of the $x$-axis between $a$ and $b$ into $n$ equal parts and erect at each point of sub-division the ordinate up to the curve; the area is thus divided into $n$ strips. We can no more calculate the area of such strips than we could that of the original surface; but if, as shown in Fig. 2, we find first the least and then the greatest value of the function $f(x)$ in each sub-interval and then replace the corresponding strip (1) by a rectangle with height equal to the least value of the function, and (2) by a rectangle with height equal to the greatest value of the function, we obtain two step-shaped figures. (In Fig. 2 above, the first of these is drawn with a solid line, the second with a broken line.) The first step-shaped figure obviously has an area which is at most equal to the area $F_a^b$ which we are trying to determine; the second has an area which is at least as large as $F_a^b$. If we denote the sum of the areas of the first set of rectangles by $\underline{F}_n$ (**lower sum**) and the sum of the areas of the second set by $\overline{F}_n$ (**upper sum**), we find

$$\underline{F}_n \leqq F_a^b \leqq \overline{F}_n.$$

If we now make the subdivision finer and finer, i.e., let $n$ increase without limit, intuition tells us that the quantities $\overline{F}_n$ and $\underline{F}_n$ approach closer and closer to each other and tend to the same limit $F_a^b$. We may therefore consider our integral as the limiting value

$$F_a^b = \lim_{n \to \infty} \underline{F_n} = \lim_{n \to \infty} \overline{F_n}.$$

Intuition also tells us the possibility of an immediate generalization. It is by no means necessary that the *n* sub-intervals should all be of the same length. On the contrary, they may have different lengths provided only that, as *n* increases, the length of the longest sub-interval tends to 0.

**2.1.2 The Analytical Definition of the Integral:** In the above section, we have considered the definite integral as a number given by an area, hence to a certain extent as previously known, and have subsequently represented it as a limiting value. We shall now reverse this procedure. We no longer take the point of view that we know by intuition how an area can be assigned to the region under a continuous curve or, indeed, that this is possible; we shall, on the contrary, begin with sums formed in a purely analytical way, like the **upper and lower sums** defined previously, and shall then prove that these sums tend to a definite limit. We take this limiting value as the **definition of the integral** and of the **area**. We are naturally led to adopt the formal **symbols** which have been used in the integral calculus since Leibnitz's time.

Let $f(x)$ be a function which is positive and continuous in the interval $a \le x \le b$ (of length $b - a$). We think of the interval as being sub-divided by ($n$—1) points $x_1, x_2, \cdots, x_{n-1}$ into $n$ equal or unequal sub-intervals and, in addition, we let $x_0=a$, $x_n=b$. In each interval, we choose a perfectly arbitrary point, which may be within the interval or at either end; suppose that in the first interval we choose the point $\xi_1$, in the second one the point $\xi_2$, $\cdots$ and in the $n$-th interval the point $\xi_n$. Instead of the continuous function $f(x)$, we now consider a discontinuous function (**step-function**) which has the constant value $f(\xi_1)$ in the first sub-interval, the constant value $f(\xi_2)$ in the second sub-interval, $\cdots$, the constant value $f(\xi_n)$ in the $n$-th sub-interval. As is shown in Fig. 3, the graph of

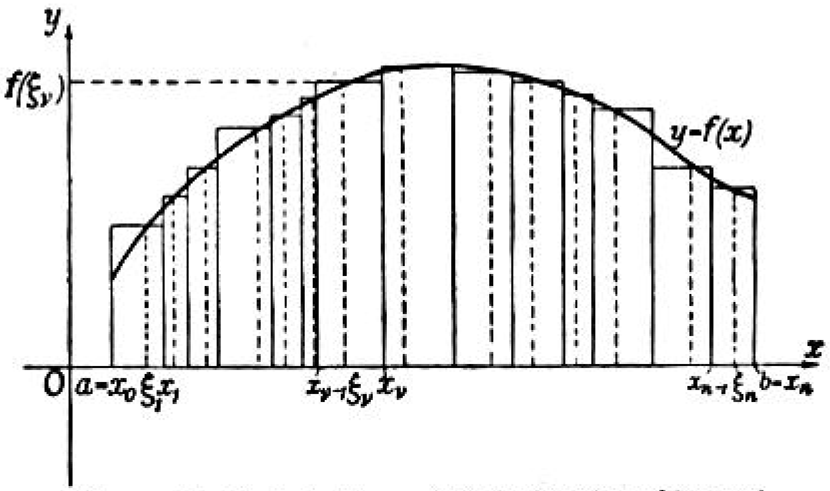



Fig. 3.—To illustrate the analytical definition of integral

this step-function defines a series of rectangles, the sum of the areas of which is given by

$$F_n = (x_1 - x_0)f(\xi_1) + (x_2 - x_1)f(\xi_2) + \ldots + (x_n - x_{n-1})f(\xi_n).$$

This expression is usually abbreviated by means of the summation sign Σ:

$$F_n = \sum_{\nu=1}^{n} (x_\nu - x_{\nu-1})f(\xi_\nu);$$

by introducing the symbol

$$\Delta x_\nu = x_\nu - x_{\nu-1}$$

we can simplify this formula to

$$F_n = \sum_{\nu=1}^{n} f(\xi_\nu)\Delta x_\nu.$$

(Here the symbol Δ is not a factor, but denotes the word **difference**. By definition, the entire inseparable symbol $\Delta x_\nu$ means the length of the $\nu$-th sub-interval.) Our basic assertion may now be stated as follows:

If we let the number of points of sub-division increase without limit and at the same time let the length of the longest sub-interval tend to 0, then the above sum tends to a limit. This limit is independent of the particular manner in which the points of division $x_1, x_2, \cdots, x_{n-1}$ and the intermediate point $\xi_1, \xi_2, \cdots, \xi_{n-1}$ are. chosen.

We shall call this limiting value the definite integral of the function $f(x)$, the **integrand**, between the limits $a$ and $b$; as we have already mentioned, we shall consider it as the definition* of the area under the curve $y = f(x)$ for $a \leq x \leq b$. Our basic assertion may then be re-worded: If $f(x)$ is continuous in $a \leq x \leq b$, its definite integral between the limits $a$ and $b$ exists.

* Of course, we may also define the notion of are in a purely geometrical way and then prove that such a definition is equivalent to the above limit=definition (cf. 5.2.1).

This theorem on the existence of the definite integral of a continuous function can be proved by purely analytical methods and without appeal to intuition. We shall nevertheless pass it over for the present and return to it in A2.1 after the use of the concept of integral has stimulated the reader's interest in constructing for it a firm foundation. For the moment, we shall content ourselves with the fact that the intuitive considerations above have made the theorem appear to be extremely plausible.

**2.1.3 Extensions, Notation. Fundamental Rules:** The above definition of the integral as the limit of a sum led Leibnitz to express the integral by the symbol

$$\int_a^b f(x)\,dx.$$

The integral symbol is a modification of a summation sign which has the shape of a long S. The passage to the limit from a sub-division of the interval into finite portions $\Delta x_\nu$ is suggested by the use of the letter $d$ in place of $\Delta$. However, we must guard ourselves against thinking of $dx$ as an **infinitely small quantity** or **infinitesimal**, or of the integral the sum of an infinite number of **infinitely small quantities**. Such a concept would be devoid of any clear meaning; it is only a naïve interpretation of what we have previously carried out with precision.

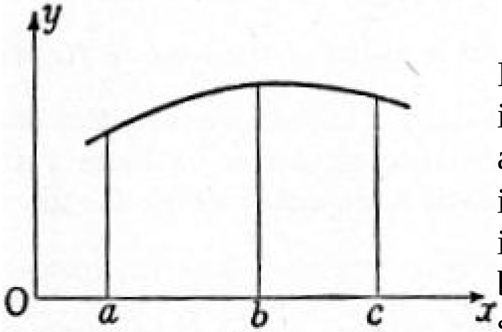



Fig. 4

In the above figures, we have assumed: (1) that the function *f*(*x*) is positive throughout the interval, and (2) that $b > a$. However, the formula which defines the integral as the limit of a sum is independent of any such assumptions. For if *f*(*x*) is negative in all or a part of our interval, the only effect is to make the corresponding factors $f(\xi_\nu)$ in our sum negative instead of positive. We shall naturally assign to the region bounded by the part of the curve below the *x*-axis a negative area, which is in agreement with the familiar convention of sign in analytical geometry. The total area bounded by a curve will thus, in general, be the sum of positive and negative terms, corresponding to the portions of the curve above and below the *x*-axis, receptively.*

* For the area of regions bounded by arbitrary closed curves, cf. 5.2.2 .

If we also omit the condition $a < b$ and assume that $a > b$, we can still retain our arithmetic definition of integral; the only change is that, if we traverse the interval from *a* to *b*, the differences $\Delta x_\nu$ are negative. We are thus led to the relation

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

which holds for all values of *a* and *b* ($a \neq b$). Correspondingly, we define $\int_a^b f(x)\,dx$ as equal to zero.

Our definition immediately yields the basic relation (cf. Fig. 4 above):

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

for $a < b < c$. By means of the preceding relations, we find at once that this equation is also true for any relative positions of the end-points *a*, *b*, *x* .

We obtain a simple but important fundamental rule by considering the function *cf*(*x*), where *c* is a constant. From the definition of the integral, we immediately obtain

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Moreover, we assert the addition rule: If

$$f(x) = \phi(x) + \psi(x),$$

then

$$\int_a^b f(x)\,dx = \int_a^b \phi(x)\,dx + \int_a^b \psi(x)\,dx.$$

The proof is quite simple.

We add a final remark about the **variable of integration**, which is perfectly obvious, but very important in applications. We have written our integral in the form $\int_a^b f(x)\,dx$. During evaluation of the integral, it does not matter whether we use the letter *x* or any other letter to denote the abscissae of the co-ordinate system, i.e., the **independent variable**. We use this particular symbol for the variable of integration and it is therefore a matter of complete indifference; instead of $\int_a^b f(x)\,dx$, we could equally well write $\int_a^b f(t)\,dt$ or $\int_a^b f(u)\,du$ or any similar expression.

## 2.2. Examples

We are now in a position to carry out the limiting process prescribed by our definition of the integral and thus actually calculate the area in question in a number of special cases; we shall do this in a series of examples, where (except in No. 5) we shall employ only the upper or the lower sum.*

*We have it as a useful exercise for the reader to prove that in the following examples we actually do arrive at the same result whether we use the upper or the lower sum.

### 2.2.1. Integration of a Linear Function:

We first consider the function $f(x) = x^n$, where $n$ is an integer greater than or equal to 0. For n = 0, i.e. for $f(x) = 1$, the result is so obvious that we simply write it down:

$$\int_a^b 1\,dx = \int_a^b dx = b - a.$$

For the function $f(x) = x$, the integration is again a triviality from the geometrical point of view. Its integral

$$\int_a^b x\,dx$$

is simply the area of the trapezoid shown in Fig. 6 below, which by an elementary formula is

$$\tfrac{1}{2}(b - a)(b + a) = \tfrac{1}{2}(b^2 - a^2).$$

We shall now verify that our limiting process leads to exactly the same result. In calculating the limit, we can restrict ourselves to the discussion of upper sums or lower sums. We subdivide the interval from $a$ to $b$ into $n$ equal parts by means of the points of sub-division

$$a + h,\quad a + 2h,\quad \ldots,\quad a + (n - 1)h,$$

where $h = (b - a)/n$. The integral must then be the limit of the following sum, which is an upper sum, if $b < a$ and a lower sum if $b > a$:

$$\begin{aligned} &h\{a + (a + h) + (a + 2h) + \ldots + (a + (n - 1)h)\} \\ &= h\{na + h + 2h + \ldots + (n - 1)h\}. \end{aligned}$$

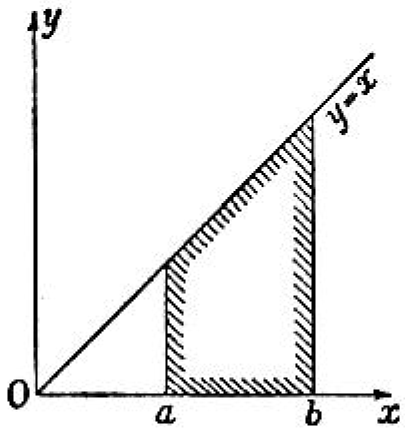


Fig. 5

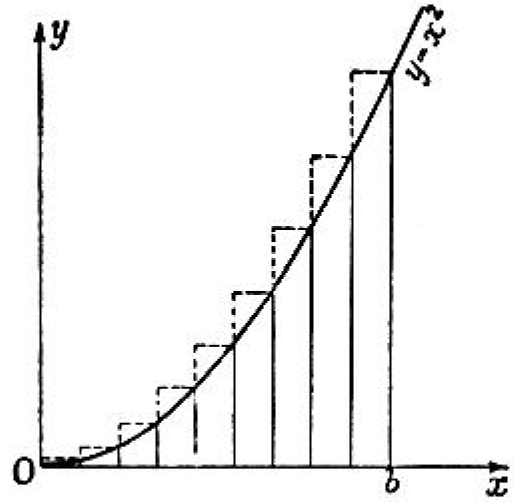


Fig. 6

By an elementary formula, we have

$$1 + 2 + \ldots + (n-1) = \tfrac{1}{2}n(n-1),$$

and our expression may therefore be rewritten in the form

$$nh\left(a + h\frac{n-1}{2}\right) = (b-a)\left(a + \frac{b-a}{2}\,\frac{n-1}{n}\right).$$

As $n$ increases, the right hand side obviously tends to the limit

$$(b-a)\{a + \tfrac{1}{2}(b-a)\} = \tfrac{1}{2}(b^2 - a^2),$$

as was to be proved.

### 2.2.2 Integration of $x^2$:

A problem which is not quite as simple as that of the integration of the function $f(x) = x^2$, or, in geometrical language, of determining the area of the region bounded by a segment of a parabola, a segment of the $x$-axis and two ordinates. For example, consider the integral

$$\int_0^b x^2\,dx,$$

where $b \geq 0$ (Fig. 6) and sub-divide the interval $0 \leq x \leq b$ into $n$ equal parts of length $h = b/n$; the area which we then wish to find is the limit of the expression (upper sum):

$$\begin{aligned} h(h^2 + 2^2h^2 + 3^2h^2 + \ldots + n^2h^2) &= h^3(1^2 + 2^2 + \ldots + n^2) \\ &= b^3(1^2 + 2^2 + \ldots + n^2)/n^3. \end{aligned}$$

However, the sum in brackets has already been found (cf. 1.4, footnote)!

$$1^2 + 2^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

If we substitute this expression and rewrite the result in a slightly different form, our sum becomes

$$\frac{b^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

As $n$ increases beyond all bounds, this expression tends to the limit $b^3/3$ and we obtain the required integral

$$\int_0^b x^2\,dx = \tfrac{1}{3}b^3.$$

Using the general result above, we immediately derive the formula

$$\int_a^b x^2\,dx = \int_0^b x^2\,dx - \int_0^a x^2\,dx = \tfrac{1}{3}(b^3 - a^3).$$

**2.2.3 Integration of $x^\alpha$, where $\alpha$ is any positive integer:** As a third example, we consider the integration of the function

$$y = f(x) = x^\alpha,$$

where $\alpha$ is any positive integer. For the computation of the integral

$$\int_a^b x^\alpha \, dx ,$$

(where we assume $0 < \alpha < b$), it would be inconvenient to subdivide the interval into $n$ equal parts.* However, the passage to the limit may be accomplished very easily, if we subdivide in **geometric progression** in the following manner. Let $\sqrt[n]{b/a} = q$ and subdivide the interval by the points

$$a, \; aq, \; aq^2, \; \ldots, \; aq^{n-1}, \; aq^n = b.$$

* We should then be obliged to base the evaluation of the integral upon the calculation of the limit of

$$\frac{1}{n^{\alpha+1}} \left(1^\alpha + 2^\alpha + \ldots + n^\alpha\right)$$

as $n \to \infty$; the reader may work this result out as has been indicated in the footnote referred to above.

The required integral is then the limit of the sum:

$$\begin{aligned} &a^\alpha (aq - a) + (aq)^\alpha (aq^2 - aq) + (aq^2)^\alpha (aq^3 - aq^2) + \ldots \\ &\quad + (aq^{n-1})^\alpha (aq^n - aq^{n-1}) \\ &= a^{\alpha+1}(q-1)\{1 + q^{\alpha+1} + q^{2(\alpha+1)} + q^{3(\alpha+1)} + \ldots + q^{(n-1)(\alpha+1)}\}. \end{aligned}$$

The terms in the last bracket form a geometric progression with the common ratio $q^{\alpha+1} \neq 1$. If we sum this progression, we obtain for the entire expression the value

$$a^{\alpha+1}(q-1)\frac{q^{n(\alpha+1)} - 1}{q^{\alpha+1} - 1}.$$

We now replace $q$ by its value $(b/a)^{1/n}$; our sum then takes the form

$$(b^{\alpha+1} - a^{\alpha+1})\frac{q-1}{q^{\alpha+1} - 1}.$$

If we now let $n$ increase without limit, the first factor retains its value. Since $q \neq 1$, we can use the formula for the sum of a geometric progression and write the second factor in the form

$$\frac{1}{q^{\alpha} + q^{\alpha-1} + \ldots + 1};$$

as the equation $q = (b/a)^{1/n}$ shows that $q$ tends to 1 as $n \to \infty$, the second factor will have the limit $1/(\alpha + 1)$. Thus, finally, the value of our integral is given by

$$\int_a^b x^{\alpha}\, dx = \frac{1}{\alpha + 1}(b^{\alpha+1} - a^{\alpha+1}).$$

In principle,the above calculation is simple, but itsdetails are somewhat complicated. We shall later on discover that it can be entirely avoided once we have become better acquainted with integration theory.

### 2.2.4 Integration of $x^{\alpha}$, where $\alpha$ is any Rational Number other than -1:

The result obtained above may be generalized considerably without essential complication of the method. Let $\alpha = r/s$ be a positive rational number, $r$ and $s$ positive integers; in the evaluation of the integral given above, nothing is changed except the evaluation of the limit $(q - 1)/(q^{n+1} - 1)$ as $q$ approaches 1. This expression is now simply $(q - 1)/(q^{(r+s)/s} - 1)$. Let us set $q^{1/s} = \tau\ (\tau \neq 1)$; then, as $q$ tends to 1, $\tau$ will also tend to 1. We have therefore to find the limiting value of $(\tau^{s} - 1)/(\tau^{r+s} - 1)$ as $\tau$ approaches 1. If we divide both the numerator and denominator by $\tau - 1$ and transform them as before, the limit simply becomes

$$\lim_{\tau \to 1} \frac{\tau^{s-1} + \tau^{s-2} + \ldots + 1}{\tau^{r+s-1} + \tau^{r+s-2} + \ldots + 1}.$$

Since both the numerator and denominator are continuous in $\tau$, this limit is at once determined it we set $\tau = 1$. We thus arrive at the limit $s/(r + s) = 1/(\alpha + 1)$ and obtain for every positive rational value of $\alpha$

$$\int_a^b x^{\alpha}\, dx = \frac{1}{\alpha + 1}(b^{\alpha+1} - a^{\alpha+1}).$$

This formula remains valid for negative rational values of $\alpha$, provided we exclude the value $\alpha = -1$ for which the formula used above for the sum of the geometric progression loses its meaning. We must now investigate the limit of the expression $(q - 1)/(q^{n+1} - 1)$ for negative values of $\alpha$, say $\alpha = -r/s$. For this purpose, we set $q^{-1/s} = \tau$ and obtain

$$q = \tau^{-s}, \quad q^{\alpha+1} = q^{-(r-s)/s} = \tau^{r-s}.$$

Hence we seek to determine the limiting value of

$$\frac{\tau^{-s} - 1}{\tau^{r-s} - 1} = \frac{1 - \tau^{s}}{\tau^{r} - \tau^{s}}.$$

We leave it to the reader to prove that this limit is again equal to $1/(\alpha + 1)$, that is, that we have the integral

$$\int_a^b x^\alpha \, dx = \frac{1}{\alpha + 1}(b^{\alpha+1} - a^{\alpha+1})$$

for the general case of rational values of either positive or negative $\alpha$, with the exception of $\alpha = -1$.

The form of the right-hand side of this equation shows that the expression is not valid for $\alpha = -1$, since both the numerator and denominator would then be zero.

It is natural to assume that the range of validity of our last formula extends also to irrational values of $\alpha$. We shall actually establish this in 2.7.2 by a simple passage to the limit.

**2.2.5 Integration of sin *x* and cos *x*:** As a last example, consider the function $f(x) = \sin x$. Also in this case, we shall employ a special device. We express the integral

$$\int_a^b \sin x \, dx$$

as the limit of the sum:

$$S_h = h\{\sin(a + h) + \sin(a + 2h) + \ldots + \sin(a + nh)\},$$

where $h = (b - a)/n$. We multiply the right-hand bracket by $2 \sin h/2$ and recall the well-known trigonometrical formula

$$2 \sin u \sin v = \cos(u - v) - \cos(u + v);$$

provided $h$ is not a multiple of $2\pi$, we obtain

$$\begin{aligned} S_h &= \frac{h}{2 \sin \frac{h}{2}} \left\{ \cos\left(a + \frac{h}{2}\right) - \cos\left(a + \frac{3}{2}h\right) + \cos\left(a + \frac{3}{2}h\right) - \cos\left(a + \frac{5}{2}h\right) \right. \\ &\qquad \left. + \dots + \cos\left(a + \frac{2n-1}{2}h\right) - \cos\left(a + \frac{2n+1}{2}h\right) \right\} \\ &= \frac{h}{2 \sin \frac{h}{2}} \left\{ \cos\left(a + \frac{h}{2}\right) - \cos\left(a + \frac{2n+1}{2}h\right) \right\}. \end{aligned}$$

Since $a + nh = b$, the integral becomes the limit of

$$\frac{h}{2 \sin \frac{h}{2}} \left\{ \cos\left(a + \frac{h}{2}\right) - \cos\left(b + \frac{h}{2}\right) \right\} \text{ as } h \to 0.$$

Now we know from Chapter I that as $h$ tends to 0, the expression $h/2/\sin h/2$ approaches the limit 1. The desired limit is then simply $\cos a - \cos b$ and we thus arrive at the integral

$$\int_a^b \sin x \, dx = -(\cos b - \cos a).$$

In the same manner, the reader may verify the formula

$$\int_a^b \cos x \, dx = \sin b - \sin a.$$

Almost every one of these examples has been attacked by means of some special method or particular device. The essential point of the systematic Integral and Differential Calculus is the very fact that we utilize instead of such special devices considerations of a general character which lead us directly to the desired result. In order to arrive at these methods, we must now turn our attention to the other fundamental concept of higher analysis, the **derivative**.

**Exercises 2.1:**

1. Find the area bounded by the parabola $y = 2x^3 + x + 1$, the ordinates $x = 1$ and $x = 3$ and the $x$-axis.

2. Find the area bounded by the parabola $y = ½x^2 + x + 1$I and the straight line $y = 3 + x$.

3. Find the area bounded by the parabola $y^2 = 5x$ and the straight line $y = 1+ x$.

4. Find the area bounded by the parabola $y = x^2$ and the straight line $y = ax + b$.

5. Using the methods developed above, evaluate the integrals

$$(a) \int_a^b (x+1)^\alpha\, dx, \quad (b) \int_a^b \sin \alpha x\, dx, \quad (c) \int_a^b \cos \alpha x\, dx,$$

where $\alpha$ is an arbitrary integer.

6. Use the formulae obtained in Example 5 along with the identities

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x, \quad \cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x,$$

to prove that

$$\int_a^b \cos^2 x\, dx = \frac{b-a}{2} + \frac{\sin 2b - \sin 2a}{4};$$
$$\int_a^b \sin^2 x\, dx = \frac{b-a}{2} - \frac{\sin 2b - \sin 2\alpha}{4}.$$

7. Use Exercise 1. in 1.4 to evaluate

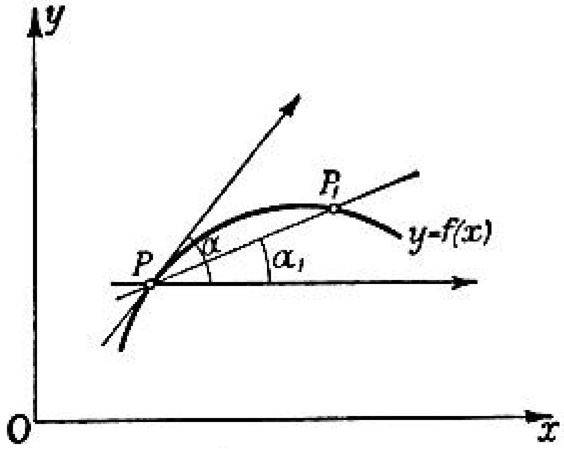


Fig. 7.—Chord and Tangent

$$\int_a^b x^3\,dx$$

by division into equal sub-intervals.

8. Evaluate

$$\int_0^1 (1-x)^n\,dx$$

(where *n* is an integer) by expansion of the bracket.

Answers and Hints

## 2.3 The Derivative

The concept of the derivative, like that of the integral, has an intuitive origin. Its sources are: (1) the problem of construction of the tangent to a given curve at a given point and (2) the problem of finding a precise definition for the velocity during an arbitrary motion.

**2.3.1 The Derivative and the Tangent:** We shall first deal with the tangent problem. If $P$ is a point on a given curve (Fig. 7), we shall, in conformity with naive intuition, define the tangent to the curve at the point $P$ by means of the following geometrical limiting process. In addition to the point $P$, we consider a second point $P_1$ on the curve. Through the two points $P$ and $P_1$, we draw a straight line, a secant of the curve. If we now let the point $P_1$ move along the curve towards the point $P$, this secant will tend to a limiting position which is independent of the direction from which it approaches $P$. This limiting position of the secant is the tangent and the statement that such a limiting position of the secant exists is equivalent to the assumption that the curve has a definite tangent or a definite direction at the point $P$. (We have used here the word assumption because we have actually made one. The hypothesis that the tangent exists is valid for most simple curves, but is by no means true for all curves or even for all continuous curves.)

Once we have represented our curve by means of a function $y = f(x)$, there arises the problem of representing our geometrical limiting process analytically, using the function $f(x)$. We take the angle, which a straight line $l$ makes with the $x$-axis, as being the angle through which the positive $x$-axis must be turned in the positive direction* in order to be for the first time parallel to the line $l$. Let $\alpha_1$ be the angle which the secant $PP_1$ forms with the positive

*x*-axis (Fig, 7) and $\alpha$ the angle which the tangent forms with the positive *x*-axis. Then, disregarding the case of a perpendicular tangent, we obviously have

$$\lim_{P_1 \to P} \alpha_1 = \alpha,$$

where the meaning of the symbols is perfectly clear. If $x$, $y$ (= $f(x)$) and $x_1$, $y_1$ (=$f(x_1)$) are the coordinates of the points $P$ and $P_1$, respectively, we immediately find**

$$\tan \alpha_1 = \frac{y_1 - y}{x_1 - x} = \frac{f(x_1) - f(x)}{x_1 - x};$$

and thus our limiting process is represented by the equation

$$\lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} = \tan \alpha.$$

* That is, in such a direction that a rotation of $\pi/2$ brings it into coincidence with the positive *y*-axis, in other words, counter-clockwise.

** In order that this equation may have a meaning, we must assume that $0 < |x - x_1| < \delta$, $\delta$ being sufficiently small. In what follows, corresponding assumptions will often be made tacitly in the steps leading to limiting processes.

The expression

$$\frac{f(x_1) - f(x)}{x_1 - x} = \frac{y_1 - y}{x_1 - x} = \frac{\Delta y}{\Delta x}$$

is called the **difference quotient** of the function $y = f(x)$, since the symbols $\Delta y$ and $\Delta x$ denote the differences of the function $y = f(x)$ and of the independent variable $x$, respectively. (As in 2.1.2, the symbol $\Delta$ is an abbreviation for the difference and is not a factor!) The tangent of $\alpha$, the direction angle of the curve,* is therefore equal to the limit to which the difference quotient of our function tends when $x_1$ tends to $x$.

* The slope or gradient of the curve is given by tan $\alpha$, whence also the term **gradient** is used for the derivative of the function represented by the curve.

We call this limit the **derivative** or **differential coefficient** of the function $y=f(x)$ at the point $x$ and, as Lagrange did, denote it by the symbol $y' = f'(x)$ or, as Leibnitz did, the symbol $dy/dx$ or $df(x)/dx$ or $d/dx\, f(x)$.* On p. 100, we shall discuss the meaning of Leibnitz's notation in greater detail; here we just point out that the notation $f'(x)$ expresses the fact that the derivative is itself a function of $x$, since it has a definite value for each value of $x$ in the interval under consideration. This fact is sometimes emphasized by the use of the terms **derived function**, **derived curve** (cf. 2.3.6). We again quote the definition of the derivative:

$$f'(x) = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x},$$

or

$$\frac{dy}{dx} = \frac{df(x)}{dx} = f'(x) = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

where we have replaced in the last expression $x_1$ by $x + h$.

***Cauchy's notation** $Df(x)$ if also occasionally found in the literature.

It is impossible to find the derivative merely by putting $x_1 = x$ in the expression for the difference quotient, for then the numerator and denominator would both be equal to 0 and we should then led to the meaningless expression 0/0. On the contrary, the actual performance of the passage to the limit in each individual case depends on certain preliminary steps (transformation of the difference quotient).

For example, for the function $f(x) = x^2$, we have

$$\frac{f(x_1) - f(x)}{x_1 - x} = \frac{x_1{}^2 - x^2}{x_1 - x} = x_1 + x.$$

This function $x_1 + x$ is not the same function as $\frac{x_1^2 - x^2}{x_1 - x}$, because the function $x_1 + x$ is defined at one point where the quotient $\frac{x_1^2 - x^2}{x_1 - x}$ is not defined, namely, at the point $x_1 = x$. For all other values of $x_1$, the two functions are equal to each other, whence in the above passage to the limit, in which we specifically required that $x_1 \neq x$, we obtain the same value $\lim_{x_1 \to x} \frac{x_1^2 - x^2}{x_1 - x}$ as for $\lim_{x_1 \to x} (x_1 + x)$. However, since the function $x_1 + x$ is defined and continuous at the point $x_1 = x$, we can do with it what we could not do with the quotient, namely, pass to the limit by simply putting $x_1 = x$. We then obtain for the derivative the expression

$$f'(x) = \frac{d(x^2)}{dx} = 2x.$$

The performance of such a process, i.e., the actual formation of the derivative is called the **differentiation of the function $f(x)$**. We shall see later on how this process of differentiation can actually be carried out in all important cases.

Now, the fact that the problem of differentiating a given function has a definite meaning apart from the geometrical intuition of the tangent is of great significance. The reader will recall that, in the case of the integral, we freed ourselves from the geometrical intuition of area and, on the contrary, based the notion of area on the definition of the integral. Now, independently of the geometrical representation of a function $y = f(x)$ by means of a curve, we shall define the derivative of the function $y = f(x)$ as being the new function $y' = f'(x)$, given by the equation above, provided always that the limit of the difference quotient exists. If this limit exists, we say that the function $f(x)$ is **differentiable**. From now on, we shall assume that every function dealt under consideration is differentiable unless specific mention is made to the contrary.* It should be observed that, if the function $f(x)$ is to be differentiable at the point $x$, the limit as $h \to 0$ of the quotient $[f(x + h) - f(x)]/h$ must exist independently of the manner in which $h$ tends to 0, whether through positive or through negative values or without restriction as to the sign.

* Examples of cases in which this assumption is not satisfied will be given later (2.3.5).

Once we have found the derivative $f'(x)$, we take the direction which makes an angle $\alpha$ with the positive $x$-axis given by the equation $\tan \alpha = f'(x)$ as the direction of the tangent to the curve at the point $(x,y)$. We thus avoid the difficulties which arise out of the indefiniteness of the geometrical view, since we base the geometrical definition on the analytical one and not *vice versa*.

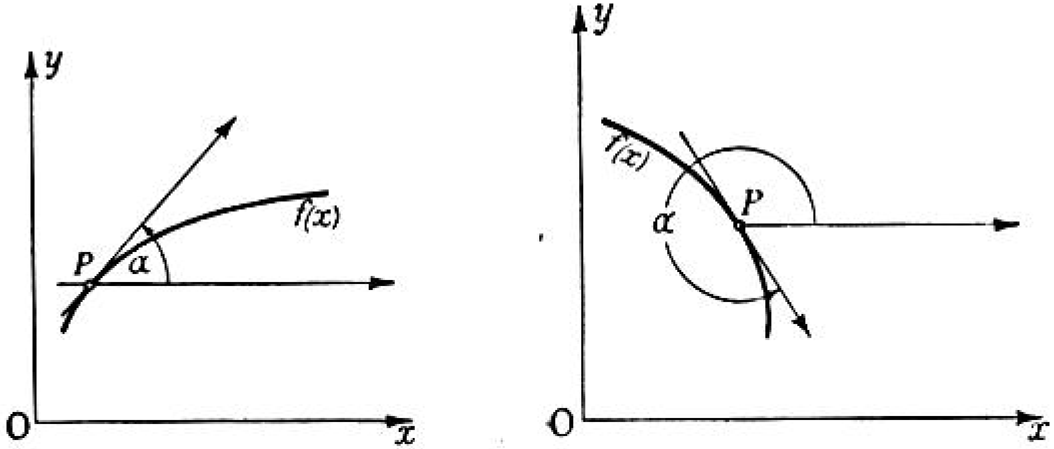


Fig. 8.—Tangents to graphs of increasing and decreasing functions

Nevertheless, the visualization of the derivative as the tangent to the curve is an important aid to understanding, even in purely analytical discussions. Accordingly, we shall at once accept the statement based on geometrical intuition: If $f'(x)$ is positive and the curve is traversed in the direction of increasing $x$, then the tangent slants upwards, and therefore at the point in question the curve rises as $x$ increases; on the other hand, if $f'(x)$ is negative, the tangent slants downwards and the curve falls as $x$ increases (Fig. 8). Analytically, this follows from the remark that the limit of $[f(x + h) - f(x)]/h$ cannot be positive unless the function is increasing at the point $x$, by which we mean that for all values of $h$ sufficiently close to 0 the value of $f(x + h)$ is greater or smaller than $f(x)$ according to whether $h$ is positive or negative. We can, of course, make a corresponding statement for the case when $f'(x)$ is negative.

**2.3.2 The Derivative as a Velocity:** Just as naive intuition led us to the notion of the direction of the tangent to a curve, so it causes us to assign a velocity to a motion. The definition of velocity leads us once again to exactly the same limiting process which we have already called differentiation.

For example, consider the motion of a point along a straight line, the position of the point being determined by a single co-ordinate $y$, which is the distance, with its proper sign, of our moving point from a fixed point on the line. The motion is given, if we know $y$ as a function of the time $t$, $y = f(t)$. If this function is a linear function $f(t) = ct + b$, we speak of **uniform motion** with the velocity $c$, and for every pair of values t and $t_1$, which are not equal to each other, we can write

$$c = \frac{f(t_1) - f(t)}{t_1 - t}.$$

The velocity is therefore the difference quotient of the function $ct + b$, and this difference quotient is completely independent of the particular pair of instants considered. But what are we to understand by the velocity of motion at an instant $t$ if the motion is no longer uniform?

In order to arrive at this definition, we consider the difference quotient $\frac{f(t_1) - f(t)}{t_1 - t}$ which we shall call the **average velocity** in the time interval between $t_1$ and $t$. If now this average velocity tends to a definite limit when we let the instant $t_1$ come closer and closer to $t$, we shall naturally define this limit as the **velocity at the time *t***. In other words: The velocity at the time $t$ is the derivative

$$f'(t) = \lim_{t_1 \to t} \frac{f(t_1) - f(t)}{t_1 - t}.$$

From this new meaning of the derivative, which in itself has nothing to do with the tangent problem, we see that it really is appropriate to define the limiting process of differentiation as a purely analytical operation independently of geometrical intuitions. Here again, the differentiability of the position-function is an assumption which we shall always make tacitly and which, in fact, is absolutely necessary if the notion of velocity is to have any meaning.

As a simple example of the connection between motion and velocity, consider the case of a freely falling body. We begin with the experimentally established law that the distance traversed in time $t$ by a freely falling body is proportional to $t^2$ and therefore can be represented by a function of the form

$$y = f(t) = at^2.$$

As before, we find immediately that the velocity is given by the expression $f'(t) = 2at$, which shows that the velocity of a freely falling body increases proportionally to the time.

**2.3.3 Examples:** We now proceed to work out a number of examples of the actual differentiation of functions and begin with the function $y = f(x)$, where $c$ is a constant. It is then always true that $f(x + h) - f(x) = c - c = 0$, so that $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0$, that is, the derivative of a constant is zero.

For a linear function $y = f(x) = cx + b$, we find that

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{ch}{h} = c.$$

Moreover, we shall differentiate the function

$$y = f(x) = x^\alpha,$$

at first assuming that $\alpha$ is a positive integer. Provided $x_1 \neq x$, we have

$$\frac{f(x_1)-f(x)}{x_1 - x} = \frac{x_1^\alpha - x^\alpha}{x_1 - x};$$

the right-hand side of this equation is equal to $x_1^{\alpha-1} + x_1^{\alpha-2}x + \cdots + x^{\alpha-1}$, as we see either by direct division or by using the formula for the sum of a geometric progression. The new expression for the right-hand side of the equation is a continuous function, whence we can carry out the passage to the limit ($x_1 \to x$) by simply replacing $x_1$ everywhere by $x$. Each term is then $x^{\alpha-1}$ and since the number of terms is exactly $\alpha$, we obtain

$$y' = f'(x) = \frac{d(x^\alpha)}{dx} = \alpha x^{\alpha-1}.$$

We arrive at the same result if $\alpha$ is a negative integer - $\beta$; however, we must assume that $x$ is not zero. We then find that

$$\frac{f(x_1)-f(x)}{x_1 - x} = \frac{\frac{1}{x_1^\beta} - \frac{1}{x^\beta}}{x_1 - x} = -\frac{x^\beta - x_1^\beta}{x - x_1} \cdot \frac{1}{x^\beta x_1^\beta}$$
$$= -\frac{x^{\beta-1} + x^{\beta-2}x_1 + \ldots + x_1^{\beta-1}}{x_1^\beta x^\beta}.$$

Once again, we can carry out the passage to the limit simply by replacing $x_1$ ] everywhere by $x$. Then, just as above, we obtain for the limit the expression

$$y' = -\beta \frac{x^{\beta-1}}{x^{2\beta}} = -\beta x^{-\beta-1}.$$

Hence, for negative integral values of $\alpha$, the derivative is again given by

$$y' = \alpha x^{\alpha-1}.$$

Finally, we shall prove the same formula, where $x$ is positive and $\alpha$ any rational number. We let $\alpha = p/q$, where both $p$ and $q$ are integers and, moreover, positive. (If one of them were negative, no essential changes in the proof would be required; for $\alpha = 0$, the result is already known, since $x^\alpha$ is then constant.) We now have

$$\frac{f(x_1) - f(x)}{x_1 - x} = \frac{x_1^{p/q} - x^{p/q}}{x_1 - x}.$$

If we now let $x^{1/q} = \xi$ and $x_1^{1/q} = \xi_1$, we obtain

$$\frac{f(x_1) - f(x)}{x_1 - x} = \frac{\xi_1^p - \xi^p}{\xi_1^q - \xi^q} = \frac{\xi_1^{p-1} + \xi_1^{p-2}\xi + \ldots + \xi^{p-1}}{\xi_1^{q-1} + \xi_1^{q-2}\xi + \ldots + \xi^{q-1}}.$$

After this last transformation, we can immediately perform the passage to the limit ($x_1 \to x$ or, what amounts to the same thing. $\xi_1 \to \xi$), and thus obtain for the limiting value

$$y = \frac{p}{q}\,\frac{\xi^{p-1}}{\xi^{q-1}} = \frac{p}{q}\xi^{p-q} = \frac{p}{q}x^{(p-q)/q} = \frac{p}{q}x^{p/q-1},$$

which is formally the same result as before. We leave it to the reader to prove that the same differentiation formula holds also for negative rational superscripts. Once we have developed the theory in a more connected form, we shall return in 2.7.2 to the differentiation of powers.

Finally, as a further example, we consider the differentiation of the trigonometric functions: sin $x$ and cos $x$. We use the elementary trigonometrical formula

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}.$$

Now, by 1.7, we know that

$$\lim_{h \to 0} \frac{\sin h}{h} = 1, \quad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

Thus, we find for the required derivative immediately

$$y' = \frac{d(\sin x)}{dx} = \cos x.$$

The function $y = \cos x$ can be differentiated in exactly the same manner. Starting with

$$\frac{\cos(x+h) - \cos x}{h} = \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h},$$

and taking the limit as $h \to 0$, we at once obtain the derivative

$$y' = \frac{d(\cos x)}{dx} = -\sin x.$$

**2.3.4. Some Fundamental Rules of Differentiation:** Just as in the case of the integral, certain simple, but fundamental rules for forming the derivative follow immediately from the definition. If $\phi(x) = f(x) + g(x)$, then $\phi'(x) = f'(x) + g'(x)$; again, if $\psi(x) = cf(x)$ (where $c$ is a constant), then $\psi'(x) = cf'(x)$. Because

$$\frac{\phi(x+h) - \phi(x)}{h} = \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

and

$$\frac{\psi(x+h) - \psi(x)}{h} = c\frac{f(x+h) - f(x)}{h},$$

our statements follow directly by passage to the limit.

For example, according to these rules, the derivative of the function $\phi(x)=f(x)+ax+b$ (where $a$ and $b$ are constants) is given by the equation

$$\phi'(x)=f'(x)+a.$$

**2.3.5 Differentiability and Continuity of Functions:** It is useful to point out that, if we know that a function can be differentiated, we need not give any special proof of its continuity.

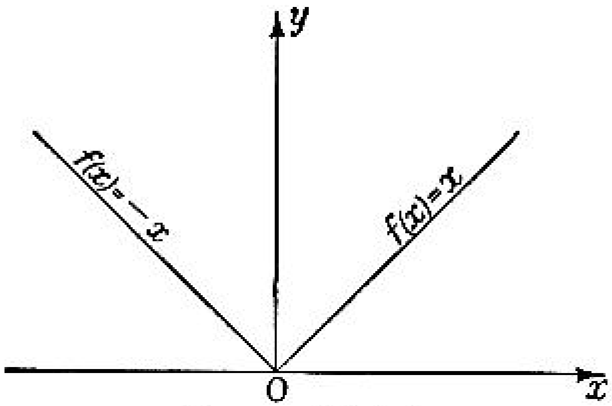


Fig. 9. $f(x)=|x|$

If a function is differentiable, then it is necessarily continuous.

In fact, if the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ approaches a definite limit as $h$ tends to zero, the numerator of the fraction, that is, $f(x+h) - f(x)$, must tend to zero with $h$, and this fact expresses the continuity of the function $f(x)$ at the point $x$.

However, the converse of this is definitely false; it is not true that every continuous function has a derivative at every point. The simplest example, which disproves this assumption, is the function $f(x) = |x|$, i.e., $f(x) = -x$ for $x \leq 0$ and $f(x) = x$ for $x \geq 0$; its graph is shown in Fig. 9. At the point $x = 0$, this function is continuous, but has no derivative. The limit of $[f(x+h)-f(x)]/h$ is equal to 1 if $h$ tends to 0 through positive values and equal to $-1$ if $h$ tends to zero through negative values; if we do not restrict the sign of $h$, There does not exist a limit. We say that the function has different right hand and and left-hand derivatives at the point $x$, where by we mean the limiting values of $[f(x+h)-f(x)]/h$ as $h$ approaches 0 through positive values only and negative values only, respectively. Thus, the differentiability of a function requires not merely that the right-hand and left-hand derivatives exist, but that they are equal. Geometrically, the inequality of the two derivatives means that the curve has a **sharp corner**.

As further examples of points where a continuous function is not differentiable, we consider the points where the derivative becomes infinite, i.e., the points at which there exists neither a right-hand nor a left-hand derivative, the difference quotient $[f(x+h)-f(x)]/h$ increasing beyond all bounds as $h \to 0$. For example, the function $y=f(x)=\sqrt[3]{x}=x^{1/3}$ is defined and continuous for all values of $x$. For all non-zero values of $x$, its derivative is given by the formula $y' = x^{-2/3}/3$. At the point $x = 0$, we have $[f(x+h)-f(x)]/h = h^{1/3}/h = h^{-2/3}$, and we see at once that,

as $h \to 0$, the expression has no limiting value, but tends to $\infty$.

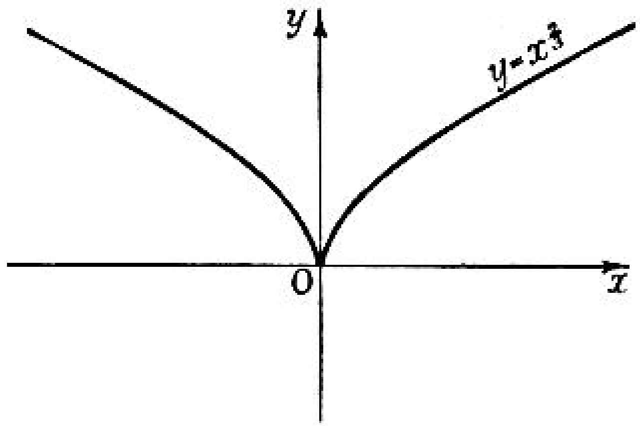


Fig. 12.—Cusp

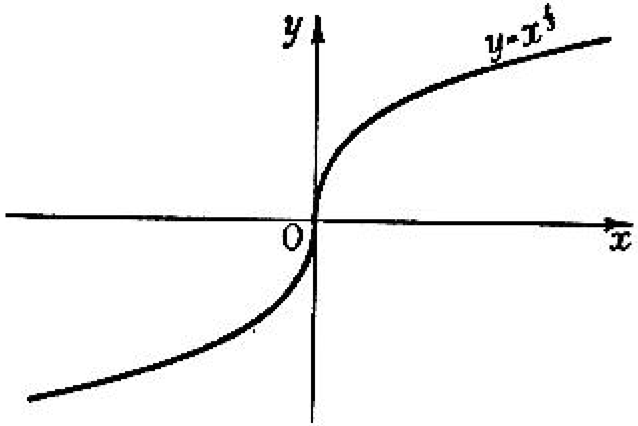


Fig. 10

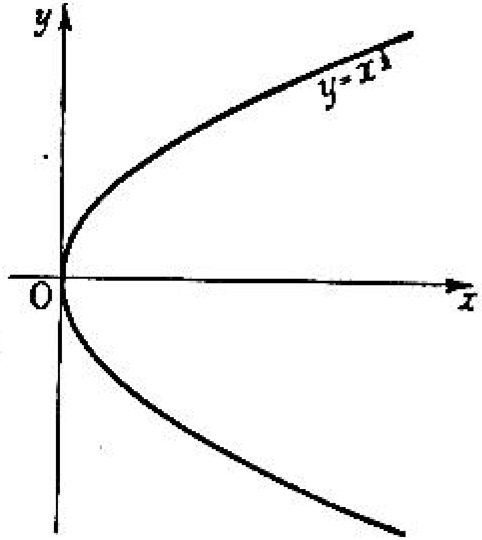


Fig. 11

This state of affairs is often briefly described by saying that the function possesses an **infinite derivative** or the derivative $\infty$ at the point in question; however, we should remember that this merely means that, as $h$ tends to 0, the difference quotient increases beyond all bounds and that the derivative in the sense, in which we have defined it, really does not exist. The geometrical meaning of an infinite derivative is that the tangent to the curve is vertical (Fig. 10 above).

The function $y = f(x) = \sqrt{x}$, which is defined and continuous for $x \geq 0$ , is also **non-differentiable** at the point $x = 0$. Since $y$ is undefined for negative values of $x$, we here consider only the right-hand derivative. The equation

$$\frac{f(h) - f(0)}{h} = \frac{1}{\sqrt{h}}$$

shows us that this derivative is infinite; the curve touches the y-axis at the origin (Fig. 11 above).

Finally, the function $y = \sqrt[3]{x^2} = x^{2/3}$ is a case in which the right-hand derivative at the point $x = 0$ is positive and infinite, while the left-hand derivative is negative and infinite, as follows from the relation

$$\frac{f(h)-f(0)}{h}=\frac{1}{\sqrt[3]{h}}.$$

As a matter of fact, the continuous curve $y = x^{2/3}$, the so-called **semi-cubical parabola** or **Neil's parabola**, has at the origin a **cusp**, perpendicular to $x$-axis(Fig. 12).

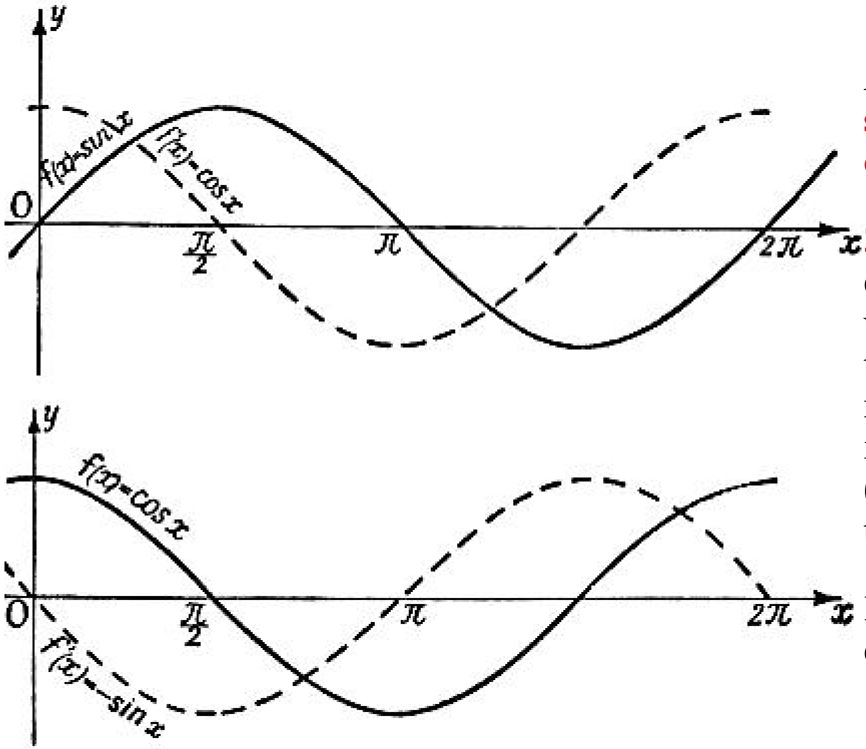


**Fig. 13.—Derived curves of sin $x$ and cos $x$**

**2.3.6. Higher Derivatives and their Significance:** The derivative $f'(x)$ of a function is itself a function of $x$, the graph of which we call the **derived curve** of the given curve. For example, the derived curve of the parabola $y = x^2$ is a straight line, represented by the function y = 2x. The derivede curve of $y$ =sin x is $y$ = cos $x$; similarly, the derived curve of $y$ = cos $x$ is $y$ = -sin $x$. (Any of these latter curves can be obtained from the others by translation in the direction of the $x$-axis.

It is now quite a natural step to form the derived curves of the derived curves, i.e., to form the derivative of the function $f'(x) = \phi(x)$. This derivative

$$\phi'(x)=\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h},$$

provided that it really exists, we shall call the **second derivative** of the function $f(x)$ and denote it by $f''(x)$.

Similarly, we may attempt to form the derivative of $f''(x)$, the so-called third derivative of $f(x)$, which we then denote by $f'''(x)$. In the case of most functions of importance, there is nothing to stop us from imagining this process repeated as many times as we like and from thus defining the ***n*-th derivative** $f^{(n)}(x)$. At times, it will be convenient to call the function $f(x)$ its own zero-th derivative.

The terms **second**, **third**, ⋯ , ***n*-th differential coefficient** are also employed.

If the independent variable is interpreted as the time $t$ and the motion of a point is represented by means of the function $f(t)$, the physical meaning of the second derivative is found to be the velocity with which the velocity $f'(t)$ changes, or, as it is usually called, the **acceleration**. Later on, we shall discuss in detail the geometrical

interpretation of the second derivative . However, we may note here the following facts: At a point where $f''(x)$ is positive, $f'(x)$ increases with $x$; on the other hand, if $f''(x)$ is negative, $f'(x)$ decreases with $x$.

**2.3.7 The Derivative and the Difference Quotient:** The fact that in the limiting process, which defines the derivative, the difference $\Delta x$ tends to 0 is sometimes expressed by saying that the quantity $\Delta x$ becomes infinitely small. This expression indicates that the passage to the limit is regarded as a process during which the quantity $\Delta x$ is never zero, yet approaches zero as closely as we please. In **Leibnitz's notation**, the passage to the limit in the process of differentiation is symbolically expressed by replacing the symbol $\Delta$ by the symbol $d$, so that we can define Leibnitz's symbol for the derivative by the equation

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

However, if we wish to obtain a clear grasp of the meaning of the differential calculus, we must beware of regarding the derivative as the quotient of two quantities which are actually infinitely small. The difference quotient $\Delta y/\Delta x$ definitely must be formed with differences $\Delta x$ which are not equal to 0. After forming this difference quotient, we must imagine the passage to the limit carried out by means of a transformation or some other device. We have no right to assume that first $\Delta x$ goes through something like a limiting process and reaches a value which is infinitesimally small but still not 0, so that $\Delta x$ and $\Delta y$ are replaced by the infinitely small quantities or infinitesimals $dx$ and $dy$, and the quotient of these quantities is then formed. Such a conception of the derivative is incompatible with the clarity of ideas demanded in mathematics; in fact, it is altogether meaningless. For a great many simple-minded people, it undoubtedly has a certain charm - the charm of mystery - which is always associated with the word infinite; and in the early days of the differential calculus, even Leibnitz himself was capable of combining these vague mystical ideas with a thoroughly clear understanding of the limiting process. It is true that this fog, which surrounded the foundations of the new science, did not prevent Leibnitz or his great successors from finding the right path. But this does not release us from the duty of avoiding every such hazy idea in our building-up of the differential and integral calculus.

However, the notation of Leibnitz is not merely attractive in itself, but is actually of great flexibility and the greatest use. The reason is that we can deal in many calculations and formal transformations with the symbols $dy$ and $dx$ in exactly the same way as if they were ordinary numbers. They enable us to give neater expression to many calculations which can be carried out without their use. In the following pages, we shall see this fact verified over and over again and shall find ourselves justified in making free and repeated use of it, provided we do not lose sight of the symbolical character of the signs $dy$ and $dx$.

Leibnitz has also devised for the second and higher derivatives a notation of great suggestiveness and practical utility. He thinks of the second derivative as the limit of the **second difference quotient** in the following manner. In addition to the variable $x$, we consider $x_1 = x + h$ and $x_2 = x + 2h$. We then take the second difference quotient as meaning the first difference quotient of the first difference quotient, i.e., the expression

$$\frac{1}{h}\left(\frac{y_2 - y_1}{h} - \frac{y_1 - y}{h}\right) = \frac{1}{h^2}(y_2 - 2y_1 + y),$$

where $y = f(x)$, $y_1 = f(x_1)$ and $y_2 = f(x_2)$. If we also write $h = \Delta x$ and $y_2 - y_1 = \Delta y_1$, $y_1 - y = \Delta y$, we may appropriately call the expression in the last bracket the difference of the difference of $y$ or the **second difference** of $y$ and write symbolically *

$$y_2 - 2y_1 + y = \Delta y_1 - \Delta y = \Delta(\Delta y) = \Delta^2 y.$$

* Here $\Delta\Delta = \Delta^2$ is not a square, but merely a symbol for difference of difference or second difference.

In this symbolic notation, the second difference quotient is then $\Delta^2 y/(\Delta x)^2$, where the denominator is really the square of $\Delta x$, while in the numerator the number 2 syrnbolically denotes the repetition of the difference process. This symbolism for the difference quotient led Leibnitz to introduce the notation

$$y'' = f''(x) = \frac{d^2y}{dx^2}, \quad y''' = f'''(x) = \frac{d^3y}{dx^3}, \quad \&c.,$$

for the second and higher derivatives, and we shall find that this notation also stands the test of use.

We must emphasize that the statement that the second derivative may be represented as the limit of the second difference quotient requires proof. For we previously defined the second derivative not in this way, but as the limit of the first difference quotient of the first derivative. In actual fact, the two definitions are equivalent, provided the second derivative is continuous; however, the proof is not given, as we have no particular need for it here.

**2.3.8 The Mean Value Theorem:** There exists between the derivative $dy/dx=f'(x)$ and the difference quotient a simple relation which is important for many purposes is known as the mean value theorem and is obtained in the following manner. We consider the difference quotient

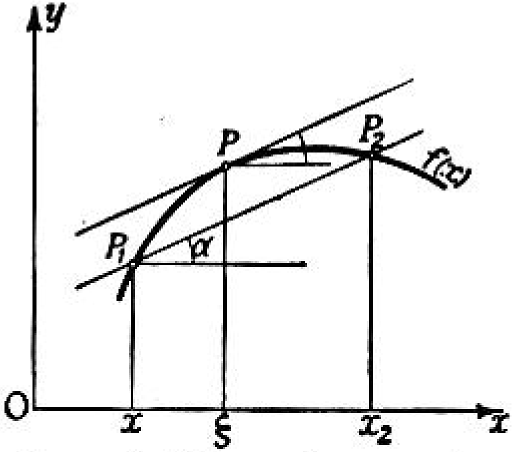


**Fig. 14.—To illustrate the mean value theorem**

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{\Delta f}{\Delta x}$$

of a function $f(x)$ and assume that the derivative exists everywhere in the interval $x_1 \le x \le x_2$ so that the graph of the curve has everywhere a tangent. The difference quotient will be represented by the direction of the secant (Fig. 14); in fact, it is the tangent of the angle $\alpha$ shown in the figure. Let us imagine that this secant is shifted parallel to itself. At least once, it will reach a position in which it is a tangent to the curve at a point between $x_1$ and $x_2$, namely at the point of the curve which is at the greatest distance from the secant. Hence there will be an intermediate value $\xi$ such that

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(\xi).$$

This statement is called the **mean value theorem of the differential calculus**. We can also express it somewhat differently by noticing that the number $\xi$ may be written in the form

$$\xi = x_1 + \theta(x_2 - x_1),$$

where $\theta$ is a certain number between 0 and 1. In applications of the mean value theorem, we shall often find that $\theta$ cannot be determined more accurately than this, but it will usually turn out that a more accurate value is not required. When accurately formulated, the mean value theorem is:

If $f(x)$ is continuous in the closed interval $x_1 \le x \le x_2$ and differentiable at every point of the open interval $x_1 < s < x_2$, then there is at least one value $\theta$, where $0 < \theta < 1$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'\{x_1 + \theta(x_2 - x_1)\}.$$

If we replace $x_1$ by $x$ and $x_2$ by $x + h$, we can express the mean value theorem by the formula

$$\frac{f(x+h) - f(x)}{h} = f'(\xi) = f'(x + \theta h), \quad x < \xi < x + h.$$

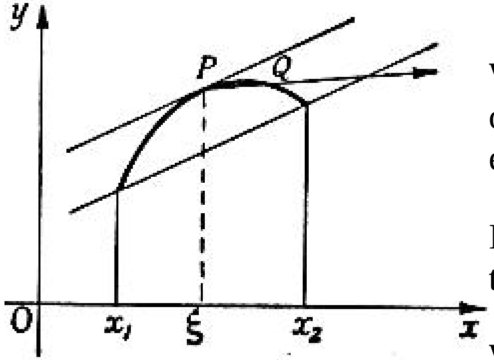


Fig. 15.—To illustrate the mean value theorem

We wish to emphasize that, while it is essential that $f(x)$ should be continuous for all points of the interval, including the end-points, we need not assume that the derivative exists at the end-points. This apparently trivial remark is actually useful in many applications.

If at any point in the interior of an interval the derivative fails to exist, the mean value theorem is not necessarily true. This is shown by the example $f(x)=|x|$.

We can complete our intuitive argument by the following consideration. There is at least one point $P$ on the curve which has the greatest possible distance from the chord joining the points on the curve with the abcissae $x_1$ and $x_2$ (Fig. 15). At this point, the curve has by assumption a definite tangent. We shall now prove that this tangent must be parallel to the chord. By definition, the tangent is the limiting position of the secant and is obtained by joining $P$ to a point $Q$ on the curve and letting the point $Q$ move towards $P$. Since, by assumption, $Q$ is not further from the chord than $P$, the line $PQ$ produced in the direction $P$ to $Q$ must either intersect the chord or run parallel to it; and this must be the case, no matter on which side of $P$ lies the point $Q$. This, however, is only possible if the limiting position is parallel to the chord. If we denote the abscissa of the point $P$ by $\xi$, the slope $f'(\xi)$ of the tangent at $P$ is then equal to the slope of the chord, $[f(x_1) - f(x_2)]/[x_1 - x_2)]$, whence we may simply take for the number $\xi$ in the theorem the abscissa of $P$.

The rigorous proof of the mean value theorem is usually developed as follows: We first establish **Rolle's theorem** - a special case of the mean value theorem: If a function $\phi(x)$ is continuous in the closed interval $x_1 \leq x \leq x_2$ and differentiable in the open interval $x_1 < x < x_2$, and, moreover, $\phi(x_1) = 0$ and $\phi(x_1) = 0$, then there exists at least one point $\xi$ in the interior of the interval at which $f'(\xi)= 0$.

In fact, there must be at least one point $\xi$, interior to the interval, at which the function $\phi(x)$ takes on its greatest or its least value; to be specific, we assume that $\xi$ is a point where $\phi(x)$ is a maximum so that for every $x$ in the interval $\phi(x)\leq\phi(\xi)$. Then it is certainly true for every number, the absolute value $|h|$ of which is small enough, that $\phi(\xi) - \phi(\xi + h) \geq 0$ If $h$ is positive,

$$\frac{\phi(\xi+h)-\phi(\xi)}{h} \leqq 0;$$

we now let $h$ tend to zero through positive values and obtain $\phi'(\xi) \leq 0$. On the other hand, if $h$ is negative,

$$\frac{\phi(\xi+h)-\phi(\xi)}{h} \geqq 0,$$

and thus, by letting $h$ tend to zero through negative values, we obtain $\phi'(\xi) \geq 0$; comparing this result with the preceding inequality, we see that $\phi'(\xi) = 0$, which establishes our theorem.

We now apply Rolle's theorem to the function

$$\phi(x) = f(x) - f(x_1) - \frac{x - x_1}{x_2 - x_1}\{f(x_2) - f(x_1)\},$$

which, apart from a factor independent of $x$, is the distance of the point $(x, f(x))$ of the curve from the secant, as the reader will readily verify. Obviously, this function satisfies the condition $\phi(x) = f(x) + ax + b = 0$ with constant coefficients $a = -[f(x_2) - f(x_1)]/[x_2 - x_1]$ and $b$. We know already that

$$\phi'(x) = f'(x) + a,$$

whence, by Rolle's theorem,

$$0 = \phi'(\xi) = f'(\xi) + a$$

for a suitably chosen intermediate value $\xi$, and therefore

$$f'(\xi) = -a = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

and the mean value theorem has been proved.

As the first of many applications of the mean value theorem,we shall prove the following: Let the function $f(x)$ be continuous in the closed interval $a \leq x \leq b$ and have the derivative $f'(x)$ at every point of the open interval $a < x < b$. Then, if $f'(x)$ is positive everywhere in $a < x < b$, the function $f(x)$ is monotonic increasing in the interval $a \leq x \leq b$; and likewise, if $f(x)$ is negative in $a < x < b$, it is monotonic decreasing.

Let $f'(x) > 0$ and $x_1$, $x_2 > 0$ be any two values of $x$ in the closed interval. Then, by the mean value theorem,

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi),$$

where $x_1 < \xi < x_2$; Since both factors on the right hand side are positive, this proves that $f(x_2) > f(x_1)$, whence $f(x)$ is monotonic increasing.

**2.3.9 The Approximate Representation of Arbitrary Functions by Linear Functions. Differentiation:** The equation

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

which defines the derivative is equivalent to the equations

$$f(x+h) - f(x) = hf'(x) + \epsilon h$$

$$\text{or} \qquad y + \Delta y = f(x + \Delta x) = f(x) + f'(x)\Delta x + \epsilon \Delta x,$$

where $\varepsilon$ is a quantity which tends to zero with $h = \Delta x$. If, for the moment, we think of the point $x$ as being fixed and the increment $\Delta x$ as being variable, then, by this formula, the increment of the function, that is, the quantity $\Delta y$, consists of two terms, namely a part $hf'(x)$, proportional to $h$ and an error which can be made as small as we please relative to $h$ by making $h$ itself small enough. Thus, the smaller is the interval about the point $x$ under consideration, the more accurately is the value of the function $f(x + h)$ (which is a function of $h$), represented by its linear part $f(x) + hf'(x)$. This approximate representation of the function $f(x + h)$ by a linear function of $h$ is expressed geometrically by the substitution of its tangent for the curve at the point $x$. In Chapter VII, we shall consider the practical application of these ideas to approximate calculations.

Here we merely remark in passing that it is possible to use this approximate representation of the increment $\Delta y$ by the linear expression $hf'(x)$ to construct a logically satisfactory definition of the notion of a differential as this was done, in particular, by Cauchy.

While the idea of the differential as an infinitely small quantity has no meaning and it is accordingly futile to define the derivative as the quotient of two such quantities, we may still try to assign a sense to the equation $f'(x) = dy/dx$ in such a way that the expression $dy/dx$ need not be thought of as purely symbolic, but as the actual quotient of two quantities $dy$ and $dx$. For this purpose, we first define the derivative $f'(x)$ by our limiting process, then think of $x$ as

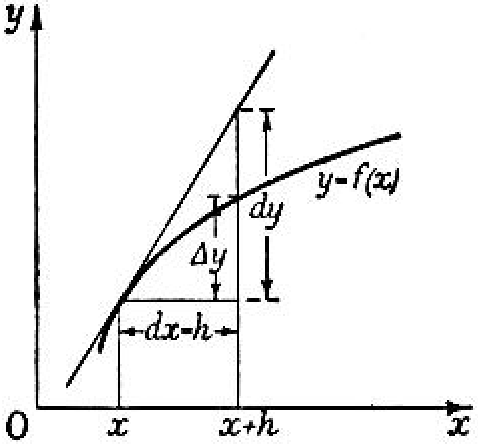


**Fig. 16.—The differential *dy***

fixed and consider the increment $h = \Delta x$ as the independent variable. We will call this quantity $h$ the **differential of *x*** and write $h = dx$. We now define the expression $dy = y'dx$ as the **differential of the function *y***; $dy$ is therefore a number which has nothing to do with infinitely small quantities. Thus, the derivative $y' = f'(x)$ is now really the quotient of the differentials $dy$ and $dx$; however, there is nothing remarkable in this statement; in fact, it is merely a restatement of the verbal definition. The differential $dy$ is accordingly the linear part of the increment $\Delta y$ (Fig. 16).

We shall not make any immediate use of these differentials. Nevertheless, it may be pointed out, for the sake of completeness, that we may also form second and higher differentials. In fact, if we think of $h$ as chosen in any manner, but always the same for every value of $x$, then $dy = hf'(x)$ is a function of $x$, of which we can again form the differential. The result will be called the **second differential of *y*** and be denoted by the symbol $d^2y = d^2f(x)$. The increment of $hf'(x)$ being $h\{f'(x+h) - f'(x)\}$, the second differential is obtained by replacing the quantity in braces by its linear part $hf''(x)$, so that $d^2y == h^2f''(x)$. We may naturally proceed further along the same lines, obtaining third, fourth, ··· differentials of y, etc., which can be defined by the expressions $h^3f'''(x)$, $h^4f^{(4)}(x)$, etc.

**2.3.10. Remarks on Applications to the Natural Sciences:**. In applications of mathematics to natural phenomena, we never have to deal with sharply defined quantities. Whether a length is exactly a metre is a question which cannot be decided by any experiment and which consequently has no physical meaning. Again, there is no immediate physical meaning in saying that the length of a material rod is rational or irrational; we can always measure it with any desired degree of accuracy in rational numbers, and the real matter of interest is whether or not we can manage to perform such a measurement using rational numbers with relatively small denominators. Just as the question of rationality or irrationality in the rigorous sense of **exact mathematics** has no physical meaning, so the actual carrying out of limiting processes in applications will usually be nothing more than a mathematical idealization.

The practical significance of such idealizations lies chiefly in the fact that, if they are used, all analytical expressions become essentially simpler and more manageable. For example, it is vastly simpler and more convenient to work with the notion of instantaneous velocity, which is a function of only one definite instant of time, than with the notion of average velocity between two different instants. Without such an idealization, every rational investigation of nature would be condemned to hopeless complications and would break down at the very outset.

However, we do not intend to enter into a discussion of the relationship of mathematics to reality. We merely wish to emphasize, for the sake of our better understanding of the theory, that we have in applications the right to replace a derivative by a difference quotient and *vice versa*, provided only that the differences are small enough to

guarantee a sufficiently close approximation. The physicist, the biologist, the engineer, or anyone else who has to deal with these ideas in practice, will therefore have the right to identify the difference quotient with the derivative within his limits of accuracy. The smaller is the increment $h = dx$ of the independent variable, the more accurately can it represent the increment $\Delta y = f(x + h) - f(x)$ by the differential $dy = hf'(x)$. As long as one keeps within the limits of accuracy required by a given problem, one is accustomed to speak of the quantities $dx = h$ and $dy = hf'(x)$ as infinitesimals. These **physically infinitesimal quantities** have a precise meaning. They are finite quantities, not equal to zero, which are chosen small enough for a given investigation, e.g., smaller than a fractional part of a wave-length or smaller than the distance between two electrons in an atom; in general, they are chosen smaller than the degree of accuracy required.

**Exercises 2.2:**

1.* Replace the statement: At the point $x = \xi$, the function $f(x)$ is not differentiable by an equivalent statement without use of any form of the word differentiable.

2. Differentiate the following functions directly by using the definition of the derivatives:

$$(a)\ \frac{1}{x+1}. \qquad (b)\ \frac{1}{x^2+2}. \qquad (c)\ \frac{1}{2x^2+1}. \qquad (d)\ \frac{1}{\sin x}.$$

$$(e)\ \sin 3x. \qquad (f)\ \cos ax. \qquad (g)\ \sin^2 x. \qquad (h)\ \cos^2 x.$$

3. Find the intermediate value $\xi$ of the mean value theorem for the following functions and illustrate graphically:

$$(a)\ 2x. \quad (b)\ x^2. \quad (c)\ 5x^3 + 2x. \quad (d)\ 1/(x^2+1). \quad (e)\ x^{1/3}.$$

4. Show that the mean value theorem fails for the following functions when the two points are taken with opposite signs, e.g., $x_1 = -1$, $x_2 = 1$:

$$(a)\ 1/x. \quad (b)\ |x|. \quad (c)\ x^{2/3}.$$

Illustrate graphically and compare with the previous example.

Answers and Hints

## 2.4 The Indefinite Integral, the Primitive Function and the Fundamental Theorems of the Differential and Integral Calculus

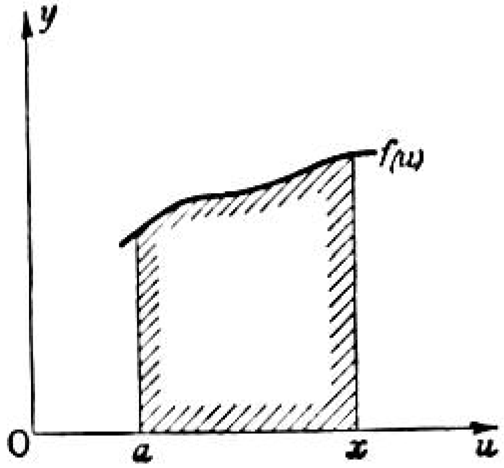


Fig. 17

As we have already mentioned above, the connection between the problems of integration and of differentiation is the corner-stone of the differential and integral calculus. We will now study this connection.

**2.4.1 The Integral as a Function of the Upper Limit:** The value of the definite integral of a function $f(x)$ depends on the choice of the two limits of integration $a$ and $b$; it is a function of the lower limit $a$ as well as of the upper limit $b$. In order to study this dependence more closely, we imagine the lower limit $a$ to be a definite fixed number, denote the variable of integration no longer by $x$ but by $u$ (Integration of a Linear Function) and the upper limit by $x$ instead of $b$, in order to suggest that we shall let the upper limit vary and wish to investigate the value of the integral as a function of the upper limit. Accordingly, we write

$$\int_a^x f(u)\,du = \Phi(x).$$

We call this function $\Phi(x)$ **an indefinite integral** of the function $f(x)$. When we speak of an and not of the indefinite integral, we suggest that instead of the lower limit $a$ any other could be chosen, in which case we should ordinarily obtain a different value for the integral. Geometrically speaking, the indefinite integral for each value of $x$ will be given by the area (shaded in Fig.17) under the curve $y=f(u)$ and bounded by the ordinates $u = a$ and $u = x$, the sign being determined by the rules given earlier.

If we choose another lower limit $\alpha$ in place of the lower limit $a$, we obtain the indefinite integral

$$\Psi(x) = \int_\alpha^x f(u)\,du.$$

The difference $\Psi(x) - \Phi(x)$ will obviously be

$$\int_\alpha^a f(u)\,du,$$

which is a constant, since $\alpha$ and $a$ are each fixed given numbers. Hence

$$\Psi(x) = \Phi(x) + \text{const.};$$

Different indefinite integrals of the same function differ only by an additive constant.

We may likewise regard the integral as a function of the lower limit and introduce the function

$$\phi(x) = \int_x^b f(u)\,du,$$

in which $b$ is a fixed number. Here again two such integrals with different upper limits $b$ and $\beta$ differ only by an additive constant

$$\int_b^\beta f(u)\,du.$$

**2.4.2 The Derivative of the Indefinite Integral:** We will now differentiate the indefinite integral $\Phi(x)$ with respect to the variable $x$. The result is the **theorem:**

The indefinite integral

$$\Phi(x) = \int_a^x f(u)\,du$$

of a continuous function $f(x)$ always possesses a derivative $\Phi'(x)$ and moreover

$$\Phi'(x) = f(x);$$

that is, differentiation of the indefinite integral of a given continuous function always gives us back that function.

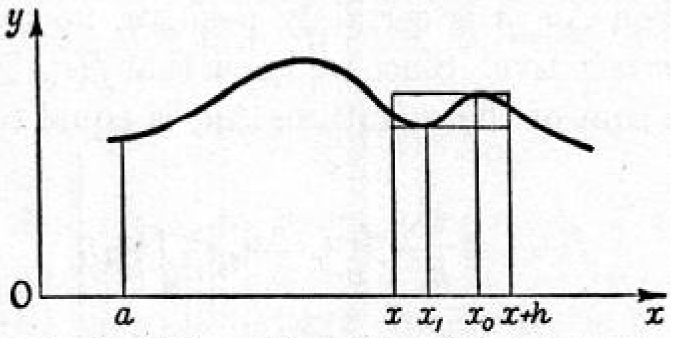


Fig. 18.—Differentiation of the indefinite integral

This is the basic idea of all of the differential and integral calculus. The proof follows extremely simply from the interpretation of the integral as an area. We form the difference quotient

$$\frac{\Phi(x+h)-\Phi(x)}{h},$$

and observe that the numerator

$$\Phi(x+h)-\Phi(x)=\int_a^{x+h} f(u)\,du-\int_a^x f(u)\,du=\int_x^{x+h} f(u)\,du$$

is the area between the ordinate corresponding to $x$ and the ordinate corresponding to $x + h$.

Now let $x_0$ be a point in the interval between $x$ and $x + h$, at which the function $f(x)$ takes its greatest value, and $x_1$ a point, at which it takes its least value in the interval (Fig. 18). Then the area in question will lie between the values $hf(x_0)$ and $hf(x_1)$ which represent the areas of rectangles with the interval from $x$ to $x+h$ as base and the altitudes $f(x_0)$ and $f(x_1)$, respectively. Expressed analytically,

$$f(x_0)\geqq\frac{\Phi(x+h)-\Phi(x)}{h}\geqq f(x_1).$$

This can also be proved directly from the definition of the integral without appealing to the geometrical interpretation (Compare also with the discussion in 2.7). For this purpose, we write

$$\int_x^{x+h} f(u)\,du = \lim_{n\to\infty} \sum_{\nu=1}^{n} f(u_\nu)\Delta u_\nu,$$

where $u_0 = x, u_1, u_2, \cdots, u_n = x + h$ are points of sub-division of the interval from $x$ to $x + h$, and the largest of the absolute values of the differences $\Delta u_\nu = u_\nu - u_{\nu-1}$ tends to zero as $n$ increases. Then $\Delta u_\nu/h$ is certainly positive, no matter whether $h$ is positive or negative. Since we know that

$$f(x_0) \geqq f(u_\nu) \geqq f(x_1)$$

and the sum of the quantities $\Delta u_\nu$ is equal to $h$, it follows that

$$f(x_0) \geqq \frac{1}{h} \Sigma f(u_\nu)\Delta u_\nu \geqq f(x_1);$$

thus, if we let $n$ tend to infinity, we obtain the above inequalities for

$$\frac{1}{h}\int_x^{x+h} f(u)\,du \quad \text{or} \quad \frac{\Phi(x+h) - \Phi(x)}{h}.$$

Now, if $h$ tends to zero, both $f(x_0)$ and $f(x_1)$ must tend to the limit $f(x)$ due to the continuity of the function, whence we see at once that

$$\Phi'(x) = \lim_{h\to 0} \frac{\Phi(x+h) - \Phi(x)}{h} = f(x),$$

as stated by the theorem.

Owing to the differentiability of $\Phi(x)$ and 2.3.5, we have the **theorem:**

The integral of a continuous function $f(x)$ is itself a continuous function of the upper limit.

For the sake of completeness, we point out that, if we regard the definite integral not as a function of its upper limit but as a function of its lower limit, the derivative is not equal to $f(x)$, but instead equal to - $f(x)$; in symbols, if we set

$$\phi(x) = \int_x^b f(u)\,du,$$

then

$$\phi'(x) = -f(x).$$

The proof follows immediately from the remark that

$$\int_x^b f(u)\,du = -\int_b^x f(u)\,du.$$

**2.4.3 The Primitive Function; General Definition of the Indefinite Integral:** The theorem, which we have just proved, shows that the indefinite integral $\Phi(x)$ at once yields the solution of the problem: Given a function $f(x)$, find a function $F(x)$ such that

$$F'(x) = f(x).$$

This problem requires us to **reverse the process of differentiation**. It is a typical inverse problem such as occurs in many parts of mathematics and such as we have already found to be a fruitful mathematical tool for generating new functions. (For example, the first extension of the idea of natural numbers was made under the pressure of the necessity for reversing certain elementary computational processes. The formation of inverse functions has led and will lead us to new kinds of functions!)

A function $F(x)$ such that $F'(x) = f(x)$ is called a **primitive function of $f(x)$**, or simply a **primitive of** $f(x)$; this terminology suggests that the function $f(x)$ arises from $F(x)$ by differentiation.

This problem of the inversion of differentiation or of the determination of a primitive function is at first sight of quite a different character from the problem of integration. However, we know from the preceding section that:

Every indefinite integral $\Phi(x)$ of the function $f(x)$ is a primitive of $f(x)$.

However, this result does not completely solve the problem of finding primitive functions, because we do not yet know whether we have found all the solutions of the problem. The question concerning the group of all primitive

functions is answered by the following theorem, sometimes referred to as the **fundamental theorem of the differential and integral calculus**:

The difference between two primitives $F_1(x)$ and $F_2(x)$ of the same function $f(x)$ is always a constant:

$$F_1(x) - F_2(x) = c.$$

Thus, from any one primitive function $F(x)$, we can obtain all the others in the form

$$F(x) + c$$

by a suitable choice of the constant $c$. Conversely, for every value of the constant $c$ the expression $F_1(x) = F(x) + c$ represents a primitive function of $f(x)$.

It is clear that for any value of the constant $c$ the function $F(x) + c$ is a primitive, provided that $F(x)$ itself is one. From 2.3.4, we have

$$\frac{\{F(x+h)+c\} - \{F(x)+c\}}{h} = \frac{F(x+h) - F(x)}{h},$$

and since, by assumption, the right-hand side tends to $f(x)$ as $h \to 0$, so does the left-hand side, whence

$$\frac{d}{dx}\{F(x)+c\} = f(x) = F'(x).$$

Thus, in order to complete the proof of the theorem, there only remains to show that the difference of two primitive functions $F_1(x)$ and $F_2(x)$ is always a constant. For this purpose, we consider the difference

$$F_1(x) - F_2(x) = G(x)$$

and form the derivative

$$G'(x) = \lim_{h \to 0} \left\{ \frac{F_1(x+h) - F_1(x)}{h} - \frac{F_2(x+h) - F_2(x)}{h} \right\}.$$

By assumption, both the expressions on the right-hand side have the same limit $f(x)$ as $h \to 0$; thus,for every value of $x$, we have $G'(n) = 0$. But a function, the derivative of which is everywhere zero, must have a graph the tangent of which is everywhere parallel to the $x$-axis, i.e., it must be a constant, whence $G(x) = c$, as has been stated above. We can prove this last fact by using the mean value theorem without relying on tuition. Applying the mean value theorem to $G(x)$, we find

$$G(x_2) - G(x_1) = (x_2 - x_1)G'(\xi);\ x_1 < \xi < x_2.$$

However, we have seen that the derivative $G'(x)$ is equal to 0 for every value of $x$, whence, in particular, it is true for the value $\xi$, and it follows immediately that $G(x_1) = G(x_2)$. Since $x_1$ and $x_2$ are arbitrary values of $x$ in the given interval, $G(x)$ must be a constant.

Combining the theorem which has just been proved with the result of 2.4.2, we can now state:

Every primitive function $F(x)$ of a given function $f(x)$ can be represented in the form

$$F(x) = c + \Phi(x) = c + \int_a^x f(u)\,du,$$

where $c$ and $a$ are constants, and conversely, for any arbitrarily chosen constant values $a$ and $c$, this expression always represents a primitive function.

It may readily be guessed that, as a rule, the constant $c$ can be omitted, since, by changing the lower limit $a$, we change the primitive function by an additive constant. However, in many cases, we should not obtain all the primitive functions, if we were to omit $c$, as is shown by the example $f(x) = 0$. For this function, the indefinite integral of 2.4.1 is always 0, independently of the lower limit; yet any arbitrary constant is a primitive function of $f(x) = 0$. A second example is the function $f(x) = \sqrt{x}$, which is only defined for non-negative values of $x$. The indefinite integral is

$$\Phi(x) = \tfrac{2}{3}x^{3/2} - \tfrac{2}{3}a^{3/2},$$

and we see that, independently of the choice of the lower limit *a*, the indefinite integral of $\Phi(x)$ is always obtained from $2x^{3/2}/3$ by addition of a constant which is less than or equal to zero, namely, by the constant $-2a^{3/2}$; yet such a function as $2x^{3/2} + 1$ is also a primitive of $\sqrt{x}$. Thus, in the general expression for the primitive function, we cannot dispense with the additive constant. The relationship which we have found suggests an extension of the idea of the indefinite integral. We shall henceforth call every expression of the form

$$c + \Phi(x) = c + \int_a^x f(u)\,du$$

an indefinite integral of $f(x)$. In other words, we shall no longer make any distinction between the primitive function and the indefinite integral. Nevertheless, if the reader is to have a proper understanding of the interrelationships of these concepts, it is absolutely necessary that he should clearly bear in mind that, in the first instance, integration and inversion of differentiation are two entirely different things, and also that it is only the knowledge of the relationship between them which allows us to apply the term indefinite integral also to the primitive function.

It is customary to represent the indefinite integral by a notation which in itself is perhaps not perfectly clear. We write

$$F(x) = c + \int_a^x f(u)\,du = \int f(x)\,dx,$$

that is, we omit the upper limit *x* and the lower limit *a* and also the additive constant *c* and use the letter *x* for the integration variable. It would really be more consistent to avoid this last change, in order to prevent confusion with the upper limit *x* which is the independent variable in *F(x)*. When using the notation $\int f(x)\,dx$, we must never lose sight of the indeterminacy connected with it, i.e., the fact that that symbol always denotes only an indefinite integral.

**2.4.4 The use of the Primitive Function in the Evaluation of Definite Integrals:**. Suppose that we know any one primitive function $F(x) = \int f(x)\,dx$ for the function $f(x)$ and that we wish to evaluate the definite integral $\int_a^b f(u)\,du.$ We know that the indefinite integral

$$\Phi(x) = \int_a^x f(u)\,du,$$

being also a primitive of $f(x)$, can only differ from $F(x)$ *by* an additive constant. Consequently,

$$\Phi(x) = F(x) + c,$$

and the additive constant $c$ is at once determined, if we recollect that the indefinite integral $\Phi(x) = \int_a^x f(u)\,du,$ must take the value 0 when $x = a$. We thus obtain

$$0 = \Phi(a) = F(a) + c,$$

whence $c = -F(a)$ and $\Phi(x) = F(x) - F(a)$. In particular, we have for the value $x=b$

$$\int_a^b f(u)\,du = F(b) - F(a),$$

which yields the important rule:

If $F(x)$ is any primitive of the function $f(x)$, the definite integral of $f(x)$ between the limits $a$ and $b$ is equal to the difference $F(b) - F(a)$.

If we use the relation $F'(x) = f(x)$, this may be written in the form

$$F(b) - F(a) = \int_a^b F'(x)\,dx = \int_a^b \frac{dF(x)}{dx}\,dx.$$

This formula can easily be proved and understood directly. We subdivide the interval $a \leq x \leq b$ into intervals $\Delta x_1$ $\Delta x_2, \cdots, \Delta x_n$ and consider the sum $\Sigma(\Delta F/\Delta x_\nu)\Delta x_\nu$. On the one hand, this sum is simply $\Sigma\Delta F = F(b) - F(a)$, independently of the particular sub-division, whence its limit is $F(b) - F(a)$. On the other hand, its limit is also equal to $\int_a^b F'(x)\,dx,$ as follows from the mean value theorem. In fact, $\Delta F/\Delta x_\nu = F'(\xi_\nu)$, where $\xi_\nu$ is a point

between the ends $x_{\nu-1}$ and $x_\nu$, of the interval $\Delta x$. The sum is therefore equal to $\Sigma \Delta x_\nu F'(\xi_\nu)$ and, by the definition of the integral, this tends to the limit $\int_a^b F'(x)\,dx$ as the subdivision is made finer, which establishes the theorem.

In applying our rule, we often employ the symbol $\big|$ to denote the difference $F(b) - F(a)$, i.e., we write

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b,$$

indicating by the vertical line that in the preceding expression first the value $b$ and then the value $a$ is to be substituted for $x$ and finally the difference of the resulting numbers found.

### 2.4.5 Examples:

We are now in a position to illustrate by a series of simple examples the relationship between the definite integral, the indefinite integral and the derivative, which we have just investigated. By virtue of the theorem of 2.4.2 , we can derive from each of the integration formulae, which have been proved directly in 2.1.2, a differentiation formula.

In 2.2.4, we have obtained the integration formula

$$\int_a^b x^\alpha\,dx = \frac{1}{\alpha+1}\left(b^{\alpha+1} - a^{\alpha+1}\right)$$

for every rational number $\alpha \neq 1$ and all positive values of $a$ and $b$; if we replace the variable of integration by $u$ and the upper limit by $x$, this may be written in the form

$$\int_a^x u^\alpha\,du = \frac{1}{\alpha+1}\left(x^{\alpha+1} - a^{\alpha+1}\right).$$

It follows from this by the fundamental theorem that the right hand side is a primitive function of the integrand, i.e., the differentiation formula

$$\frac{d}{dx} x^{\alpha+1} = (\alpha + 1)x^{\alpha}$$

is valid for every rational value of $\alpha \neq -1$ and all positive values of $x$. By direct substitution, we find that this last formula is also true for $\alpha = -1$, if $x > 0$. The result obtained exactly agrees with what we have already found in 2.3.3 by direct differentiation. Thus, by using the fundamental theorem after having carried out the integration, we could have saved ourselves the trouble of that differentiation.

Moreover, it follows from the integration formula

$$\int_a^x \cos u \, du = \sin x - \sin a,$$

given in 2.2.5 that $d/dx \sin x = \cos x$, in agreement with the result found at the end of 2.3.3.

However, conversely, we may regard every directly proved differentiation formula $F'(x) = f(x)$ as a link between a primitive function $F(x)$ and a derived function $f(x)$, that is, we may regard it as a formula for indefinite integration and then obtain from it the definite integral of $f(x)$ as in 2.4.4. This very method is frequently employed, as we shall see in 4.1. In particular, we may start from the results of 2.3.3 and obtain the integral formula of 2.1.3 by virtue of the fundamental theorem. For example, we know from 2.3.3 that $dx^{\alpha+1}/dx = (\alpha+1)x^{\alpha}$, whence $x^{\alpha+1}/(\alpha+1)$ is a primitive function or indefinite integral of $x^{\alpha}$, provided that $\alpha \neq -1$, and thus, by 2.4.4 arrive at the above integration formula.

### Exercises 2.3:

1. From the differentiations performed in Examples 2, 3 above, set up the corresponding integrations.

2. Evaluate

$$(a) \int_0^1 \frac{dx}{(x+1)^2}. \qquad (b) \int_0^1 \frac{2x\,dx}{(x^2+1)^2}.$$

3. Using Example 2, prove from the definition of the definite integral that

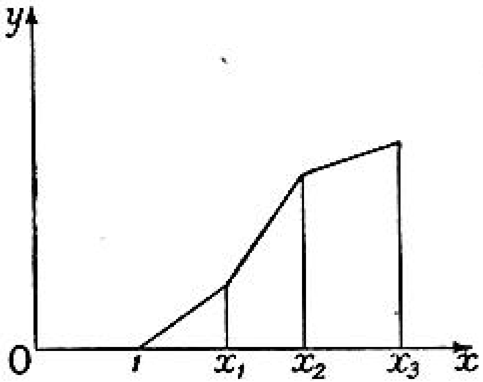



Fig. 19.—Graphical integration

$$\text{(a)}\ \lim_{n\to\infty} n\left[\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\dots+\frac{1}{(2n)^2}\right]=\frac{1}{2}.$$

$$\text{(b)}\ \lim_{n\to\infty} n^2\left[\frac{1}{(n^2+1)^2}+\frac{2}{(n^2+2^2)^2}+\dots+\frac{n}{(n^2+n^2)^2}\right]=\frac{1}{4}.$$

Answers and Hints

## 2.5 Simple Methods of Graphical Integration

Since an indefinite integral or primitive function of $f(x)$ is a function $y = F(x)$, which not only can be visualized as an area, but, like any other function, can be represented graphically by a curve. Our definition immediately suggests the possibility of constructing this curve approximately and thus obtaining a graph of the integral function. To begin with, we must remember that this last curve is not unique, but on account of the additive constant can be shifted parallel to itself in the direction of the $y$-axis. We can therefore require that the integral curve shall pass through an arbitrarily selected point, e.g., if $x = 1$ belongs to the interval of definition of $f(x)$, through the point with the coordinates $x = 1$, $y = 0$. The curve is thereafter determined by the requirement that for each value of $x$ its direction is given by the corresponding value of $f(x)$. In order to obtain an approximate construction of a curve which satisfies these conditions, we seek to construct not the curve $y = F(x)$ itself, but a **polygonal path** (broken line) the corners of which lie vertically above previously assigned points of division of the $x$-axis and the segments of which have approximately the same direction as the portion of the integral curve between the same points of sub-division. For this purpose, we subdivide our interval of the $x$-axis by means of the points $x = 1$, $x_1$, $x_2$, $\cdots$ into a certain number of parts, not necessarily all of the same length,

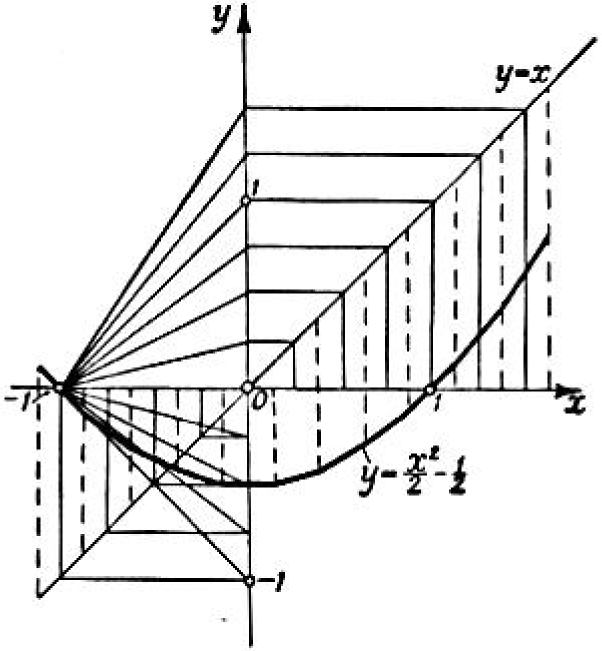



Fig. 21.—Graphical integration of $x$

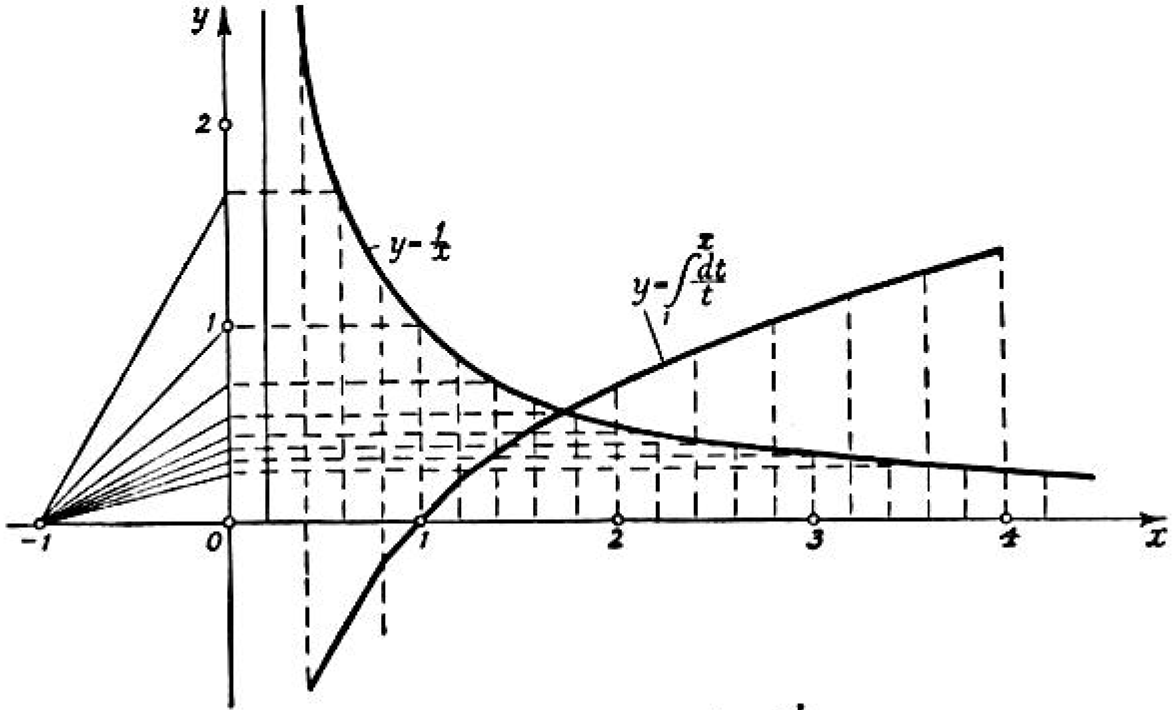



Fig. 20.—Graphical integration of $\frac{1}{x}$

and at each point of subdivision we draw a parallel to the $y$-axis (Fig 19 above). We then draw through the point $x$= 1, $y$ =0 the straight line the slope of which is equal to $f$(l); through the intersection of this line with the line $x = x_1$, we draw the line with the slope $f(x_1)$; through the intersection of this line with $x = x_2$, we draw the line with the slope $f(x_2)$, and so on. In the actual construction of these lines, we erect at each point of sub-division the ordinate to the curve y=$f(x)$ and project these ordinates onto any parallel to the $y$-axis; in order to be specific, let us suppose that they are projected onto the $y$-axis itself. We then obtain the direction of the integral curve by joining the points with coordinates $x = 0$ and $y=f(x)$ to the point $x = -1$, $y = 0$. By transferring these directions parallel to themselves, we obtain a polygonal path the corners of which lie vertically above the given points of sub-division of the $x$-axis and the directions of which agrees with the direction of the integral curve at the starting point of each interval. This polygonal path can be made to represent the integral curve with any desired degree of accuracy by making the subdivisions of the interval fine enough. We can frequently improve the accuracy of the construction by choosing for the direction of each segment of the polygon that direction which does not belong to the starting but to the central point of the corresponding interval (Figs. 20 and 21).

We mention here in passing that graphical integration (that is, finding the graph of a primitive $F\{x)$ of a function $f(x)$ which itself is given by a graph) can also be performed by means of a mechanical device, the so-called integraph. In this mechanism, a pointer

is moved along the given curve and a pen automatically traces one of the curves $y = F(x)$ for which $F'(x) = f(x)$. The indeterminacy of the constant of integration is expressed by a certain arbitrariness in the initial position of the instrument. For integrating devices, cf. B. Williamson, Integral Calculus, pp. 214-217 (Longmans); Dictionary of Applied Physics, Vol. Ill, pp. 460-467 (Macmillan, 1923).

In Fig. 21, the construction described above is carried out for the function $f(x)=x$. By graphical integration, we obtain an approximation to the integral curve, which is the parabola $y = \frac{1}{2}x^2 - \frac{1}{2}$. In addition, Fig. 20 shows an approximation to the integral function of the function $f(x) = 1/x$. We shall study this integral later in greater detail - it will turn out to be the **logarithmic function**. Finally, the reader would he well advised to work out some other examples on his own, e.g., the graphical integration of the functions sin $x$ and cos $x$.

Exercises 2.4:

1. Construct by graphical integration with the interval $h = 1/10$ the following integral curves:

$$(a)\ \int_0^x x^2\,dx\ (0 \leqq x \leqq 2). \qquad (b)\ \int_1^x \frac{1}{x^2}\,dx\ (1 \leqq x \leqq 2).$$

$$(c)\ \int_0^x \frac{1}{1+x^2}\,dx\ (0 \leqq x \leqq 1).$$

In particular, evaluate

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

Answers and Hints

## 2.6. Further Remarks on the Connection between the Integral and the Derivative

Before we pursue systematically the relationships found in 2.4, we shall illustrate them from another point of view, which is closely related to the intuitive idea of density and other physical concepts.

**2.6.1 Mass Distribution and Density; Total Quantity and Specific Quantity**: We assume that any mass is distributed along a straight line, the $x$-axis, the distribution being continuous, but not necessarily uniform. For example, we may think of a vertical column of air standing on a surface of area 1; we take as $x$-axis a line pointing

vertically upwards and as origin the point on the Earth's surface. The total mass between two abscissae $x_1$ and $x_2$ is then determined in the following manner by means of a so-called sum-function $F(x)$. We measure the distance along the line from the initial point of the mass-distribution $x = 0$ and denote by $F(x)$ the total mass between the abscissa 0 and the abscissa $x$. The increment of mass from the abscissa $x_1$ to the abscissa $x_2$ is then given simply by

$$F(x_2) - F(x_1);$$

thus a sign is assigned to the increment and this sign changes if $x_1$ and $x_2$ are interchanged.

The average mass per unit length in the interval $x_1$ to $x_2$ is

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1}.$$

If we assume that the function $F(x)$ is differentiable, then, as $x_2 \to x_1$, this value tends to the derivative $F'(x_1)$. This quantity is precisely what is usually called the **specific mass** or **density** of the distribution at the point $x_1$; as a rule, of course, its value depends on the particular point chosen. There exists accordingly between the density $f(x)$ and the sum-function $F(x)$ the relation

$$F(x) = \int_0^x f(u)\,du; \quad f(x) = F'(x).$$

The sum-function is a primitive function of the density, or, what amounts to the same thing, the mass is the integral of the density; conversely, the density is the derivative of the sum-function.

Exactly the same relation is very frequently encountered in physics. For example, if we denote by $Q(t)$ the total amount of heat needed to raise the unit mass of a substance from the temperature $t_0$ to the temperature $t$, then to raise the temperature from $t_1$ to $t_2$ requires the amount of heat

$$Q(t_2) - Q(t_1)$$

Between $t_1$ to $t_2$, the average amount of heat used per unit increase in temperature is then

$$\frac{Q(t_2) - Q(t_1)}{t_2 - t_1}.$$

If we assume once again differentiability of the function $Q(t)$, we obtain in the limit the function

$$q(t) = \lim_{t_1 \to t} \frac{Q(t) - Q(t_1)}{t - t_1},$$

which we call the **specific heat** of the substance. In general, this specific heat is to be regarded as a function of the temperature. Here again, there exists between the specific heat and the total quantity of heat the characteristic relationship of integral and derivative

$$\int_a^b q(t)\,dt = Q(b) - Q(a).$$

We shall encounter the same relations in all cases where total and specific quantities are interrelated, e.g., electric charge with density of charge, or total force acting on a surface with force-density or pressure.

In Nature, usually what we know directly is not density or specific quantity, but total quantity, whence it is the integral which is primary (as the name primitive suggests) and the specific quantity is only arrived at after a limiting process, namely, differentiation.

Incidentally, it may be noted that if the masses considered are by their nature positive, the sum-function $F(x)$ must be a monotonically increasing function of $x$, and consequently the specific quantity, the density $f(x)$, must be non-negative. However, nothing stops us from considering also negative quantities (for example, negative electricity); then our sum-functions $F(x)$ need no longer be monotonic.

**2.6.2 The Question of Applications:** Perhaps, the relationship of the primitive sum-function to the density distribution becomes clearer when it is realized that, from the point of view of physical facts, the limiting processes of integration and differentiation represent idealizations and that they do not express anything exact in nature. On the contrary, in the realm of physical reality, we can form in place of the integral only a sum and in place of the derivative only a difference quotient of very small quantities. The quantities $\Delta x$ remain different from 0; the passage to the limit $\Delta x \to 0$ is merely a mathematical simplification, in which the accuracy of the mathematical representation of the reality is not essentially impaired.

As an example, we return to the vertical column of air. According to the atomic theory, we find that we cannot think of the mass distribution as a continuous function of *x*. On the contrary, we will assume (and this, too, is a simplifying idealization) that the mass is distributed along the *x*-axis in the form of a large number of point-molecules lying very close to each other. Then the sum-function $F(x)$ will not be continuous, but it will have a constant value in the interval between two molecules and will take a sudden jump as the variable *x* passes the point occupied by a molecule. The amount of this jump will be equal to the mass of the molecule, while the average distance between molecules, according to results established in atomic theory, is of the order of $10^{-8}$ cm. Now, if we are performing upon this air column some measurement in which masses of the order $10^4$ molecules are to be considered negligible, our function cannot be distinguished from a continuous function. In fact, if we choose two values *x* and as $x + \Delta x$, the difference $\Delta x$ of which is less than $10^{-4}$ cm, then the difference between $F(x)$ and $F(x+ \Delta x)$ will be the mass of the molecules in the interval; since the number of these molecules is of the order of $10^4$, the values of $F(x)$ and $F(x+ \Delta x)$ are equal as for as our experiment is concerned. We consider simply as density of distribution the difference quotient

$$\frac{\Delta F(x)}{\Delta x} = \frac{F(x+\Delta x) - F(x)}{\Delta x};$$

it is an important physical assumption that we do not obtain measurably different values for this quotient when $\Delta x$ is allowed to vary between certain bounds, say between $10^{-4}$ and $10^{-5}$ cm. Now, imagine that $F(x)$ *is* measured and plotted for a large number of points about $10^{-4}$ cm. apart and that the points thus found are joined by straight lines; we obtain a polygon and, by rounding off the corners, obtain finally a curve with a continually turning tangent. This curve is the graph of some function, say $F_1(x)$. This new function cannot be distinguished within the limits of experimental accuracy from $F(x)$ and its derivative is within the same limits equal to $\Delta F/\Delta x$; we thus have found a continuous differentiable function which for the purposes of physics is the function $F(x)$.

It is perhaps appropriate to discuss yet another example of the concepts of sum-function and distribution density. In statistics, e.g., in the kinetic theory of matter or in statistical biology, these concepts frequently occur in a form in which the nature of the mathematical idealization is particularly clear. For example, let us consider the molecules of a gas confined in a vessel and observe their velocities at a given instant of time. Let the number of molecules be *N* and the number of those with velocities less than *x* be $N\Phi(x)$. Then $\Phi(x)$ denotes the ratio of the number of molecules moving with velocities between 0 and *x* to the total number of molecules. Of course, this sum-function is not continuous, but is sectionally constant (cf. Chapter IX) and suddenly increases by $1/N$ when *x*, as it increases, passes a value which is equal to the velocity of some molecule.

The idealization which we shall make here is that we shall think of the number $N$ as increasing beyond all bounds. We assume that, in this passage to the limit $N \to \infty$, the sum-function $\Phi(x)$ tends to a definite continuous limit function $F(x)$. That this is really the case, i.e., that we can with sufficient accuracy replace $\Phi(x)$ by this continuous function $F(x)$, is obviously an important physical assumption; and it is another such assumption to assume that this sum-function $F(x)$ possesses a derivative $F'(x) = f(x)$, which we then call the density-distribution. The sum-function is connected with the density distribution by the equations

$$F(x) = \int_0^x f(u)\,du; \; F(b) - F(a) = \int_a^b f(x)\,dx.$$

The density distribution is occasionally referred to as the **specific probability** that a molecule possesses the velocity $x$. The idealization we have just carried out has a great role in the **kinetic theory of gases** of Maxwell; it appears in exactly the same mathematical form in many problems of mathematical statistics.

## 2.7 The estimation of Integrals and the mean Value Theorem of the Integral Calculus

We close this chapter with some considerations of a matter of general significance, the full importance of which will not appear until somewhat later on. The point in question is the **estimation of integrals**.

**2.7.1 The Mean Value Theorem of the Integral Calculus:** The first and simplest of these estimation roles runs as follows: If in an interval $a \leq x \leq b$ the continuous function $f(x)$ is everywhere non-negative (is either positive or zero), then the definite integral

$$\int_a^b f(x)\,dx$$

is also non-negative. Similarly, the integral is not positive, if the function is nowhere positive in the interval. The proof of this theorem follows directly from the definition of the integral. This leads to the theorem: If

$$f(x) \geqq g(x)$$

everywhere in the interval $a \leq x \leq b$, then also

$$\int_a^b f(x)\,dx \geqq \int_a^b g(x)\,dx\,.$$

In fact, by our first remark, the integral of the difference $f(x) - g(x)$ is non-negative and, by our addition rule,

$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Let $M$ be the greatest and $m$ the smallest value of the function $f(x)$ in the interval $ab$. The function $M$-$f(x)$ is non-negative in the interval and the same is true for the function $f(x)$-$m$, whence we obtain immediately the double inequality

$$\int_a^b m\,dx \leqq \int_a^b f(x)\,dx \leqq \int_a^b M\,dx.$$

However,

$$\int_a^b m\,dx = m\int_a^b dx = m(b-a)$$

and likewise

$$\int_a^b M\,dx = M(b-a),$$

whence

$$m(b-a) \leqq \int_a^b f(x)\,dx \leqq M\,(b-a).$$

Hence the integral under consideration can be represented as the product of $(b$-$a)$ and some number $\mu$ between $m$ and $M$:

$$\int_a^b f(x)\,dx = \mu(b-a), \qquad m \leqq \mu \leqq M.$$

As a rule, there is no need to state the exact value of this mean value $\mu$. However, we may say that it will be assumed by the function at least at one point $\xi$ of the interval $a \leq \xi \leq b$, since in its interval of definition a continuous function assumes all values between its greatest and smallest values. As in the case of the mean value theorem of the differential calculus, the exact statement of the value $\xi$ is in many cases unimportant. Hence we may set $\mu = f(\xi)$, where $\xi$ is an intermediate value of $x$, and find then

$$\int_a^b f(x)\,dx = (b-a)f(\xi), \qquad a \leqq \xi \leqq b.$$

This last formula is called the **mean value theorem of the integral calculus**.

We can generalize the theorem somewhat by considering instead of the integrand $f(x)$ an integrand of the form $f(x)p(x)$, where $p(x)$ is an arbitrary/ non-negative function which, like $f(x)$, is assumed to be continuous. Since $mp(x) \leq f(x)p(x) \leq Mp(x)$, we find immediately

$$m\int_a^b p(x)\,dx \leqq \int_a^b f(x)p(x)\,dx \leqq M\int_a^b p(x)\,dx,$$

or, as a single equation,

$$\int_a^b f(x)p(x)\,dx = f(\xi)\int_a^b p(x)\,dx,$$

where $\xi$ is again a number between $a$ and $b$.

Thus, we have proved the theorem:

If $f(x)$ and $p(x)$ are continuous functions in $a \leq x \leq b$ and $p(x) \geq 0$, then

$$\int_a^b f(x)p(x)\,dx = f(\xi)\int_a^b p(x)\,dx,$$

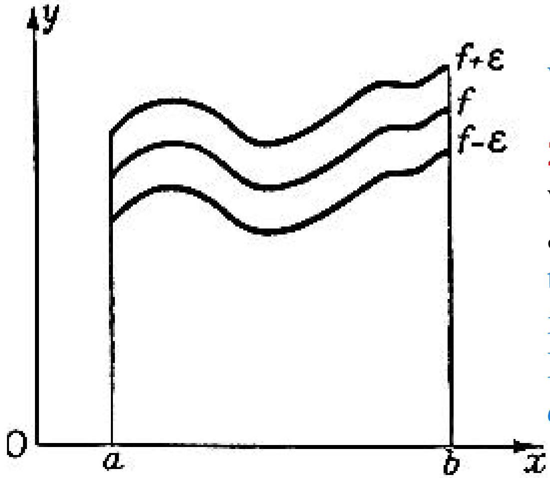


Fig. 22.—To illustrate the continuity of an integral

where $a \leq \xi \leq b$.

**2.7.2 Applications. The Integration of $x^\alpha$ for any Irrational Value of $\alpha$:** The mean value theorem and the equivalent integral estimates give us immediately an insight into an intuitive and easily understood fact: The value of an integral changes very little, if the function itself is everywhere changed very little. In precise language: If in the entire interval $a \leq x \leq b$ the absolute value of the difference of two functions $f(x)$ and $g(x)$ is less than $\varepsilon$, then the absolute value of the difference of their integrals is less than $\varepsilon(b - a)$. In symbols: If $|f(x - g(x) < \varepsilon$ throughout the interval $a \leq x \leq b$, then

$$\left| \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| < \epsilon(b-a)$$

or

$$-\epsilon(b-a) + \int_a^b g(x)\,dx < \int_a^b f(x)\,dx < \int_a^b g(x)\,dx + \epsilon(b-a).$$

Fig. 22 illustrates very clearly this theorem. We draw for the curve $y = f(x)$ the **parallel curves** $y=f(x)+\varepsilon$ and $y=f(x)-\varepsilon$. By assumption, the function $g(x)$ keeps within the strip bounded by these **parallel curves**. It is clear from this that the areas, which are bounded by the curves $f(x)$ and $g(x)$, differ from each other by less than half the area of the strip, and the area of the strip is just

$$\int_a^b \{f(x) + \epsilon\}\,dx - \int_a^b \{f(x) - \epsilon\}\,dx = 2\epsilon(b-a).$$

There is no need for intuition. Since

$$-\epsilon + g(x) < f(x) < \epsilon + g(x)$$

it follows, by arguments analogous to those in 2.7.1,

$$\int_a^b \{-\epsilon + g(x)\}\,dx < \int_a^b f(x)\,dx < \int_a^b \{g(x) + \epsilon\}\,dx,$$

which, as the result of the fundamental rules of integration, takes the form

$$-\epsilon(b-a)+\int_a^b g(x)\,dx<\int_a^b f(x)\,dx<\int_a^b g(x)\,dx+\epsilon(b-a);$$

here we have merely replaced the integral of a sum by the corresponding sum of integrals and have taken into consideration that

$$\int_a^b \epsilon\,dx=\epsilon(b-a).$$

As an indication of the importance of this theorem, we shall show that with its help we can integrate the function $x^\alpha$ for any irrational value of $\alpha$, or more precisely, calculate the definite integral $\int_a^b x^\alpha\,dx.$ Here we assume that $0 < \alpha < b$.

We represent the index $\alpha$ as the limit of a sequence of rational numbers $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ so that $\alpha = \lim\limits_{n\to\infty} \alpha_n;$ we can here assume that none of the values $\alpha_n$ is equal to -1, since $\alpha_n$ itself is different from -1. Now, we use for the power $x^\alpha$ the definition

$$x^\alpha = \lim_{n\to\infty} x^{\alpha_n}$$

and note that, no matter how small a positive number $\varepsilon$ we choose, we can always find an $n$ so large that in the entire interval $\tau \le x \le b$ * we have $|x^\alpha - x^{\alpha_n}| < \varepsilon.$

* This can be proved quite simply as follows (cf. A1.3). Remembering that $x^\alpha$ is monotonic and setting $\delta_n = \alpha_n - \alpha$, we have

$$|x^\alpha - x^{\alpha_n}| = x^\alpha |1 - x^{\delta_n}| \le (a^\alpha + b^\alpha)(|1 - a^{\delta_n}| + |1 - b^{\delta_n}|);$$

in fact, $x^\alpha$ lies between $a^\alpha$ and $b^\alpha$, so that $x^\alpha \le a^\alpha + b^\alpha$, and likewise $1 - x^{\delta_n}$ lies between $1 - a^{\delta_n}$ and $1 - b^{\delta_n}$, so that $|1 - x^{\delta_n}| \le (|1 - a^{\delta_n}| + |1 - b^{\delta_n}|)$. From $\lim\limits_{n\to\infty} a^{\delta_n} = \lim\limits_{n\to\infty} b^{\delta_n} = 1,$ it follows that

$$\lim_{n\to\infty} |1 - a^{\delta_n}| = \lim_{n\to\infty} |1 - b^{\delta_n}| = 0;$$

hence, if $n$ is chosen large enough, the right-hand side of the inequality is less than $\varepsilon$. This yields $|x^{\alpha_n} - x^{\alpha}| < \varepsilon$ simultaneously for all values of $x$ in the interval $a \le x \le b$.

Now we need only apply the relationship, referred to above, to the functions $f(x) = x^{\alpha}$ and $g(x) = x^{\alpha_n}$, yielding

$$-\varepsilon(b-a) + \int_a^b x^{\alpha_n}\,dx < \int_a^b x^{\alpha}\,dx < \int_a^b x^{\alpha_n}\,dx + \varepsilon(b-a).$$

However, the integrals on the right-hand and left-hand sides may be evaluated in accordance with. earlier results, which yields

$$-\varepsilon(b-a) + \frac{1}{\alpha_n + 1}(b^{\alpha_n+1} - a^{\alpha_n+1}) < \int_a^b x^{\alpha}\,dx < \frac{1}{\alpha_n+1}(b^{\alpha_n+1} - a^{\alpha_n+1}) + \varepsilon(b-a).$$

If we now let the number $\varepsilon$ decrease steadily and tend to 0, the corresponding values of $n$ increase beyond all bounds; the numbers $\alpha_n$, $a^{\alpha_n}$, and $b^{\alpha_n}$ must then converge to $\alpha$, $a^{\alpha}$ and $b^{\alpha}$, respectively, and we immediately obtain the result

$$\int_a^b x^{\alpha}\,dx = \frac{1}{\alpha+1}(b^{\alpha+1} - a^{\alpha+1}).$$

In other words, the integration formula, which holds for rational values of $\alpha$, also holds for irrational values of $\alpha$.

It follows from this, by virtue of the fundamental theorem, that, for positive values of $x$, the differentiation formula for rational values

$$\frac{d}{dx}x^{\alpha+1} = (\alpha+1)x^{\alpha}$$

is also valid for irrational values of $\alpha$.

**Exercises 2.5:**

1. Find the intermediate value $\xi$ of the mean value theorem of the integral calculus for the following integrals and interpret them geometrically:

$$(a) \int_a^b 1\,dx. \qquad (b) \int_a^b x\,dx.$$

$$(c) \int_a^b x^n\,dx. \qquad (d) \int_a^b \frac{dx}{x^2}$$

2. Let $f(x)$ be continuous. Prove, using the mean value theorem of the integral calculus, that the derivative of the indefinite integral of $f(x)$ is equal to $f(x)$.

3. (a) Evaluate $I_n = \int_0^a x^{1/n}\,dx.$ What is $\lim_{n\to\infty} I_n$? Interpret it geometrically. (b) Do the same for $I_n = \int_0^a x^n\,dx.$

4.* Let the function $f(\xi)$ be continuous for all values of $\xi$ and let $F(x)$ be defined by

$$F(x) = \frac{1}{2\delta}\int_{-\delta}^{\delta} f(x+t)\,dt,$$

where $\delta$ is an arbitrary positive number. Prove that:

(a) the function $F(x)$ possesses a continuous derivative for all values of $x$,
*b)* in any fixed interval $a \le x \le b$, we can make $|F(x) - f(x)| < \varepsilon$, where $\varepsilon$ *is* an arbitrary pre-assigned positive number, by choosing $\delta$ small enough.

5. **Schwartz's Inequality for integrals:** Prove that for all continuous functions $f(x)$, $g(x)$

$$\int_a^b (f(x))^2\,dx \int_a^b (g(x))^2\,dx \geqq \left(\int_a^b f(x)g(x)\,dx\right)^2.$$

Answers and Hints

# Appendix to Chapter II

**A2.1 The Existence of the Definite Integral of a Continuous Function:** We must still give a proof of the fact that there always exists the definite integral of a continuous function between the limits $a$ and $b$ ($a < b$). For this purpose, we recall the earlier discussed notation and consider the sum

$$F_n = \sum_{\nu=1}^{n} f(\xi_\nu)\Delta x_\nu.$$

It is certainly true that

$$\underline{F}_n = \sum_{\nu=1}^{n} f(v_\nu)\Delta x_\nu \leqq F_n \leqq \sum_{\nu=1}^{n} f(u_\nu)\Delta x_\nu = \overline{F}_n,$$

where $f(v_\nu)$ denotes the least and $g(u_\nu)$ the largest value of the function in the $\nu$-th sub-interval. The problem is to prove that $F_n$ tends to a definite limit independently of the particular manner of subdivision and of the particular choice of the quantities $\xi_\nu$, provided that, as $n$ increases, the length of the longest subinterval tends to zero. In order to establish this, it is obviously necessary and sufficient to show that the two expressions $\underline{F}_n$ and $\overline{F}_n$ converge to one and the same limit.

No matter how small the positive number $\varepsilon$ is chosen, we know from the uniform continuity of $f(x)$ that in every sufficiently small interval the oscillation $|f(u_\nu) - f(v_\nu)|$ is less than $\varepsilon$ so that, if the subdivision is fine enough, we certainly must have

$$0 \leqq \overline{F}_n - \underline{F}_n = \sum_{\nu=1}^{n} \Delta x_\nu \{f(u_\nu) - f(v_\nu)\} < \epsilon(b-a).$$

Hence we see that, as $n$ increases, this difference must tend to zero, and so we can be content with proving that one of the sums, say $\overline{F}_n$, converges. This convergence will have been proved as soon as we show that $|\overline{F}_n - \overline{F}_m|$ can be made as small as desired by requiring that the corresponding subdivisions (which we shall refer to as subdivision $n$ and subdivision $m$, respectively, exceed a certain degree of fineness. This degree of fineness is characterized by the property that for both subdivisions the oscillation of the function in each subinterval is less than $\varepsilon$ ($\varepsilon$>0). We continue to a third subdivision the points of subdivision of which consist of all the points of subdivision $n$ and of subdivision $m$ taken together. This new subdivision, which has, say $l$ points of subdivision, we

denote by the subscript *l* and consider the corresponding upper sum $\overline{F}_l$ . We shall now estimate the value of $|\overline{F}_n - \overline{F}_m|$ ,first obtaining estimates for the expressions $|\overline{F}_n - \overline{F}_l|$ and $|\overline{F}_m - \overline{F}_l|$. We assert that the following two relationships hold:

$$\underline{F}_n \leqq \overline{F}_l \leqq \overline{F}_n \quad \text{and} \quad \underline{F}_m \leqq \overline{F}_l \leqq \overline{F}_m.$$

The proof follows at once from the meaning of our expressions. Let us consider, say, the $\nu$-th subinterval of the subdivision *n*. This subinterval will consist of one or several subintervals of the subdivision *l*; the terms corresponding to these intervals will each consist of two factors, one of which is a difference $\Delta x$ and the other certainly not greater than $f(u_\nu)$ and not less than $f(v_\nu)$. The sum of the lengths $\Delta x$ of those intervals of the subdivision *l* which lie in the $\nu$-th subinterval of the coarser subdivision *n* is, however, exactly $\Delta x_\nu$. Hence we see that the corresponding contribution to the sum $\overline{F}_l$ must lie between the limits $f(u_\nu)\Delta x_\nu$ and $f(v_\nu)\Delta x_\nu$. If we now sum over all the *n* subintervals, we obtain the first of the above inequalities; the second is obtained in exactly the same manner, if we consider the subdivision *m* instead of the subdivision *n*.

We have already seen that $\overline{F}_n - F_n < \epsilon(b-a)$; it is likewise true that $\overline{F}_m - \underline{F}_m < \epsilon(b-a)$. Hence,by the inequalities for $\overline{F}_l$ proved above, one has

$$0 \leqq \overline{F}_n - \overline{F}_l < \epsilon(b-a) \quad \text{and} \quad 0 \leqq \overline{F}_m - \overline{F}_l < \epsilon(b-a).$$

Thus, it is also certain that

$$|\overline{F}_n - \overline{F}_m| = |(\overline{F}_n - \overline{F}_l) - (\overline{F}_m - \overline{F}_l)| < 2\epsilon(b-a).$$

Since we can choose $\varepsilon$ as small as we please, this relation shows, by Cauchy's convergence test, that the sequence of numbers $\overline{F}_n$, actually converges. At the same time, we see at once from our argument that the limiting value is completely independent of the manner of sub-division.

The proof of the existence of the definite integral of a continuous function is thus complete.

Our method of proof teaches us yet more. It shows us that, in many cases, we are also led to the integral by a somewhat more general limiting process. If, for example, $f(x) = \phi(x)\psi(x)$ and the interval from $a$ to $b$ is divided into $n$ parts by the points $x_\nu$, we consider instead of the sum $\Sigma f(\xi_\nu)\Delta x_\nu$ the more general sum

$$\Sigma \phi(\xi_\nu')\psi(\xi_\nu'')\Delta x_\nu,$$

where $\xi'_\nu$ and $\xi''_\nu$ are two not necessarily coincident points of the $\nu$-th subinterval. This sum will also tend to the integral

$$\int_a^b f(x)\,dx = \int_a^b \phi(x)\psi(x)\,dx$$

as $n$ increases, provided that the length of the longest subinterval tends to zero. A corresponding statement holds for all sums formed in an analogous manner; for example, the sum

$$\sum_{\nu=1}^{n} \sqrt{\{\phi(\xi_\nu')^2 + \psi(\xi_\nu'')^2\}}\,\Delta x_\nu$$

tends to the integral

$$\int_a^b \sqrt{\{\phi(x)^2 + \psi(x)^2\}}\,dx.$$

The proof of these facts follows along exactly the same lines like those of the above proof and hence need not be worked out in detail.

**A2.2 The Relation between the Mean Value Theorem of the Differential Calculus and the Mean Value Theorem of the Integral Calculus:** There exists between the mean value theorems of the differential and the integral calculus a simple relation which is arrived at by way of the fundamental theorem and which we give as an instructive example of the use of that theorem. We take the mean value theorem of the integral calculus in its more special form

$$\int_a^b f(x)\,dx = (b-a)f(\xi).$$

If we set put $\int f(x)\,dx = F(x)$, so that $f(x) = F'(x)$, this formula assumes the form

$$F(b) - F(a) = (b - a)\,F'(\xi).$$

Obviously, we can choose here for $F(x)$ any function the first derivative $F'(x)=f(x)$ of which is continuous, and thus, for such functions, the mean value theorem of the differential calculus has been proved.

If we consider the more general form of the mean value theorem of the integral calculus

$$\int_a^b f(x)\,p(x)\,dx = f(\xi)\int_a^b p(x)\,dx,$$

where $p(x)$ is a function which in our interval is continuous and positive and $f(x)$ is an arbitrary continuous function, we are led to a correspondingly more general mean value theorem of the differential calculus. We set

$$\int f(x)\,p(x)\,dx = F(x), \quad \text{i.e. } f(x)\,p(x) = F'(x),$$

and

$$\int p(x)\,dx = G(x), \quad \text{i.e. } p(x) = G'(x);$$

the above mean value formula then takes the form

$$F(b) - F(a) = \{G(b) - G(a)\}\,f(\xi),$$

or, since $f(x) = F'(x)/G'(x)$,

$$\frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(\xi)}{G'(\xi)},$$

where $a \neq b$.

This formula, in which $\xi$ once again denotes a number between $a$ and $b$, is the **generalized mean value theorem of the differential calculus**. For this to be valid, it is obviously sufficient to assume that $F(x)$ and $G(x)$ are continuous functions with continuous first derivatives and that, in addition, $G'(x)$ is everywhere positive (or everywhere negative). In fact, with these assumptions, the entire process can be reversed.

Finally, it should be observed that in the present discussion of the mean value theorem of the differential calculus we have had to make assumptions more stringent than the theorems themselves require. (cf. 2.3.8 and 3.3.3)

**Exercise 2.6:**

1. Show that if $f(x)$ has a continuous derivative in the interval $a \le x \le b$, then $f(x)$ can be represented as the difference of two monotonic functions.

Answer and Hint

## Chapter III

# Differentiation and Integration of the Elementary Functions

### 3.1. The Simplest Rules for Differentiation and their Applications

In higher analysis and its applications, it is usually the case that the problems of integration are more important than those of differentiation, although differentiation offers less difficulty than integration. Consequently, the natural method of constructing the integral and differential calculus is first to learn to differentiate the widest possible classes of functions and then, by virtue of the fundamental theorem, make the results thus obtained available for the solution of integration problems. In the following sections, it will be our task to carry out this programme. To a certain extent, we shall make a fresh start, since we shall work out the most important differentiations and integrations systematically without referring to the results of the last chapter. In this development of the subject, certain rules for differentiation, with the first of which we are already acquainted , will have an important role.

**3.1.1 Rules for Differentiation:**

We assume that in the interval under consideration the functions $f(x)$ and $g(x)$ are differentiable; our rules then are:

Rule 1: **Multiplication by a constant**.

If $c$ is a constant and $\phi(x) = cf(x)$, then $\phi(x)$ is differentiable and

$$\phi'(x) = cf'(x).$$

This follows immediately from the relation

$$\frac{\phi(x+h)-\phi(x)}{h} = c\,\frac{f(x+h)-f(x)}{h}$$

if we take the limit $h \to 0$.

**Rule 2: Derivative of a sum.**

If $\phi(x) = f(x) + g(x)$, then $\phi(x)$ is differentiable, and

$$\phi'(x) = f'(x) + g'(x);$$

that is, the processes of differentiation and addition are interchangeable. The same holds for the sum of any finite number $n$ of terms

$$\phi(x) = \sum_{\nu=1}^{n} f_\nu(x),$$

for which we obtain

$$\phi'(x) = \sum_{\nu=1}^{n} f_\nu'(x).$$

We may pass over the proof, which, by 2.3, is fairly obvious.

**Rule 3: Derivative of a product.**

If $\phi(x) = f(x)g(x)$, then $\phi(x)$ is differentiable and

$$\phi'(x) = f(x)g'(x) + g(x)f'(x).$$

The proof follows from the equation

$$\begin{aligned}\frac{\phi(x+h)-\phi(x)}{h} &= \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\ &= \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\ &= f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}.\end{aligned}$$

The passage to the limit $h \to 0$ can here be carried out directly and yields the result shown.

This formula takes a still more elegant form, if we divide throughout by $\phi(x)=f(x)g(x)$. We then obtain

$$\frac{\phi'(x)}{\phi(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

where we, of course, assume that $\phi(x)$ is nowhere equal to zero.

By repeated application of this product formula, we obtain, by induction, for the derivative of a product of $n$ factors an expression consisting of $n$ terms, each of which consists of the derivative of one factor multiplied by all the other factors of the original product. In symbols, this is

$$\begin{aligned}\phi'(x) &= \frac{d}{dx}\{f_1(x)f_2(x)\ldots f_n(x)\} \\ &= f_1'(x)f_2(x)\ldots f_n(x) + f_1(x)f_2'(x)f_3(x)\ldots f_n(x) \\ &\qquad\qquad\qquad + \ldots + f_1(x)f_2(x)\ldots f_n'(x) \\ &= \sum_{\nu=1}^{n} f_\nu'(x)\frac{\phi(x)}{f_\nu(x)},\end{aligned}$$

or on division by $\phi(x) = f_1(x)f_2(x)\ldots f_n(x)$, assuming again that $\phi(x)$ does not vanish anywhere,

$$\frac{\phi'(x)}{\phi(x)} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \ldots + \frac{f_n'(x)}{f_n(x)} = \sum_{\nu=1}^{n}\frac{f_\nu'(x)}{f_\nu(x)}.$$

**Rule 4: Derivative of a quotient.**

We have for a quotient

$$\phi(x) = \frac{f(x)}{g(x)}$$

the rule: The function $\phi(x)$ is differentiable at every point at which $g(x)$ does not vanish and

$$\phi'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{\{g(x)\}^2}.$$

If $\phi(x) \neq 0$, this can be rewritten

$$\frac{\phi'(x)}{\phi(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

$$\phi'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{\{g(x)\}^2}.$$

If we accept the differentiability of $\phi(x)$ as a hypothesis, we can apply the product rule to $f(x)=\phi(x)g(x)$ and conclude that

$$f'(x) = \phi(x)g'(x) + g(x)\phi'(x).$$

By substituting $f(x)/g(x)$ for $\phi(x)$ on the right hand side and solving for $\phi'(x)$, we obtain the above rule. In order to prove the differentiability of $\phi(x)$ as well as the rule, we use the method: We write

$$\frac{\phi(x+h) - \phi(x)}{h} = \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \frac{g(x)\dfrac{f(x+h)-f(x)}{h} - \dfrac{g(x+h)-g(x)}{h}f(x)}{g(x)g(x+h)}.$$

If we now let $h$ tend to 0, we arrive at the result stated, because, by assumption, the two terms obtained by performing the division on the right hand side have definite limits, which are

$$\frac{g(x)f'(x)}{\{g(x)\}^2} \text{ and } \frac{g'(x)f(x)}{\{g(x)\}^2},$$

respectively. This proves immediately the existence of the limit on the left hand side as well as the differentiation formula.

**3.1.2 Differentiation of the Rational Functions:** To begin with, we shall again deduce the differentiation formula

$$\frac{d}{dx}x^n = nx^{n-1}$$

for every positive integer $n$, basing the proof on the rule for differentiation of a product. We think of $x^n$ as a product of $n$ factors, $x^n = x, \cdots . x$ and hence obtain

$$\frac{d}{dx} x^n = 1 . x^{n-1} + 1 . x^{n-1} + \ldots + 1 . x^{n-1} = nx^{n-1}.$$

If we use this formula and the first rule of differentiation, the second derivative of the function $x^n$ becomes:

$$\frac{d^2}{dx^2} x^n = n(n-1)x^{n-2}.$$

Continuing this process, we obtain

$$\frac{d^3}{dx^3} x^n = n(n-1)(n-2)x^{n-3}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d^n}{dx^n} x^n = 1 . 2 \ldots n = n!$$

It is clear from the last of these formulae that the ($n$+ l)-th derivative of $x^n$ vanishes everywhere.

By virtue of our first two rules, a knowledge of the differentiation of powers enables us at once to differentiate any polynomial

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

We have simply

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + na_n x^{n-1},$$

and

$$y'' = 2a_2 + 3 . 2 a_3 x + 4 . 3 a_4 x^2 + \ldots + n(n-1) a_n x^{n-2},$$

and so on.

The differentiation of any rational function now follows with the help of the quotient rule. In particular, we shall again derive the differentiation formula for the function $x^n$, where $n = -m$ is a negative integer. The application of the quotient rule together with the fact that the derivative of a constant is equal to zero yields

$$\frac{d}{dx}\left(\frac{1}{x^m}\right) = -\frac{mx^{m-1}}{x^{2m}} = -\frac{m}{x^{m+1}},$$

or, if we take $m = -n$,

$$\frac{d}{dx}x^n = nx^{n-1},$$

which agrees formally with the result for positive values of $n$ and with the results given earlier.

**3.1.3 Differentiation of the Trigonometric Functions:** For the trigonometric functions sin $x$ and cos $x$, we have already obtained the differentiation formulae

$$\frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x.$$

The quotient rule now enables us to differentiate the functions

$$y = \tan x = \frac{\sin x}{\cos x} \quad \text{and} \quad y = \cot x = \frac{\cos x}{\sin x}.$$

According to the rule, the derivative of the first of these functions is

$$y' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x},$$

and we obtain the result

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} = \sec^2 x = 1 + \tan^2 x.$$

Similarly, we obtain

$$\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x = -(1 + \cot^2 x).$$

## 3.2 The corresponding Integral formulae

**3.2.1. General Rules for Integration:** The fundamental theorem of and the definition of the indefinite integral reveal the possibility of writing down an integration formula corresponding to each differentiation formula. The following rules of integration (of which the first two have already been mentioned) are completely equivalent to the first three rules of differentiation.

**Multiplication by a constant:** If $c$ is a constant, then

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

**Integration of a sum:** It is always true that

$$\int \{f(x) + g(x)\}\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

To the third rule of differentiation corresponds the rule for the **integration of a product** or, as it is usually called, the rule for **integration by parts**. On integration, the product rule yields

$$\int \{f(x)g(x)\}'\,dx = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx.$$

The indefinite integral on the left hand side is obviously $f'(x)g(x)$ (except possibly for an additive constant), whence we can write the rule for integration by parts in the form:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

This last integration formula, the counterpart of the rule for the differentiation of a product, has been given here only for the sake of completeness; it will not become important for us until 4.4.

**3.2.2 Integration of the Simplest Functions:**. Corresponding to the differentiation formulae for special functions which we have found above, we now set down the equivalent integration formulae. The formula

$$\frac{d}{dx}x^n = nx^{n-1},$$

expressed as an integration formula, becomes

$$\int x^{n-1}\,dx = \frac{x^n}{n}, \qquad n \neq 0.$$

In fact, this formula merely means that the derivative of the right hand side is equal to the integrand on the left hand side. If we replace $n$ by $n$ +1, we obtain the formula

$$\int x^{n-1}\,dx = \frac{x^n}{n}, \qquad n \neq 0.$$

This formula holds for every integral index $n$ (when $n < 0$, it holds of course only if $x \neq 0$) except when $n = -1$, when the denominator $n + 1$ would vanish. In 3.6, this exceptional case will be studied in detail. The fundamental theorem of the integral calculus allows at once to use our integral formulae for the determination of areas, that is, the values of definite integrals. By 2.4.5, we immediately obtain

$$\int_a^b x^n\,dx = \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right), \qquad n \neq -1,$$

where, if $n$ is negative, we assume that $a$ and $b$ have the same sign, since otherwise the integrand would be discontinuous in the interval of integration.

There correspond to the differentiation formulae for sin $x$, cos $x$ the integration formulae:

$$\int \cos x\, dx = \sin x, \qquad \int \sin x\, dx = -\cos x,$$

$$\int \frac{1}{\cos^2 x}\, dx = \tan x, \quad \int \frac{1}{\sin^2 x}\, dx = -\cot x.$$

They yield by way of the fundamental rule of 2.4 the value of the definite integral between any limits, an only restriction being that when the last two formulae are used, the interval of integration must not contain any point of discontinuity of the integrand. For example,

$$\int_a^b \cos x\, dx = \sin x \Big|_a^b = \sin b - \sin a.$$

It hardly need be emphasized that, with the help of the first two rules of integration, we are now in a position to integrate any polynomial in $x$ and, in fact, any linear combination with arbitrary constant coefficients of the functions integrated here. However, note that, according to the fundamental theorem, rules of integration and differentiation must be equivalent to each other; it is therefore possible to prove the general integration rules of this section and then to read off the differentiation rules of the preceding section. The reader is well advised to carry out this suggestion.

**Exercises 3.1:**

1. Find the numerical values of all the derivatives of $x^5 - x^4$ at $x = 1$.

2. What is the numerical value of the eleventh derivative of

$$317x^9 - 202x^7 + 76 \quad \text{at} \quad x = 13\tfrac{1}{2}?$$

3. Differentiate the following functions and write down the corresponding integral formulae:

(a) $ax + b$.

(b) $25cx^7$.

(c) $a + 2bx + cx^2$.

(d) $\dfrac{ax+b}{cx+d}$.

(e) $\dfrac{ax^2 + 2bx + c}{\alpha x^2 + 2\beta x + \gamma}$.

(f) $\dfrac{1}{1-x^2} - \dfrac{1}{1+x^2}$.

(g) $\dfrac{(x^8 - \sqrt{8}x^4 + 4)(x^8 + \sqrt{8}x^4 + 4)}{x^{16} + 16}$.

4. Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.
($a$) Calculate the polynomial $F(x)$ from the equation $F(x) - F'(x) = P(x)$.
*(b)** Calculate $F(x)$ from the equation $c_0F(x) + c_1F'(x) + c_2F''(x) = P(x)$.

6. Differentiate the following functions and write down the corresponding integration formulae:

(a) $2 \sin x \cos x$.

(b) $\dfrac{1}{1+\tan x}$.

(c) $x \tan x$.

(d) $\dfrac{\sin x + \cos x}{\sin x - \cos x}$.

(e) $\dfrac{\sin x}{x}$.

Recalling that sec $x$ = 1/cos $x$, cosec $x$ = 1/sin $x$, find the derivatives of the functions 6. to 9.:

6. $\dfrac{d^2}{dx^2} \sec x$.

7. $\dfrac{d^3}{dx^3} \sec x \tan x$.

8. $\dfrac{d^3}{dx^3} \operatorname{cosec} x$.

9. $\dfrac{d^4}{dx^4} \tan x \sin x$.

10. Find the limit as $n \to \infty$ of the absolute value of the $n$-th derivative of $1/x$ at the point $x = 2$.

Evaluate:

11. $\int (ax + b)\,dx.$

12. $\int (ax^2 + 2bx + c)\,dx.$

13. $\int (9x^8+7x^6+5x^4+3x^2+1)\,dx.$

14. $\int \left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4}\right)dx.$

15. $\int \left(x^2 + \frac{1}{x^2}\right)dx.$

16. $\int \left(a \cos x + \frac{b}{\sin^2 x}\right)dx.$

17. $\int \left(3x+7\sin x+\frac{5}{x^3}-\frac{9}{\cos^2 x}\right)dx.$

18. $\int \sec x \tan x \,dx.$

Answers and Hints

## 3.3 The Inverse Function and its derivative

**3.3.1 The General Formula for Differentiation:** We have seen earlier (1.2.4 and A1.2.4) that a continuous function $y = f(x)$ has a continuous inverse in every interval in which it is monotonic. More exactly:

If $a \le \xi \le b$ is an interval in which the continuous function $y = f(x)$ is monotonic, and if $f(a) = \alpha$ and $f(b) = \beta$, then $x$ is a function of $y$ which in the interval between $\alpha$ and $\beta$ is single-valued, continuous and monotonic.

As we have already shown in 2.3.1, the concept of the derivative gives us simple means for recognizing that a function is monotonic and therefore has an inverse. In fact, a differentiable function is certainly always monotonic increasing, if $f'(x)$ is greater than zero throughout the corresponding interval, and similarly is monotonic decreasing, if $f'(x)$ is everywhere less than zero in the interval.

We shall now prove the theorem: If in the interval $a < x < b$ the function $y=f(x)$ is differentiable and in that interval either $f'(x) > 0$ or else $f'(x) < 0$ everywhere, then the inverse function $x = \phi(y)$ also has a derivative at every point of its interval of definition and there exists, for corresponding values of $x$ and $y$, between the derivative of the given function $y = f(x)$ and that of the inverse function $x = \phi(y)$ the relationship $f'(x) \cdot \phi'(y) = 1$, which we can also write in the form

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

In this formula, we observe again the flexibility of Leibnitz's notation. It is just as if the symbols *dy* and *dx* were quantities which could be operated upon like actual numbers. The proof of this formula is correspondingly simple, if we regard the derivative as the limit of the difference quotient

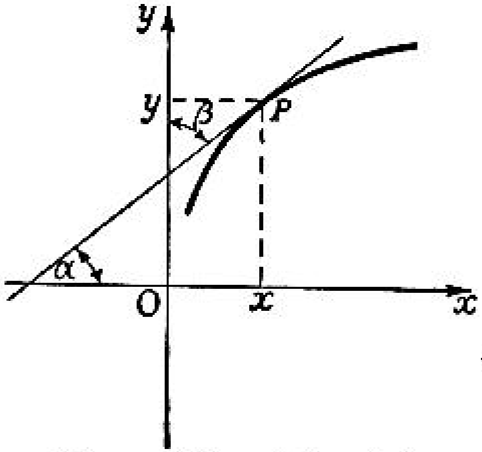


Fig. 1.—Differentiation of the inverse function

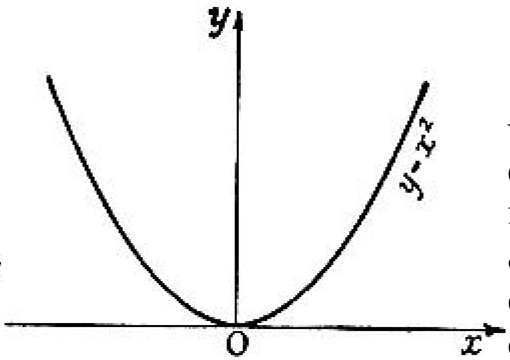


Fig. 2.—Parabola

$$y' = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x} \frac{y_1 - y}{x_1 - x},$$

where $x$ and $y = f(x)$, and $x_1$ and $y_1 = f(x_1)$, respectively, denote pairs of corresponding values. By assumption, the first of these limiting values is not equal to zero. On account of the continuity of $y=f(x)$ and $x=\phi(y)$, the equation $\lim \Delta x = 0$ is equivalent to $\lim \Delta y = 0$, and consequently the relations $y_1 \to y$ and $y_1 \to x$ are also equivalent. Therefore the limiting value

$$\lim_{x_1 \to x} \frac{x_1 - x}{y_1 - y} = \lim_{y_1 \to y} \frac{x_1 - x}{y_1 - y}$$

exists and is equal to $1/f'(x)$. On the other hand, the limiting value is by definition the derivative $\phi'(y)$ of the inverse function $\phi(y)$, and the formula is proved.

This formula has a simple geometrical meaning, which is clearly shown in Fig. 1. The tangent to the curve $y = f(x)$ or $x = \phi(y)$ forms with the positive $x$-axis an angle $\alpha$, with the positive $y$-axis an angle $\beta$ and from the geometrical meaning of the derivative

$$f'(x) = \tan \alpha, \quad \varphi'(y) = \tan \beta.$$

However, since the sum of the angles $\alpha$ and $\beta$ is $\pi/2$,. $\tan \alpha \tan \beta = 1$ and this relationship is exactly equivalent to the differentiation formula.

We have hitherto expressly assumed that either $f'(x) > 0$ or $f'(x) < 0$, i.e., that $f'(x)$ is never zero. What will happen if $f'(x) = 0$? If $f'(x)=0$ everywhere in an interval, the function is constant there, whence it has no inverse, since the same value of $y$ must correspond to all values of $x$ in the interval. If the equation $f'(x) = 0$ is true only at isolated points and if, for the sake simplicity, $f'(x)$ is assumed to be continuous, then we must distinguis between whether

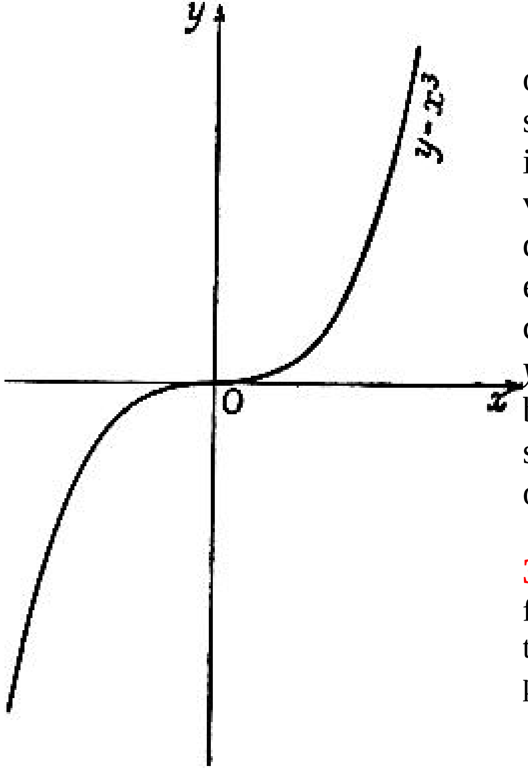



Fig. 3.—Cubical parabola

on passing through these points $f'(x)$ changes its sign or not. In the first case, this point separates a point, where the function is monotonic increasing, from another point, where it is monotonic decreasing. In the neighbourhood of such a point, there cannot be a singe-valued inverse function. In the second case, the vanishing of the derivative does not destroy the monotonic character of the function $y=f(x)$, so that a single-valued inverse exists. However, the inverse function will no longer be differentiable at the corresponding point; in fact, its derivative will be infinite there. The functions $y = x^2$ and $y = x^3$ at the point $x=0$ are examples of the two types. Figs. 2 and 3 illustrate the behaviour of the two functions, where they pass through the origin, and at the same time show that one of the functions, namely $y = x^3$, has a single-valued inverse, but that the other function, $y = x^2$ has not.

**3.3.2 The Inverse of the Power Function:**. The simplest example of an inverse function is the function $y = x^n$ for positive integers $n$ and, as we will at first assume, positive values of $x$. Under these conditions $y'$ is always positive, so that we can form for all positive values of $y$ a unique positive inverse function

$$x = \sqrt[n]{y} = y^{1/n}.$$

The derivative of this inverse function is immediately obtained in accordance with the general rule above by the calculations

$$\frac{d(y^{1/n})}{dy} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{nx^{n-1}} = \frac{1}{n}\,\frac{1}{y^{(n-1)/n}} = \frac{1}{n}\,y^{1/n-1},$$

and if we now denote the independent variable by $x$, we may finally write

$$\frac{d\sqrt[n]{x}}{dx} = \frac{d}{dx}(x^{1/n}) = \frac{1}{n}\,x^{1/n-1},$$

which agrees with the result obtained directly in 2.3.3.

The point $x = 0$ demands special consideration.. If $x$ approaches 0 through positive values, , for n > 1, $d(x^{1/n})/dx$ will obviously increase beyond all bounds! This corresponds to the fact that, for $n > 1$, the derivative of the $n$-th power $f(x) = x^n$ vanishes at the origin. Geometrically speaking, this means that the curves $y = x^{1/n}$, $n > 1$ touch the $y$-axis at the origin (**Fig. 17,** 1.5.7).

For the sake of completeness, it should be noted that, for odd values of $n$, the assumption that $x>0$ can be omitted and the function $y = x^n$ can be considered for all values of $x$ without loss of its monotonic character or of the uniqueness of its inverse. The differentiation formula $d(y^{1/n})/dy=y^{1/n-1}/n$ still holds for negative values of $y$; for $x = 0$, $n > 1$, we have $d(x^n)/dx = 0$, which corresponds to an infinite derivative ($dy/dx$) of the inverse function at the point y = 0.

**3.3.3 The Inverse Trigonometric Functions:** In order to form the inverses of the trigonometric functions, we once again consider the graphs of sin $x$, cos $x$, tan $x$ and cot $x$. We see at once from Figs. 14 and 15 that it is necessary to select for each of these functions a definite interval, if we are to speak of a unique inverse; in fact, the lines $y = c$ parallel to the $x$-axis cut the curves, if at all, at an infinite number of points.

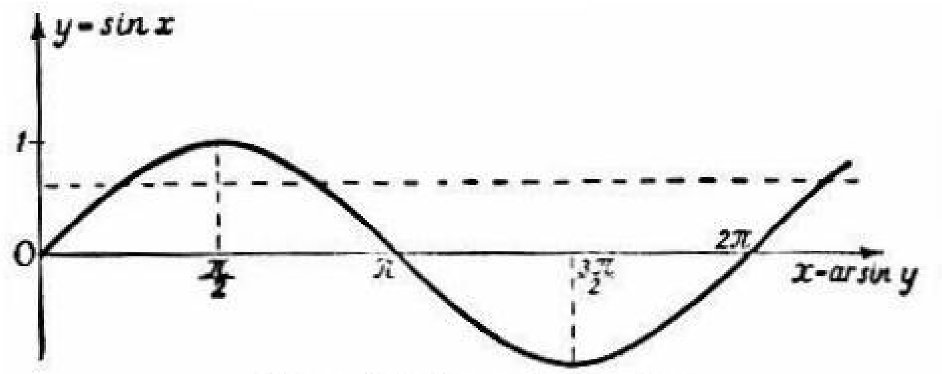



Fig. 4.—The inverse sine function

For example, for the function $y = \sin x$, the derivative $y' = \cos x$ will be positive in the interval $-\pi/2<x<\pi/2$. Hence, in this interval, sin $x$ has an inverse function: We write the inverse function of sin $x$ as

$$x = \operatorname{arsin} y$$

(read arsin $y$; it is the angle the sine of which has the value $y$; **English texts** also use $x = \sin^{-1} y$). This function runs monotonically from $-\pi/2$ to $+\pi/2$ as $y$ traverses the interval -1 to +1. If we wish to emphasize especially that we are considering the inverse function of the sine for this very interval, we speak of the **principal value** of the arsine. If we form the inverse function for some other interval in which sin $x$ is monotonic, e.g., the interval $+\pi/2 < x < 3\pi/2$, we obtain another **branch** of arsin; without an exact statement of the interval, in which the values of the function must lie, arsine is a **multi-valued function**; in fact, it has an **infinite** number of values.

In general, the fact that arsin $y$ is multip-valued is expressed by the statement that there corresponds to anyone value $y$ of the sine not only the angle $x$, but also the angle $(2k + 1)\pi + x$, as well as the angle $(2k + 1)\pi - x$, where $k$ is any integer (Fig. 4).

The differentiation of the function $x = \text{arsin } y$ is performed by our general rule as follows:

$$\frac{dx}{dy} = \frac{1}{y'} = \frac{1}{\cos x} = \frac{1}{\pm\sqrt{(1-\sin^2 x)}} = \frac{1}{\pm\sqrt{(1-y^2)}},$$

where the square root is positive, if we confine ourselves to the first interval above.

If we had chosen instead the interval $\pi/2 < x < 3\pi/2$, corresponding to the substitution of $x + \pi$ for $x$, we should have had to use the negative sqare root, since $\cos x$ is negative in this interval.

If the independent variable is finally changed back from y to $x$, the differentiation formula for the function arsin $x$ is

$$\frac{d}{dx}\text{arsin } x = \frac{1}{\sqrt{(1-x^2)}}.$$

It is assumed here that arsin $x$ lies between $-\pi/2$ and $+\pi/2$ and the square root sign is chosen positive.

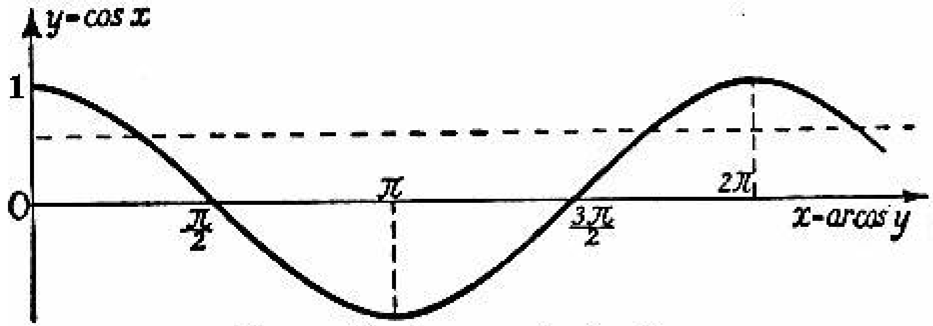


Fig. 5.—The inverse cosine function

For the inverse function of y= cos $x$, denoted by arcos $x$, we obtain the differentiation formula

$$\frac{d}{dx}\operatorname{arcos} x = -\frac{1}{\sqrt{(1-x^2)}}$$

in exactly the same way. Here we take the positive sign of the root if the value of arcos $x$ is taken in the interval between 0 and $\pi$ (not as in the case of arc sin $x$, between $-\pi/2$ and $+\pi/2$).(Fig. 5).

There remains to say something about the end-points $x = -1$ and $x = +1$. The derivatives become infinite as these end-points are approached, corresponding to the fact that the graphs of the inverse sine and inverse cosine must have vertical tangents at these points.

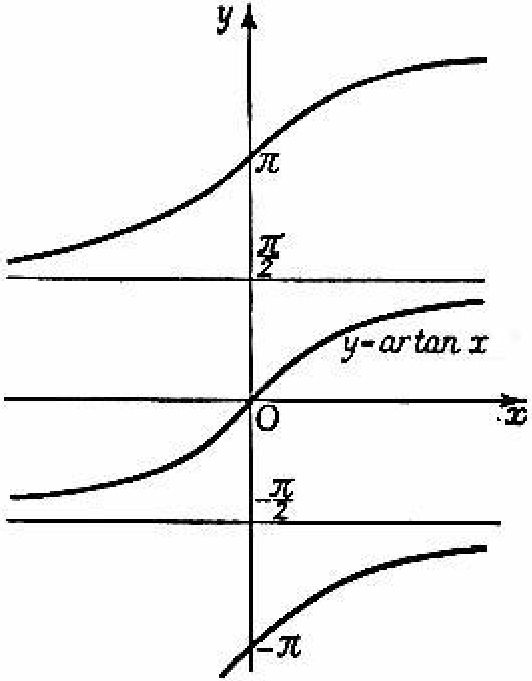



Fig. 6.—The inverse tangent function

We can deal with the inverse functions of the tangent and cotangent in an analogous manner. The function $y = \tan x$, the derivative $1/\cos^2 x$ of which is everywhere positive for $x \neq \pi/2 + k\pi$, has a unique inverse in the interval $-\pi/2 < x < \pi/2$. We call inverse function $x = \operatorname{artan} y$ or (by interchange of the letters $x$ and $y$=artan $x$. We see at once from Fig. 6 that the original multi-valuedness of the inverse function - i.e., the multi-valuedness which occurs if the interval of the values of the function is not fixed - is expressed by the fact that for each $x$ we could have chosen instead of y any of the values $y + k\pi$ (where $k$ is an integer). For the function $y = \cot x$, the inverse $x = \operatorname{arcot} y$, (by interchange of $x$ and $y$) $y = \operatorname{arcot} x$, is uniquely determined, if we require that its value shall lie in the interval from 0 to $\pi$; the many-valuedness of arcot $x$ is otherwise the same as for artan $x$.

The differentiation formulae are

$$x = \operatorname{artan} y, \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2};$$

$$x = \operatorname{arcot} y, \quad \frac{dx}{dy} = -\sin^2 x = -\frac{1}{1+\cot^2 x} = -\frac{1}{1+y^2};$$

or, finally, if we denote the independent variable by $x$,

$$\frac{d}{dx}\operatorname{artan} x = \frac{1}{1+x^2}, \quad \frac{d}{dx}\operatorname{arcot} x = -\frac{1}{1+x^2}.$$

**3.3.4 The Corresponding Integral Formulae:** Expressed in the language of the indefinite integral, the formulas which we have just derived become

$$\int \frac{1}{\sqrt{(1-x^2)}}\,dx = \operatorname{arsin} x, \quad \int \frac{1}{\sqrt{(1-x^2)}}\,dx = -\operatorname{arcos} x,$$

$$\int \frac{1}{1+x^2}\,dx = \operatorname{artan} x, \quad \int \frac{1}{1+x^2}\,dx = -\operatorname{arcot} x.$$

There arises no contradiction between the pair of formulae on the left and on the right hand sides, which express each indefinite integral in the form of two functions, which appear to be entirely different. We must remember that, in the case of the indefinite integral, an arbitrary additive constant remains at at our disposal. If we choose these constants so that they differ by $\pi/2$ and recall that $\pi/2$ - arcos $x$ = arsin$x$ and likewise $\pi/2$ - arcot $x$ = artan $x$, this formal disagreement is immediately removed. The indefiniteness simply depends on the fact that the indefinite integral is not a single definite function, but an entire family of functions which differ from each other by arbitrary additive constants. The equation for an indefinite integral does not specify the value, but only one value of it. As we have already remarked, it would be more correct to express this fact by always including the undetermined constant, thus not writing

$$\int f(x)\,dx = F(x),$$

but

$$\int f(x)\,dx = F(x) + c.$$

However, for the sake of convenience, this more detailed form is usually avoided; the reader should therefore be all the more careful to bear in mind the indefiniteness which is always associated with the shorter form.

There follow immediately from the formulae for indefinite integration formulae for definite integration. In particular,

$$\int_a^b \frac{dx}{1+x^2} = \operatorname{artan} x \Big|_a^b = \operatorname{artan} b - \operatorname{artan} a.$$

If we set $a = 0$, $b = 1$ and recall that tan 0 = 0 and tan $\pi/4 = 1$, we obtain the remarkable formula

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\, dx.$$

The number $\pi$, which originally arose from the study of the circle, is by this formula brought into a very simple relationship to the rational function $1/(1+x^2)$ and is expressed and defined by the area shown in Fig. 7.

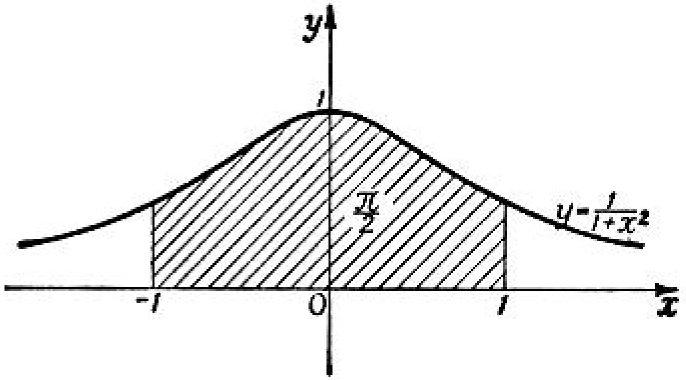



Fig. 7.—$\pi/2$ illustrated by an area

**Exercises 3.2:**

1. If $y = x^4/4$, $y = 16$ corresponds to $x = 8$. Find $dy/dx$ for $x = 8$; solve $y=x^2/4$ for $x$ and find $dx/dy$ for $y = 16$ and show that the values of these derivatives are consistent with the rule for inverse functions.

2. Prove that

(a) $\arcsin\alpha + \arcsin\beta = \arcsin(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2})$;

(b) $\arcsin\alpha + \arcsin\beta = \arccos(\sqrt{1-\alpha^2}\sqrt{1-\beta^2} - \alpha\beta)$;

(c) $\arctan\alpha + \arctan\beta = \arctan\dfrac{\alpha+\beta}{1-\alpha\beta}$.

Differentiate the 3. - 10. and write down the corresponding integral formulae:

3. $\dfrac{\sqrt{x}}{1+x}$.

4. $\sqrt{x}\cos^2 x$.

5. $\dfrac{1+\sqrt{x}}{1-\sqrt{x}}$.

6. $\dfrac{\sqrt{x}}{1-\tan x}$.

7. $\arcsin x . \arccos x$.

8. $\dfrac{1+\arctan x}{1-\arctan x}$;

9. $\dfrac{\arcsin x}{\arctan x}$.

10. $5\,\mathrm{arc\,cot}\,x + \dfrac{1}{\arccos x}$.

11. Using graph paper, plot y = 1/(1 + $x^2$) on a large scale. By counting squares, find $\int_0^1 \frac{1}{1+x^2}\,dx$, thus obtaining an estimate for $\pi/4$ (cf. Example 1. in 2.5).

Answers and Hints

## 3.4 Differentiation of a function of a function

**3.4.1 The Chain Rule:** The preceding rules for differentiation enable us to differentiate every function which can be expressed as a rational expression, the terms of which are functions with known derivatives. However, we can take yet another important step forward and differentiate all functions, obtained by compounding functions with known derivatives. Let $\phi(x)$ be a function which is differentiable in an interval $a \le x \le b$ and assumes all values in the interval $\le \phi \le \beta$. We now wish to consider a second differentiable function $g(\phi)$ of the independent variable $\phi$, in which the variable $\phi$ ranges over an interval from $\alpha$ to $\beta$. We can now regard the function $g(\phi) = g\{\phi(x)\} = f(x)$ as a function of $x$ in the interval $\alpha \le \phi \le \beta$. The function f($x$) = $g\{\phi(x)\}$ will then be called a **compound function** of $x$ - compounded from the functions $g$ and $\phi$ – or a **function of a function**.

For example, if $\phi(x) = 1 - x^2$ and $g(\phi) = \sqrt{\phi}$, this compound function is simply $f(x) = \sqrt{(1 - x^2)}$. For the interval $a \le x \le b$, we take here the interval $0 \le x \le 1$. The values of the function $\phi(x)$ exactly fill up the interval $0 \le \phi \le 1$; the compound function $f(x) = \sqrt{(1 - x^2)}$ is therefore defined in the interval $0 \le x \le 1$.

Another example of the compounding of functions is the function $f(x) = \sqrt{(1-x^2)}$, where the compounding process may be indicated by the equations

$$\varphi(x) = 1 + x^2, \quad g(\varphi) = \sqrt{\varphi}$$

and where the value of the function $\phi(x)$ runs through all positive numbers $\ge 1$, so that the function $f(x) = g\{\phi(x)\}$ can be formed for all values of $x$.

When compounding functions in this way, we must naturally be careful to restrict ourselves to intervals $a \le x \le b$ for which the compound function is defined. For example, the compound function $\sqrt{(1 - x^2)}$ is only defined for values of $x$ in the region $-1 \le x \le 1$, and not in the region $1 < x \le 2$, because, when $x$ is in this last interval, the values of the function $\phi(x)$ consist of negative numbers, for which the function $g(\phi)$ is not defined.

Just as we can compound two functions with one another, we can and must consider functions in which the compounding process is performed more than once. Such a function is

$$\sqrt{(1 + \operatorname{artan} x^2)}$$

which can be built up by the compounding process

$$\varphi(x) = x^2, \quad \psi(\varphi) = 1 + \operatorname{artan} \varphi, \quad g(\psi) = \sqrt{\psi(\varphi)} = f(x).$$

For the differentiation of compound functions, we have the fundamental theorem - the **chain rule of the differential calculus**:

The function $f(x) = g\{\phi(x)\}$ is differentiable and its derivative is given by the equation

$$f'(x) = g'(\phi) \cdot \phi'(x),$$

or, in Leibniz's notation, by

$$\frac{dy}{dx} = \frac{dy}{d\phi} \cdot \frac{d\phi}{dx}.$$

In words: The derivative of the compound function is the product of the derivatives of the constituent functions.

The proof of this formula follows readily, if we recall the meaning of the derivative. There exist for any arbitrary $\Delta x \neq 0$ and corresponding values of $\Delta\phi$ and $\Delta g$ two quantities $\varepsilon$ and $\eta$, tending to 0 with $\Delta x$, such that

$$\Delta g = g'(\phi)\Delta\phi + \epsilon\Delta\phi \quad \text{and} \quad \Delta\phi = \phi'(x)\Delta x + \eta\Delta x;$$

we have only to calculate $\eta$ from the second equation and $\varepsilon$, where $\Delta\phi \neq 0$, from the first equation, while, if $\Delta\phi = 0$, we set $\varepsilon = 0$. If we now substitute in the first of these equations the value of $\Delta\phi$ from the second equation, we obtain

$$\Delta g = g'(\phi)\phi'(x)\Delta x + \{\eta g'(\phi) + \epsilon\phi'(x) + \epsilon\eta\}\Delta x,$$

or

$$\frac{\Delta g}{\Delta x} = g'(\phi)\phi'(x) + \{\eta g'(\phi) + \epsilon\phi'(x) + \epsilon\eta\}.$$

However, in this equation, we can let $\Delta x$ tend to 0 and obtain at once the result stated, since the bracket on the right hand side tends to zero with $\Delta x$. Consequently, the left hand side of our equation has a limit $f'(x)$ and this limit is equal to the first term on the right hand side, as stated above.

We could also have proved the rule by carrying out the passage to the limit $\Delta x \to 0$, and consequently $\Delta\phi \to 0$, in the equation $\frac{\Delta g}{\Delta x} = \frac{\Delta g}{\Delta\phi} \cdot \frac{\Delta\phi}{\Delta x}$. However, the method in the text is to be preferred. because it avoids the necessity for considering specially the case $\phi'(x) = 0$.

By successive application of our formula, we can immediately extend it to functions which arise from **compounding of more than two functions**, for example,

$$y = g(u), \quad u = \phi(v), \quad v = \psi(x),$$

we can think of $y = f(x)$ as a function of $x'$ with the derivative given by the rule

$$\frac{dy}{dx} = y' = g'(u)\phi'(v)\psi'(x) = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

The case of a function compounded by an arbitrary number of functions is essentially similar. The proof may be left to the reader.

**3.4.2 Examples:** As a very simple example, we consider the function $y = x^{\alpha}$, where we $\alpha = p/q$, $q$ being a positive integer and $p$ a positive or negative integer, so that $\alpha$ is an arbitrary, positive or negative rational number. Let $x$ be positive. By the chain rule with

$$y = \varphi^p, \quad \varphi = x^{1/q},$$

we have the formula

$$y' = p\varphi^{p-1} \cdot \frac{1}{q} x^{(1-q)/q} = \frac{p}{q} x^{p/q-1},$$

so that for arbitrary, rational values of $\alpha$ we obtain the differentiation formula

$$\frac{d}{dx} x^{\alpha} = \alpha x^{\alpha-1},$$

*in* agreement with the result already found in another way in 2.3.3. *As* a second example, we consider

$$y = \sqrt{(1 - x^2)} \quad \text{or} \quad y = \sqrt{\varphi},$$

where $\varphi = I - x^2$ and $-1 < x < 1$. The chain rule yields

$$y' = \frac{1}{2\sqrt{\varphi}} \cdot (-2x) = -\frac{x}{\sqrt{(1 - x^2)}}.$$

Further examples are given in brief calculations:

1. $y = \arcsin\sqrt{(1-x^2)}$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{\{1-(1-x^2)\}}} \cdot \frac{d\sqrt{(1-x^2)}}{dx}$$

$$= \frac{1}{|x|} \cdot \frac{-x}{\sqrt{(1-x^2)}} = \mp \frac{1}{\sqrt{(1-x^2)}}.$$

2. $$y = \sqrt{\left(\frac{1+x}{1-x}\right)},$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\left(\frac{1+x}{1-x}\right)}} \cdot \frac{d\left(\frac{1+x}{1-x}\right)}{dx}$$

$$= \frac{\sqrt{(1-x)}}{2\sqrt{(1+x)}} \cdot \frac{2}{(1-x)^2} = \frac{1}{(1+x)^{1/2}(1-x)^{3/2}}.$$

The chain rule for differentiation can also be expressed in the form of an integration formula in agreement with the fact that there corresponds to each differentiation formula a completely equivalent integration formula. Nevertheless, we will pass over this formula for the present, since we have no immediate need of it here and, moreover, it is discussed in detail lin in 4 .2.

**3.4.3 Further Remarks on the Integration and Differentiation of $x^\alpha$, when $\alpha$ is Irrational**: In view of the elementary definition of the power $x^\alpha$ by the equation

$$x^\alpha = \lim x^{r_n},$$

where the numbers $r_n$ form a sequence of rational numbers with the limit $\alpha$, we might be tempted to effect the differentiation of $x^\alpha$ by direct passage to the limit in the differentiation formula

$$\frac{d}{dx} x^{r_n} = r_n x^{r_n - 1}.$$

We are not entitled to do so unless we have the might conclude that there follows from the relation $x^{r_n} \to x^\alpha$ the relation $\frac{d}{dx} x^{r_n} \to \frac{d}{dx} x^\alpha$. However, there is a very serious objection to such a passage to the limit. In fact, in any

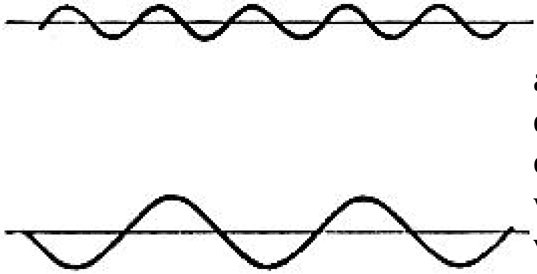


Fig. 8.—Approximation to a straight line by wavy curves

arbitrarily small neighbourhood of a given curve one may draw other curves the direction of which at arbitrarily selected points differs from the direction of the original curve by any desired amount; for example, we may approximate to a straight line by a wave lying arbitrarily near to it, the angle between the wave and the line reaching a value as large as 45° (Fig. 8). In other words, the above example shows that from the fact that two functions differ only very little from each other, we cannot immediately conclude that their derivatives also are everywhere nearly equal to one another. This objection forbids us to perform the apparently obvious passage to the limit in the absence of further justification .

However, in this respect, the integral behaves quite differently from the derivative. We have already observed that, if two functions differ by less than $\varepsilon$ throughout the interval from $a$ to $b$, their integrals must differ by less than $\varepsilon(b-a)$. We used there this result to establish the validity of the differentiation formula

$$\frac{1}{\alpha+1}\frac{d}{dx}x^{\alpha+1}=x^{\alpha},$$

or, replacing $\alpha+1$ by $\alpha$,

$$\frac{d}{dx}x^{\alpha}=\alpha x^{\alpha-1}.$$

Hence, in this indirect way, there is verified the relation $\frac{d}{dx}x^{r_n}\to\frac{d}{dx}x^{\alpha}$ given above. This discussion is a characteristic example of the interrelation of the differential and integral calculus. Yet, in principle, it is preferable to replace (as we shall do in 3.6.5) the elementary definition of $x^{\alpha}$ by another, essentially simpler definition which will lead us once more to the same result and this time directly,

## Exercises 3.3:

Differentiate the functions:

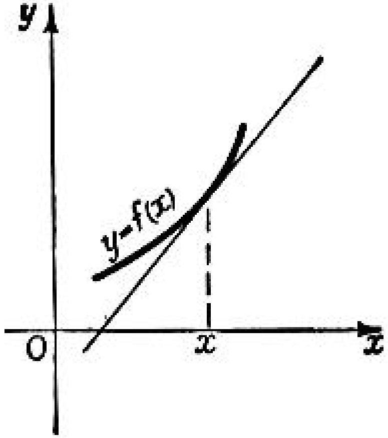


Fig. 9a.— $f''(x) > 0$

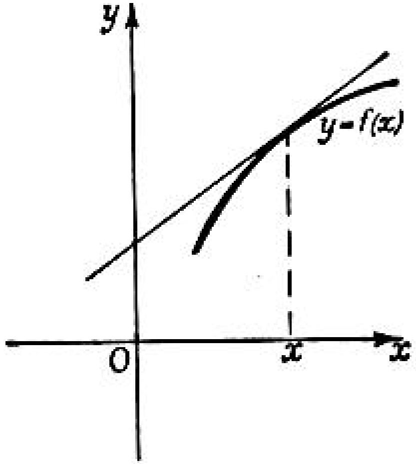


Fig. 9b.— $f''(x) < 0$

1. $(x + 1)^3$.
2. $(3x + 5)^2$.
3. $(x^9 - 3x^6 - x^3)^5$.
4. $\dfrac{1}{1 + x}$.
5. $\dfrac{1}{1 - x^2}$.
6. $(ax + b)^n$ ($n$ an integer).
7. $\dfrac{1}{x + \sqrt{(x^2 - 1)}}$.
8. $\sqrt{\left(\dfrac{ax^2 + bx + c}{lx^2 + mx + n}\right)}$.
9. $(\sqrt{(1 - x)^{2/3}})^5$.
10. $\sin^2 x$.
11. $\sin(x^2)$.
12. $\sqrt{(1 + \sin^2 x)}$.
13. $x^2 \sin \dfrac{1}{x^2}$.
14. $\tan \dfrac{1 + x}{1 - x}$.
15. $\sin(x^2 + 3x + 2)$.
16. $\arcsin(3 + x^3)$.
17. $\arcsin(\cos x)$.
18. $\sin(\arccos \sqrt{(1 - x^2)})$.
19. $x^{\sqrt{2}} - x^{-\sqrt{2}}$.
20. $[\sin(x + 7)]^{\sqrt[6]{5}}$.
21. $[\arcsin(a \cos x + b)]^a$.

Answers and Hints

## 3.5 Maxima and Minima

Now that we have attained a certain mastery of the problem of differentiating the elementary functions and the functions compounded from them, we are in a position to make a variety of applications. We shall consider here the simplest of these applications - the **theory of maxima and minima of a function** - in conjunction with a geometrical discussion of the second derivative - and then, in the next section, we shall again take up the general theory.

**3.5.1 Convexity or Concavity of Curves:** By definition, the derivative $df(x)/dx$ of a function $f(x)$ gives the slope of the curve $y=f(x)$. This slope itself can be represented by a curve $y=df(x)/dx$ - the derived curve of the given curve. The slope of this last curve will be given by $df'(x)/dx = d^2f(x)/dx^2 = f''(x)$ - the second derivative of $f(x)$ - and so on. If the second derivative $f''(x)$ is positive at a point $x$ - so that owing to continuity (which we here assume) it is positive in a certain neighbourhood of the point $x$ - then the derivative $f'(x)$ must increase as it passes this point in the direction of increasing values of $x$. Hence, the curve $y = f(x)$ turns its convex side towards the direction of

decreasing values of $y$. The opposite is true if $f'(x)$ is negative. Hence, in the first case, the curve in the neighbourhood of the point lies above the tangent, in the second case below the tangent (Figs. 9*a* and *b*).

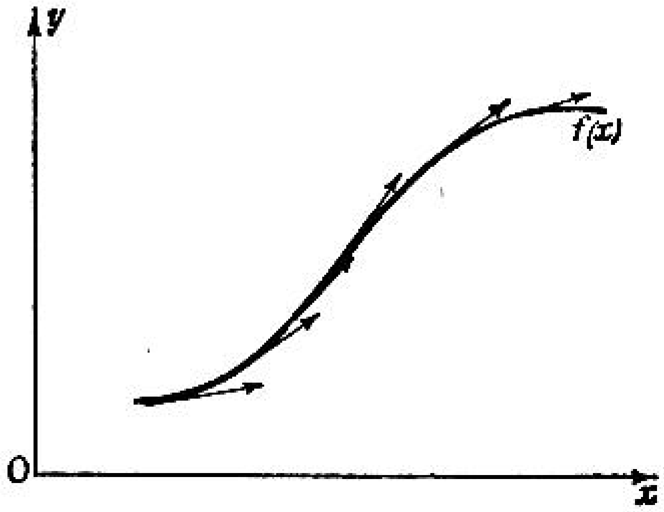



Fig. 10.—Point of inflection

Special consideration is required only in the case of points where $f''(x) = 0$. As a rule, on passing through such a point, the second derivative $f''(x)$ will change its sign. Such a point will then be a point of transition between the two cases indicated above; that is, the tangent will on one side be above the curve and on the other side below it, so that, besides touching the curve, it will also cross it (Fig. 10). This is a **point of inflection** of the curve, and the corresponding tangent is called an **inflectional tangent**.

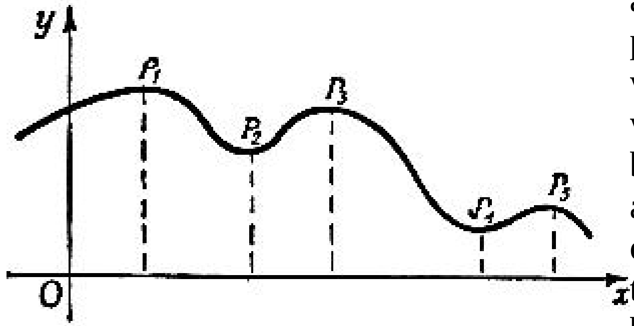



Fig. 11.—Maxima and minima

The simplest example is given by the function $y=a^3$, the cubical parabola, for which the x-axis itself is an inflectional tangent at the point $x = 0$. Another example is given by the function $f(x) = \sin x$, for which

$$f'(x) = d(\sin x)/dx = \cos x \text{ and } f''(x)=d^2(\sin x)/dx = -\sin x.$$

Consequently, $f'(0) = 1$ and $f''(0) = 0$; since the sign of $f''(x)$ changes at $x = 0$, the sine curve has at the origin an inflectional tangent, inclined at an angle of 45° to the $x$-axis.

However, it must be noted that points can exist where $f''(x) = 0$, although the tangent does not cut the curve, but remains entirely on one side of it. For example, the curve $y = x^4$ lies entirely above the $x$-axis, although the second derivative $f''(x)$ vanishes for $x = 0$.

**3.5.2 Maxima and Minima:** We say that a continuous function or a curve $f(x)$ has a maximum **(**minimum**)** at a point $\xi$, if in at least some neighbourhood of the point $x = \xi$ the values of the function $f(x)$ for $x \neq \xi$ are all less than $f(\xi)$ (greater than $f(\xi)$ ).We mean by a **neighbourhood** of a point an interval $\alpha \leq x \leq \beta$ which contains the point $\xi$ in its interior. Geometrically speaking, such maxima and minima are the **wave-crests** and **wave-troughs** of the curve, respectively. A glance at Fig. 11 tells us that the value of the maximum at the point $P_5$ may very well be less than

the value of the minimum at another point $P_2$; thus, on account of the restriction to a certain neighbourhood, the concept of maximum and minimum is always relative to some extent.

If we wish to focus on the actual greatest or least value of a function, we must employ special means for deciding how this value is to be selected from among the maxima or minima.

Our objective at present is to find the (relative) maxima or minima, or, to use a word that covers both maxima and minima, the **relative extreme values** (**extrema**) of a given function or curve. This problem, which is *very* frequently encountered in geometry, mechanics, and physics and which occurs in many other applications, was in the Seventeenth Century one of the principal incentives for the development of the differential and integral calculus.

The expressions **turning value**, **turning point**, are also used. On the other hand, the terms **stationary value, stationary point**, include inflections as well as maxima and minima.

We see at once that, if a function is assumed to be differentiable, the tangent to its curve at an extreme value $\xi$ must be horizontal. Hence, the condition

$$f'(\xi) = 0$$

is a necessary condition for an extreme value; by solving this equation for the unknown $\xi$, we obtain the points at which an extreme value may possibly occur. Our condition, however, is by no means a sufficient condition for an extreme value; there may be points at which the derivative vanishes, i.e., at which the tangent is horizontal, although the curve has there neither a maximum nor a minimum. This occurs, if at the given point the curve has a horizontal inflectional tangent cutting it, as in the above example of the function $y = x^3$ at the point $x = 0$.

However, if we have found a point at which $f'(x)$ vanishes, we may immediately conclude that the function has a maximum at that point, if $f''(\xi) < 0$, a minimum, if $f''(\xi) > 0$. In fact, in the first case, the curve in the neighbourhood of this point lies completely below the tangent, in the second case, completely above the tangent.

Instead of basing the deduction of our necessary condition on intuition, we could, of course, have given an easy proof by purely analytical methods (cf. the exactly analogous considerations for Rolle's theorem. If the function $f(x)$ has a maximum at the point $\xi$, then, for all sufficiently small values of $h$ other than 0, the expression $f(\xi) - f(\xi + h)$ must be positive, whence the quotient $[f(\xi+h)-f(\xi)]/h$ will be positive or negative, according to whether $h$ is negative or positive. Thus, if $h$ tends to zero through negative values, the limit of this quotient cannot be negative,

while if $h$ tends to zero through positive values, the limit cannot be positive. However, since we have assumed that the derivative exists, these two limits must be equal to each other and, in fact, to $f'(\xi)$, which therefore can only have the value zero; we must have $f'(\xi)=0$. A similar proof holds for the case of a minimum.

We can also formulate and prove analytically conditions, which are necessary and sufficient for the occurrence of a *maximum* or a minimum, without involving the second derivative. We assume that the function $f(x)$ is continuous and has a continuous derivative $f'(x)$ which vanishes only at a finite number of points.

Then $f(x)$ has a maximum or a minimum at the point $x = \xi$, if and only if the derivative $f'(x)$ changes sign on passing through this point; in particular, the function has a minimum, if the derivative is negative to the left of $\xi$ and positive to the right of it, while, in the opposite case, it has a maximum.

We prove this by using the **mean value theorem**. First, we observe that there exist to the left and right of $\xi$ intervals $\xi_1 < x < \xi$ and $\xi < x < \xi_2$ (extending to the nearest points at which $f'(x) = 0$) at the each of which $f'(x)$ has only one sign. If the signs of $f'(x)$ in these two intervals differ, then $f(\xi+h)-f(\xi)=hf'(\xi+\theta h)$ has the same sign for all numerically small values of $h$, whether $h$ is positive or negative, so that $f(\xi)$ is an extreme value. If $f(x)$ has the same sign in both intervals, then $hf'(\xi+\theta h)$ changes sign with $h$ so that $f(\xi + \theta h)$ is greater than $f(\xi)$ on one side and less than $f(\xi)$ on the other side, whence there is no extreme value. Our theorem has thus been proved.

At the same time, we see that the value $f(\xi)$ is a greatest or least value of the function in every interval, containing the point $\xi$, in which the only change of sign of $f'(x)$ occurs at $\xi$ itself.

The mean value theorem, on which this proof is based, can still be used even if $f(x)$ is not differentiable at an end-point of the interval to which it is applied, provided that $f(x)$ is differentiable at all the other points of the interval; for example, the above proof still holds if $f'(x)$ does not exist at $x = \xi$. This leads us to the following more general result: If the function $f(x)$ is continuous in an interval containing the point $\xi$ and everywhere in this interval, with the possible exception of $\xi$ itself, has a derivative $f'(x)$ which vanishes at not more than a finite number of points, then $f(x)$ has an extreme value at the point $x = \xi$ if, and only if, the point $\xi$ separates two intervals in which $f'(x)$ has different signs. For example, the function $y = |x|$ has a minimum at $x = 0$, since $y' > 0$ for $x > 0$ and $y' < 0$ for $x < 0$ (cf. Chapter II, Fig. 9). The function $y = \sqrt[3]{x^2}$ likewise has a minimum at the point $x=0$, even though its derivative $2x^{-1/2}/3$ is infinite there (Chapter II. Fig. 12).

In addition, we make the following remark on the general theory of maxima and minima: The finding of maxima and minima is not directly equivalent to the finding of the greatest and least values of a function in a closed

interval. In the case of a monotonic function, these greatest and least values will be assumed at the ends of the interval and are therefore not maxima and minima in our sense; in fact, this latter concept refers to a **complete neighbourhood** of the location in question. For example, the function $f(x) = x 0 \leq x \leq 1$ in the interval assumes its greatest value at the point $x = 1$, and its least value at $x = 0$, and a corresponding statement holds for every monotonic function. The function $y$=artan $x$ with the derivative $1/(1 + x^2)$ is monotonic for $-\infty \leq x \leq \infty$ and has in that open interval neither a maximum nor a minimum nor a greatest or a least value.

If, after finding the zeros of $f'(x)$, we wish to make sure that we have thereby found the points at which the function has its extreme values, we can often employ the criterion:

A point $\xi$, at which $f'(x)$ vanishes, yields the least or greatest value of the function $f(x)$ in an entire interval, if throughout that interval $f''(x) > 0$ or $f''(x) < 0$, respectively.

In fact, if both, $\xi$ and $\xi + h$, belong to the interval, by the mean value theorem

$$f'(\xi + h) = f'(\xi + h) - f'(\xi) = hf''(\xi + \theta h),$$

Hence, at the point $x = \xi + h$, the derivative $f'(x)$ has the same sign as $h$ or the opposite sign, according to whether $f''(x) > 0$ $f''(x) < 0$; the statement then follows from the remark following the theorem above.

### 3.5.3 Examples of Maxima and Minima:

1. Of all rectangles of given area, find that with the least perimeter.

Let $a^2$ be the area of the rectangle and $x$ the length of one side (we must consider here $x$ as ranging over the interval $0 < x < \infty$ ), then the length of the other side is $a^2/x$ and half the perimeter is given by

$$f(x) = x + \frac{a^2}{x}.$$

We have

$$f'(x) = 1 - \frac{a^2}{x^2}, \qquad f''(x) = \frac{2a^2}{x^3}.$$

The equation $f'(\xi) = 0$ has the single positive root $\xi = a$. For this value, $f''(x)$ is positive (as it is for any positive value of $x$), whence it yields the required least value and we obtain the very plausible result that of all rectangles of given area the square has the smallest perimeter.

2. Of all triangles with given base and area, find that with the least perimeter.

In order to solve this problem, we place the $x$-axis along the given base $AB$ and the central point of $AB$ as the origin. If $C$ is the vertex of the triangle, $h$ its altitude (which is fixed) and $(x,h)$ are the co-ordinates of the vertex, then the sum of the two sides $AC$ and $BC$ of the triangle, which are to be determined, will be given by

$$f(x) = \sqrt{\{(x+a)^2 + h^2\}} + \sqrt{\{(x-a)^2 + h^2\}},$$

where $2a$ is the length of the base, whence

$$f'(x) = \frac{x+a}{\sqrt{\{(x+a)^2 + h^2\}}} + \frac{x-a}{\sqrt{\{(x-a)^2 + h^2\}}},$$

$$\begin{aligned} f''(x) &= \frac{-(x+a)^2}{\sqrt{\{(x+a)^2 + h^2\}^3}} + \frac{1}{\sqrt{\{(x+a)^2 + h^2\}}} + \frac{-(x-a)^2}{\sqrt{\{(x-a)^2 + h^2\}^3}} \\ &\quad + \frac{1}{\sqrt{\{(x-a)^2 + h^2\}}} \\ &= \frac{h^2}{\sqrt{\{(x+a)^2 + h^2\}^3}} + \frac{h^2}{\sqrt{\{(x-a)^2 + h^2\}^3}}. \end{aligned}$$

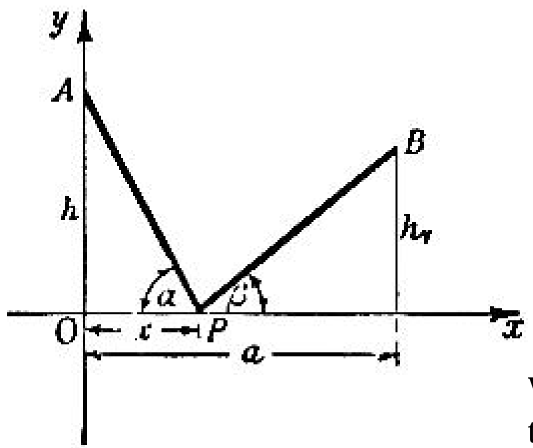



Fig. 12.—Law of reflection

We see at once (1) that $f'(0)$ vanishes, (2) that $f''(x)$ is always positive, whence at $x = 0$ there is a least value. In fact, since $f''(x) > 0$, the first derivative $f'(x)$ always increases and therefore cannot be equal to zero at any other point, so that the point $x = 0$ must really yield the least value of $f(x)$. Hence, this least value is given by the **isosceles triangle**.

Similarly, we find that of all triangles with given perimeter and given base the isosceles triangle has the greatest area.

3. Find a point on a given straight line such that the sum of its distances from two given fixed points is a minimum.

Let there be given a straight line and two fixed points *A* and *B* on the same side of the line. We wish to find a point *P* on the straight line such that the distance *PA* + *PB* has the least possible value.

We take the x-axis as the given line and use the notation of Fig. 12. Then the distance in question is given by

$$f(x) = \sqrt{(x^2 + h^2)} + \sqrt{\{(x-a)^2 + h_1^2\}},$$

and we obtain

$$f'(x) = \frac{x}{\sqrt{(x^2+h^2)}} + \frac{x-a}{\sqrt{\{(x-a)^2+h_1^2\}}},$$

$$\begin{aligned} f''(x) &= \frac{-x^2}{\sqrt{(x^2+h^2)^3}} + \frac{1}{\sqrt{(x^2+h^2)}} + \frac{-(x-a)^2}{\sqrt{\{(x-a)^2+h_1^2\}^3}} \\ &\quad + \frac{1}{\sqrt{\{(x-a)^2+h_1^2\}}} \\ &= \frac{h^2}{\sqrt{(x^2+h^2)^3}} + \frac{h_1^2}{\sqrt{\{(x-a)^2+h_1^2\}^3}}. \end{aligned}$$

Hence. the equation $f'(\xi) = 0$ yields

$$\frac{\xi}{\sqrt{(\xi^2+h^2)}} = \frac{a-\xi}{\sqrt{\{(\xi-a)^2+h_1^2\}}} \quad \text{or} \quad \cos\alpha = \cos\beta,$$

which means that the two lines *PA* and *PB* must form equal angles with the given line. The positive sign of $f''(x)$ shows us that we really have a least value.

The solution of this problem is closely linked to the optical law of reflection. By an important principle of optics known as **Fermat's principle of least time**, the path of a light ray is determined by the property that the time taken by the light to go from a point *A* to at point *B* under known conditions must be the least possible. If the condition is imposed that a ray of light shall on its way from *A* to *B* pass through some point on a given straight line (say, on a

mirror), we see that the shortest time will be taken along the ray for which the **angle of incidence is equal to the angle of reflection**.

## 4. The Law of refraction

Let there be given two points *A* and *B* on opposite sides of the *x*-axis. Which path from *A* to *B* corresponds to the shortest possible time, if the velocity on one side of the *x*-axis is $c_1$ and on the other side $c_2$?

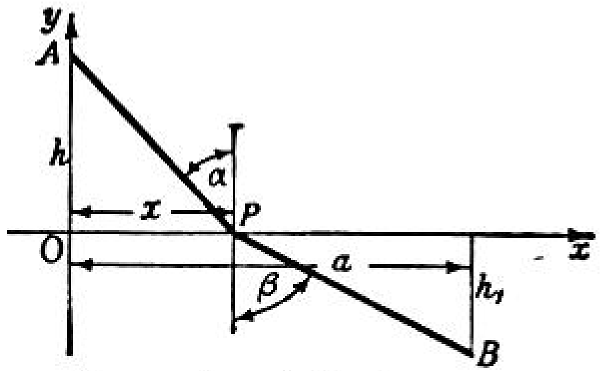



Fig. 13.—Law of refraction

It is clear that the shortest path must lie along two portions of straight lines meeting each other at a point *P* on the *x*-axis. Using the notation of Fig. 13, we obtain the two expressions

$$\sqrt{(h^2 + x^2)} \text{ and } \sqrt{\{h_1^2 + (a - x)^2\}}$$

for the lengths *PA*, *PB*, respectively, and we find the time of passage along this path by dividing the lengths of the two segments by the corresponding velocities and adding the results. This yields for the time taken

$$f(x) = \frac{1}{c_1}\sqrt{(h^2 + x^2)} + \frac{1}{c_2}\sqrt{\{h_1^2 + (a - x)^2\}}.$$

By differentiation, we obtain

$$f'(x) = \frac{1}{c_1}\frac{x}{\sqrt{(h^2+x^2)}} - \frac{1}{c_2}\frac{a-x}{\sqrt{\{h_1^2+(a-x)^2\}}},$$

$$f''(x) = \frac{1}{c_1}\frac{h^2}{\sqrt{(h^2+x^2)^3}} + \frac{1}{c_2}\frac{h_1^2}{\sqrt{\{h_1^2+(a-x)^2\}^3}}.$$

As we readily see from the figure, the equation $f'(x) = 0$, i.e.,

$$\frac{1}{c_1}\frac{x}{\sqrt{(h^2+x^2)}} = \frac{1}{c_2}\frac{a-x}{\sqrt{\{h_1^2+(a-x)^2\}}},$$

is equivalent to the condition

$$\frac{1}{c_1}\sin\alpha = \frac{1}{c_2}\sin\beta, \text{ or } \quad \frac{\sin\alpha}{\sin\beta} = \frac{c_1}{c_2}.$$

We leave it to the reader to prove that there is only one point which satisfies this condition and that this point actually yields the required least value. The physical meaning of our example is again given by the **optical principle of least time**. A ray of light travelling between two points describes the path of shortest time. If $c_1$ and $c_2$q are the velocities of light on either side of the boundary of two optical media, the path of the light will be that given by our result, which accordingly yields **Snell's law of refraction**.

**Exercises 3.4:**

1. Find the maxima, minima and points of inflection of the following functions. Graph them, and determine the regions of increase, decrease and of convexity, concavity:

$$(a)\ x^3 - 6x + 2. \quad (b)\ x^{2/3}(1-x). \quad (c)\ 2x/(1+x^2).$$
$$(d)\ x^3/(x^4+1). \quad (e)\ \sin^2 x.$$

2. Determine the maxima, minima, and points of inflection of $x^3 + 3px + q$. Discuss the nature of the roots of $x^3 + 3px + q = 0$.

3. Which point of the hyperbola $y^2 - x^2/2 = 1$ is nearest to the point $x=0, y=3$.

4. Let $P$ be a fixed point with coordinates $x_0,y_0$ in the first quadrant of a rectangular coordinate system. Find the equation of the line through $P$ such that the length intercepted between the axes is a minimum.

6. A 12 ft high statue stands on a 15 ft high pillar. At what distance must a 6ft high man stand in order that the statue may subtend the greatest possible angle at his eyes?

6. Two sources of light of intensities $a$ and $b$ are at a distance $d$ apart. At which point of the line joining them is the illumination least? (Assume that the illumination is proportional to the intensity and inversely proportional to the square of the distance.)

7. Of all rectangles with a given area, find ($a$) the one with the smallest perimeter; ($b$) the one with the shortest diagonal.

8. In the ellipse $x^2/a^2 + y^2/b^2 = 1$ inscribe the rectangle of greatest area.

9. Two sides of a triangle are $a$ and $b$. Determine the third side so that the area is a maximum.

10. A circle of radius $r$ is divided into two segments by a line $g$ at a distance $h$ from the centre. Inscribe in the smaller of these segments the rectangle of greatest possible area.

11. Find of all circular cylinders with a given volume the one with the least area.

12. Given the parabola $y^2 = 2px$, $p > 0$, and a point $P(x = \xi, y = \eta)$ inside it ($\eta^2<2p\xi$), find the shortest path (consisting of two line segments) leading from $P$ to a point $Q$ on the parabola and then to the focus $F(x = p/2, y = 0)$ of the parabola. Show that the angle $FQP$ is bisected by the normal to the parabola and that $QP$ is parallel to the axis of the parabola. (**Principle of the parabolic mirror**)

13.* A prism deflects a beam of light travelling in a plane perpendicular to the edge of the pane. What must the relative position of the prism and the beam be for the deflection to be a minimum?

14. Given $n$ fixed numbers $a_1, \cdots, a_n$, determine $x$ so that $\sum_{i=1}^{n} (a_i - x)^2$ is a minimum.

15. Prove that if $p > 1$ and $x > 0$, $x^p - 1 \geq p(x - 1)$.

16. Prove the inequality $1 \geq \sin x/x \geq 2/\pi$, $0 \leq x \leq \pi/2$.

17. Prove that (*a*) $\tan x \geq x$, $0 \leq x \leq \pi/2$, (*b*) $\cos x \geq 1 - x^2/2$.

18.* Given $a_1 > 0, a_2 > 0, \cdots, a_n > 0$, determine the minimum of

$$\frac{\dfrac{a_1 + \ldots + a_{n-1} + x}{n}}{\sqrt[n]{a_1 a_2 \ldots a_{n-1} x}}$$

for $x > 0$. Use the result to prove by mathematical induction that

$$\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + \ldots + a_n}{n}.$$

Answers and Hints

## 3.6 The Logarithm and the Exponential Function

The systematic relations between the differential and the integral calculus lead naturally to a convenient method of approach to the exponential function and the logarithm. Although we have already investigated these functions in 1.3.4 and A1.3, we now define them afresh and redevelop their theory without making any use of our previous definition and the results based on it. We shall begin with the logarithm and then obtain the exponential function as its inverse.

**3.6.1 Definition of the Logarithm. The Differentiation Formula:** We have seen that, in general, indefinite integration of the power $x^n$ for integral indices $n$ leads to a power of $x$. The only exception is the function $1/x$, which does not appear as the derivative of any of the functions which we have dealt with so far. It is natural to assume that the indefinite integral of the function $1/x$ represents a new sort of function; thus, pursuing this idea, we will investigate the function

$$y = \int_1^x \frac{d\xi}{\xi} = f(x)$$

for $x > 0$. We call it the logarithm of $x$ or, more accurately, the **natural logarithm of $x$**, and write it $y=\log_e x$ or $y = \ln x$. We have denoted the variable of integration by $\xi$, in order to avoid confusion with the upper limit $x$.

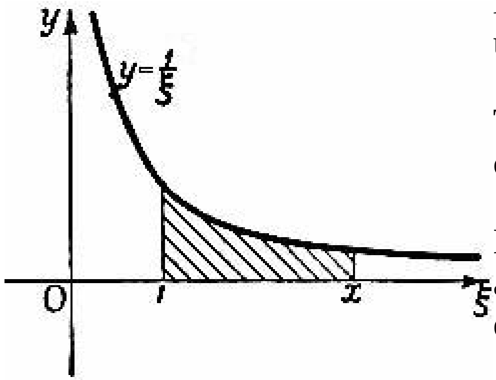


Fig. 14.—Log x illustrated by an area

The choice of the number 1 as lower limit is arbitrary; however, it will soon prove its convenience.

During the following argument, it will appear that the logarithm, thus defined, is the same as the logarithm which we defined previously in an elementary way. But, as we once more emphasize, the results of the following study are independent of those obtained earlier.

Geometrically, our logarithmic function means the area shown shaded Fig. 14; it is bounded above by the rectangular hyperbola $y=1/\xi$, below *by* the $\xi$-axis, and on the sides by the lines $\xi=1$ and $\xi=x$. This area is to be reckoned positive, if $x>1$, negative if $x<1$. For $x = 1$, the area vanishes, whence we have log 1 = 0.

According to the above definition, the derivative of the logarithm is given by the formula

$$\frac{d(\log x)}{dx} = \frac{1}{x}.$$

We emphasize specially that we assume throughout that the argument $x$ is positive; the logarithm of 0 or of any negative value cannot be formed in accordance with the formula above, because the integrand $1/\xi$ becomes infinite when $\xi = 0$. On the other hand, if we choose some negative number, say -1, as the lower limit, we can form the integral with a negative upper limit $x$, i.e. we can consider the expression

$$\int_{-1}^{x} \frac{d\xi}{\xi} \qquad (x < 0).$$

Owing to the significance of the integral as the limit of a sum or as an area, we see that for $x < 0$

$$\int_{-1}^{x} \frac{d\xi}{\xi} = \int_{1}^{-x} \frac{d\xi}{\xi} = \int_{1}^{|x|} \frac{d\xi}{\xi} = \log |x|.$$

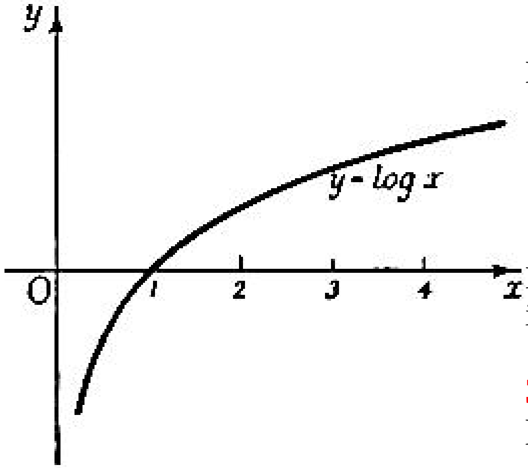


Fig. 15

Hence we can, in general, write the formula for indefinite integration as

$$\int \frac{dx}{x} = \log |x|.$$

Naturally, the logarithm can be represented by means of a graph. This graph, the logarithmic curve, is shown in Fig.15. We have already seen in 2.4.5 how to construct it.

**3.6.2 The Addition Theorem:** The logarithm defined as above obeys the fundamental law:

$$\log(ab) = \log a + \log b.$$

The proof of this addition theorem follows directly from the differentiation formula. In fact, writing $z = \log(ax)$ and applying the chain rule, we obtain

$$\frac{dz}{dx} = \frac{1}{ax} \cdot a = \frac{1}{x}.$$

However,

$$\frac{d}{dx} \log x = \frac{1}{x},$$

and, since the functions $z$ and $\log x$ have the same derivative, they differ only by a constant, so that $z = \log x + c$ or

$$\log ax = \log x + c.$$

This being true for all positive values of $x$, we first put $x = 1$ to find $c$; since
$\log 1 = 0$, this yields

$$\log a = c.$$

Substituting this value for $c$, we have

$$\log ax = \log x + \log a,$$

whence, for $x = b$,

$$\log ab = \log a + \log b,$$

which was to be proved.

For arbitrary positive numbers $a_1, a_2, \cdots, a_n$, the equation

$$\log(a_1 a_2 \ldots a_n) = \log a_1 + \log a_2 + \ldots + \log a_n$$

follows from the addition theorem for the logarithm.

In particular, if all the numbers $a_1, a_2, \cdots, a_n$ are equal to one and the same number $a$, we have

$$\log a^n = n \log a.$$

Similarly, it follows that

$$\log a + \log \frac{1}{a} = \log 1 = 0,$$

so that

$$\log a = -\log \frac{1}{a}.$$

Moreover, if we set $\sqrt[n]{a} = \alpha$, it follows that $\log a = n \log \alpha$, or

$$\log \sqrt[n]{a} = \log a^{1/n} = \frac{1}{n} \log a.$$

Hence, by repeated use of the addition theorem, we find that, when $m$ is a positive integer,

$$\frac{m}{n}\log a = \log \sqrt[n]{a^m} = \log a^{m/n}.$$

The equation

$$\log a^r = r \log a$$

is thus proved for all positive rational values of $r$, and it is obviously correct for $r = 0$. It is also valid for negative rational values of $r$, because then

$$\log a^r = \log \frac{1}{a^{-r}} = -\log a^{-r} = r \log a.$$

**3.6.3 Monotonic Character and Values of theLogarithm:** Obviously, the value of the logarithm increases and decreases with $x$, whence the logarithm is a monotonic function.

Since the derivative $l/x$ becomes smaller and smaller as $x$ increases, the function increases more and more slowly as $x$ increases. Nevertheless, as $x$ increases beyond all bounds, the function log $x$ does not tend to a positive limit, but becomes infinite, that is, for every positive number $A$, no matter how large, there are values of $x$ for which $\log x > A$. This fact follows very readily from the **addition theorem**. In fact, $\log 2^n = n \log 2$, and since log 2 is a positive number, we can, by taking $x = 2^n$ with sufficiently large values of $n$. *make* log $x$ as large as we please.

Since $\log (1/2^n) = - n \log 2$, we see that, as $x$ tends to zero through positive values, log $x$ is negative and increases numerically beyond all bounds.

In summary, we have: The function log $x$ is a monotonic function which assumes all values between $-\infty$ and $+\infty$ as the independent variable $x$ ranges over the continuum of positive numbers.

**3.6.4 The Inverse Function of the Logarithm (the Exponential Function):** Since the function $y=\log x$ $(x > 0)$ is a monotonic function of $x$ which assumes all real values, its inverse function, which we shall at first denote by $x=E(y)$, must be a single-valued monotonic function defined for every real value of $y$; it is differentiable, since log $x$ itself is differentiable. We interchange the notation for the dependent and independent variables and study the function $E(x)$ in detail. In the first place, obviously, it must be positive for every value of $x$. Moreover, we must have

$$E(0) = 1;$$

In fact, this equation is equivalent to the statement that log 1 = 0.

From the addition theorem for the logarithm follows immediately the multiplication theorem

$$E(\alpha)E(\beta) = E(\alpha + \beta).$$

In order to prove this, we merely need note that the equations

$$E(\alpha) = a, \quad E(\beta) = b, \quad E(\alpha + \beta) = c.$$

are equivalent to

$$E(\alpha) = a, \quad E(\beta) = b, \quad E(\alpha + \beta) = c.$$

Since, by the addition theorem for the logarithm, $\alpha + \beta = \log ab$, it must be true that $c = ab$, which proves the multiplication theorem.

We derive from this theorem a fundamental property of the function $y = E(x)$, which gives us the right to call our function the exponential function and to write it symbolically in the form

$$y = e^x.$$

In order to obtain this property, we observe that there must be a number - which we shall call $e$ - for which

$$\log e = 1.$$

Its identity with the number $e$ considered in in 1.6.5 will be proved in 3.6.6.

This is equivalent to the definition

$$E(1) = e,$$

Using the multiplication theorem for the function $E(x)$, we have

$$E(n) = e^n,$$

and, in the same way, for positive integers $m$ and $n$,

$$E\left(\frac{m}{n}\right) = e^{m/n},$$

which we could also have found directly from the addition theorem for the logarithm.

The equation $E(r) = e^r$ thus proved for positive rational numbers $r$ holds also for negative rational numbers by virtue of the equation

$$E(r)E(-r) = E(0) = 1.$$

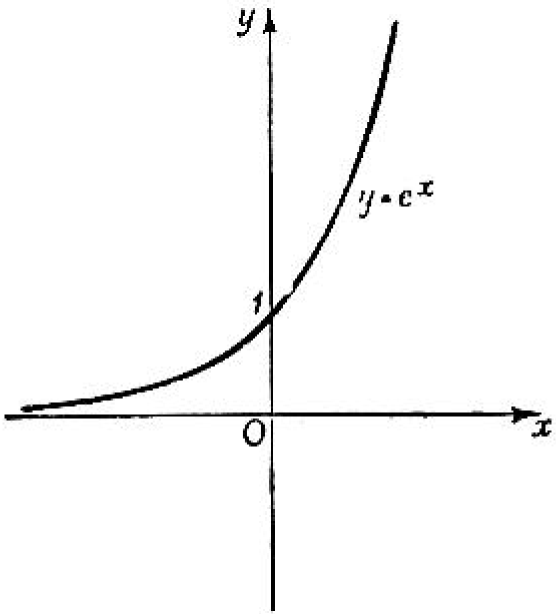



Fig. 16.—The exponential function

The function $E(x)$ is therefore a function which is continuous for all values of $x$ and for rational values of $x$ coincides with $e^x$. These facts give us the right to call our function $e^x$ also for arbitrary irrational values of $x$ . (It should be noted that here the continuity of $e^x$ is an immediate consequence of its definition as the inverse function of a continuous, monotonic function, while, if the elementary definition is adopted, the continuity must be proved.)

The exponential function is differentiated according to the formula

$$\frac{d}{dx} e^x = e^x \quad \text{or} \quad y' = y.$$

This formula expresses the important fact that the derivative of the exponential function is the function itself.

The proof is very simple. In fact, we have $x = \log y$, whence, by the formula for the differentiation of the logarithm, we have $dx/dy = 1/y$, and then, by the rule for inverse functions,

$$\frac{dy}{dx} = y = e^x,$$

as stated.

The graph of the exponential function $e^x$ - the so-called **exponential curve** - is obtained by reflection of the logarithmic curve in the line which bisects the first quadrant (Fig. 16).

### 3.6.5 The General Exponential Function $a^x$ and the General Power $x^\alpha$:

The exponential Function $a^x$ for an arbitrary positive base $a$ *is* now simply defined by the equation

$$y = a^x = e^{x \log a},$$

which agrees with the earlier definition by virtue of the relation

$$e^{\log a} = a.$$

By the chain rule, we immediately obtain

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \log a} = e^{x \log a} \cdot \log a,$$

$$= a^x \log a.$$

The inverse function of the exponential function y = $a^x$ is called the **logarithm to the base *a*** and is written

$$x = \log_a y,$$

while the logarithmic function previously introduced, when it is required to distinguish it, is spoken of as the **natural logarithm**, or the **logarithm to the base *e***.

It follows immediately from the definition that

$$\log y = x \log a = \log_a y \cdot \log a,$$

which shows that the logarithm of $y$ for an arbitrary positive base $a \neq 1$ is obtained by multiplying the natural logarithm of $y$ by the reciprocal of the natural logarithm of $a$, the **modulus of the system of logarithms to the base *a*.**

If we take $a = 10$, we obtain the ordinary **Briggian logarithm**s, which have already been encountered in elementary mathematics and which are advantageous for use in numerical computations.

Instead of our previous definition of the general power $x^{a}$' ($x > 0$), we shall now define this power by the equation

$$x^{\alpha} = e^{\alpha \log x}.$$

The rule for differentiating the power $x^{\alpha}$ follows immediately from the definition, using the chain rule; for

$$\frac{d}{dx} x^{\alpha} = e^{\alpha \log x} \cdot \frac{\alpha}{x} = \alpha x^{\alpha - 1},$$

in agreement with our previous result .

**3.6.6. The Exponential Function and the Logarithm represented as Limits:** We are now in a position to state important limiting relations referring to the quantities introduced above. We begin with the formula for differentiating the function $f(x) = \log x$:

$$\frac{1}{x} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\log(x+h) - \log x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \log\left(1 + \frac{h}{x}\right).$$

If we set $1/x = z$, this becomes

$$\lim_{h \to 0} \frac{1}{h} \log(1 + zh) = z.$$

Since the function $e^{x}$ is continuous for all values of $x$, this implies that

$$e^z = \lim_{h \to 0} e^{\{\log(1+zh)/h\}} = \lim_{h \to 0} (1 + zh)^{1/h}. \quad . \quad . \quad (a)$$

In particular, if we give $h$ the sequence of values 1, 1/2, 1/3, ··· , 1/$n$, ··· , we have

$$\lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n = e^z. \quad . \quad . \quad . \quad . \quad (b)$$

If we assign to $z$ the value 1, Equation (a) states the important fact:

As $h$ tends to zero, the expression $(1 + h)^{1/h}$ tends to the number $e$:

$$\lim_{h \to 0} (1 + h)^{1/h} = e.$$

Equation ($b$) yields

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e,$$

which proves that the number $e$ is the same as the number denoted earlier by the symbol $e$.

It follows from the differentiation formula for $a^x$

$$a^x \log a = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

that for $x = 0$

$$\log a = \lim_{h \to 0} \frac{a^h - 1}{h},$$

which expresses the logarithm of $a$ directly as a limit.

We append to this equation the remark that we can complete by it the earlier obtained relation

$$\int_a^b x^a\,dx = \frac{1}{a+1}\left(b^{a+1} - a^{a+1}\right).$$

We have always been obliged to exclude the case *a* = -l. However, we can now discover what happens when the number *a* tends to the **limit -1**. If we put *a*=1, the left-hand side will, by our definition of the logarithm, have the limit

$$\int_1^b \frac{dx}{x} = \log b;$$

whence the right-hand side has the same limit when $a \to -1$.

We have here carried out the passage to the limit $\to -1$ under the integral sign without further investigation; cf. the earlier discussion.

Moreover, this fact agrees with the formula

$$\log b = \lim_{h \to 0} \frac{b^h - 1}{h};$$

we only need write $a + 1 = h$.

We have thus cleared up the exceptional case $a = -1$ in the integration formula which we have so often used. The formula above is still meaninglesS when $a = -1$, but it retains as a limit formula its significance as $a \to -1$.

**3.6.7 Final Remarks:** We review here briefly the train of thought followed in this section, We have first defined the natural logarithm $y = \log x$ as for $x > 0$ by means of an integral, whence we immediately deduced the differentiation formula, the addition theorem, and the existence of an inverse. We then investigated the inverse function $y = e^x$, where the number *e* was seen to be the number the logarithm of which is 1, and derived its differentiation formula as well as limit expressions for it and for the logarithm. The introduction of the functions $y = x^\alpha = e^{\alpha \log x}$ and $y = a^x = e^{x \log a}$ followed naturally.

In the discussion given here, in contrast to the **elementary treatment**, the question of continuity causes no difficulty, since the logarithm is defined as an integral and therefore as a continuous and differentiable function, the inverse function of which is also continuous.

## Exercises 3.6:

1. Sketch the function $y = 1/x$ $(1 \le x \le 2)$ on a large scale, using graph paper, and find $\log_e 2$ by counting squares.

Differentiate the functions:

2. $x(\log x - 1)$.

3. $\log \log x$.

4. $\log\{x + \sqrt{(1 + x^2)}\}$.

5. $\log\{\sqrt{(1 + \log x)} - \sin x\}$.

6. Differentiate $\log \dfrac{\sqrt{(x^2 + 1)}}{\sqrt[3]{(2 + x)}}$ (*a*) by using the chain rule and the quotient rule, without preliminary simplification, (*b*) first simplifying by means of the theorems on logarithms.

7. (*a*) Differentiate $y = \dfrac{\sqrt[3]{(7x^2 + 1)}}{\sqrt[4]{(x - 2)}\sqrt{(x^4 + 1)}}$,
(*b*) Differentiate the same function, first taling logarithms and simplifying it.

8.* Given $\lim\limits_{n \to \infty} \varepsilon_n = 0$, prove that $\lim\limits_{n \to \infty} \left(1 + \varepsilon_n \cdot \dfrac{x}{n}\right)^n = 1.$

9. Show that the function $y = e^{-ax}(a \cos x + b \sin x)$ satisfies the equation

$$y'' + 2ay' + (a^2 + 1)y = 0$$

for all values of *a* and *b*.

10.* Show that $\dfrac{d^n}{dx^n}(e^{-1/x^2}) = \dfrac{P_n(x)}{x^{3n}} e^{-1/x^2}$, when $x \neq 0$, where $P_n(x)$ is a polynomial of degree $2n$ -2. Establish the **recurrence relation**

$$P_{n+1}(x) = (2 - 3nx^2)\, P_n(x) + x^3 P_n'(x).$$

11. Find the maximum of $y = x\lambda^{\alpha}e^{-\lambda x}$, where $\lambda$ and $\alpha$ are constants. Find the locus of this maximum when $\lambda$ is allowed to vary.

12. Differentiate $a^{(a^x)}$ $(a > 0)$.

13. Differentiate $a^{\sin x(\log x)\cdot\sin x(\log x)}$.

Answer and Hints

## 3.7 Some Applications of the Exponential Function

We shall consider now some different problems involving the exponential function and thus gain insight into the fundamental importance of this function for all kinds of applications.

**3.7.1 Definition of the Exponential Function by Means of a Differential Equation:** We can define the exponential function by a simple theorem the use of which will save us many detailed investigations of particular cases:

If a function y = $f(x)$ satisfies an equation of the form

$$y' = ay,$$

where $a$ is a constant other than zero, then $y$ has the form

$$y = f(x) = ce^{ax},$$

where $c$ if also a constant; conversely, every function of the form $ce^{\alpha x}$ satisfies the equation $y' = ay$.

The last equation is usually briefly referred to as a **differential equation**, since it expresses a relation between the function and its derivative. In order to make the theorem clear, we note first of all that in the simplest case $a$ =1 the above equation becomes $y' = y$. We know that $y = e^x$ satisfies this equation and it is clear that the same is also true of $y = ce^x$, where $c$ is an arbitrary constant. Conversely, we can easily see that no other function satisfies the differential equation. For if $y$ is such a function, we consider the function $u = ye^{-x}$. We must then have

$$u' = y'e^{-x} - ye^{-x} = e^{-x}(y' - y).$$

But the right-hand side vanishes, since we have assumed that $y' = y$, whence $u' = 0$, so that, by 2.4.3, $u$ is a constant $c$ and $y = ce^x$, as we wished to prove.

The case of any non-zero value of $\alpha$ can be treated in exactly the same manner as the special case $\alpha = 1$. If we introduce the function $u = ye^{-ax}$, we obtain the equation $u' = y'e^{-ax} — ae^{-ax}$. Hence we find from the assumed differential equation that $u' = 0$, so that $u = c$ and $y = ce^{ax}$. The converse is clear.

We will now apply this theorem to a number of examples and thereby make it more intelligible.

**3.7.2 Interest, Compounded Continuously. Radio-active Disintegration:** A capital sum, or **principal**, which has its interest added to it at regular periods of time, increases by jumps at these interest periods in the following manner. If $100\alpha$ is the rate of interest per cent and moreover the interest accrued is added to the principal at the end of each year, then after $x$ years the accumulated amount of an original principal of 1 will be

$$(1 + \alpha)^x.$$

However, if the principal had the interest added to it not at the end of each year, but at the end of each $n$-th part of a year, then after $x$ years the principal would amount to

$$\left(1 + \frac{\alpha}{n}\right)^{nx}.$$

Taking $x = 1$, for the sake of simplicity, i.e, reckoning the interest at $100\alpha$ per cent for one year, we find that, if the interest is computed in this latter way, the principal 1 amounts after one year to

$$\left(1 + \frac{\alpha}{n}\right)^{n}.$$

If we now let $n$ increase beyond all bounds, i.e., if we let the interest be calculated at shorter and shorter intervals, the limiting case will signify in a sense that the interest is compounded continuously, at each instant; and we see that the total amount after one year will be $e^{\alpha}$ times the original principal. Similarly, if the interest is calculated in this manner, an original principal of 1 will have grown after $x$ years to an amount $e^{\alpha x}$, where $x$ may be any number, integral or otherwise.

The discussion in 3.7.1 forms a framework within which examples of this type are readily understood. We consider a quantity $y$, which increases (or decreases) with time. Let the rate at which this quantity changes be proportional to the total quantity. Then, if we take time as the independent variable $x$, we obtain a law of the form $y' = \alpha y$ for the rate of increase, where $\alpha$, the factor of proportionality, is positive or negative, according to as the quantity is increasing or decreasing, Then, in accordance with 3.7.1, the quantity $y$ itself will be given by a formula

$$y = ce^{\alpha x},$$

where the meaning of the constant $c$ is immediately obvious, if we consider the instant $x = 0$. At that instant, $e^{\alpha x} = 1$ and we find that $c = y_0$ is the. quantity at the beginning of the time under consideration, so that we may write

$$y = y_0 e^{\alpha x}.$$

A characteristic example of the use of these ideas is the case of radioactive disintegration. The rate at which the total quantity $y$ of a radioactive substance is diminishing at any instant is proportional to the total quantity present at that instant; this is *a priori* plausible, as each portion of the substance decreases as rapidly as every other portion. Hence the quantity $y$ of the substance, expressed as a function of the time, satisfies a relation of the form $y' = -ky$, where $k$ is to be taken as positive, since we are dealing with a diminishing quantity. The quantity of substance is thus expressed as a function of the time by $y = 3.8y_0e^{-kx}$ , where $y_0$ is the amount of the substance at the beginning of the time (time $x = 0$).

After a certain time $\tau$, the radioactive substance will be diminished to half its original quantity. This so-called **half-period** is given by the equation

$$\tfrac{1}{2}y_0 = y_0 e^{-k\tau},$$

whence we immediately obtain $\tau = \dfrac{\log 2}{k}.$

**3.7.3 Cooling or Heating of a Body by a Surrounding Medium:** Another typical example of the occurrence of the exponential function is offered by the cooling of a body, e.g., a metal plate, which is immersed in a very large bath of a given temperature. In considering this cooling, we assume that the surrounding bath is so large that its temperature is unaffected by the cooling process. Moreover, we assume that at each instant all parts of the immersed body are at the same temperature and that the rate at which the temperature changes is proportional to the difference between the temperature of the body and that of the surrounding medium (**Newton's law of cooling**).

If we denote the time by $x$ and the temperature difference by $y = y(x)$, this law of cooling is expressed by the equation

$$y' = -ky,$$

where $k$ is a positive constant the value of which depends on the body itself. From this instantaneous relationship, which expresses the effect of the cooling process at a given instant, we now wish to derive an integral law which will allow us to find the temperature at an arbitrary time $x$ from the temperature at an initial time $x = 0$. The theorem of 3.7.1 immediately gives us this integral law in the form

$$y = ce^{-kx},$$

where $k$ is the above-mentioned constant depending on the body. This shows that the temperature decreases exponentially and tends to become equal to the external temperature. The rapidity with which this happens is expressed by the number $k$. As before, we find the meaning of the constant $c$ by considering the instant $x = 0$; this yields $y_0 = c$, so that our law of cooling can finally be written is the form

$$y = y_0e^{-kx}.$$

Obviously, the same discussion will also apply to the heating of a body. The only difference is that the initial difference of the temperature $y_0$ is in this case negative instead of positive.

**3.7.4 Variation of the Atmospheric Pressure with the Height above the Surface of the Earth:** As another example of the occurrence of the exponential formula we shall derive the law according to which the atmospheric pressure varies with the height. We use here: (1) the physical fact that the atmospheric pressure is equal to the weight of the column of air vertically above a surface of area 1 and (2) **Boyle's law**, according to which the pressure of the air ($p$) at a given constant temperature is proportional to the density of the air ($\sigma$). Boyle's law, expressed in symbols, is $p = \alpha\sigma$, where $\alpha$ is a constant which depends on a specific physical property of the air and, in addition, it is proportional to the absolute temperature—here, we are not concerned with this, as we shall assume that the temperature is constant. Our problem is to determine $p = f(h)$ as a function of the height ($h$) above the surface of the Earth.

If we denote by $p_0$ the atmospheric pressure at the surface of the Earth, i.e., the total weight of the air column supported by unit area, and by $\sigma(\lambda)$ the density of the air at the height $\lambda$ above the Earth, the weight of the column up to the height $h$ will he given by the integral $\int_0^h \sigma(\lambda)\,d\lambda.$. Hence the pressure at height $h$ will be

$$p = f(h) = p_0 - \int_0^h \sigma(\lambda)\,d\lambda.$$

By differentiation, this yields the relationship between the pressure $p = f(h)$ and the density $\sigma(h)$:

$$\sigma(h) = -f'(h) = -p'.$$

We now use Boyle's law to eliminate the quantity $\sigma$ from this equation and obtain an equation

$$p' = -\frac{1}{a} p$$

which only involves the unknown pressure function. It follows from 3.7.1 that

$$p = f(h) = p_0 e^{-h/a}.$$

If we denote, as above, the pressure at the Earth's surface, i.e., $f(0)$, by $p_0$, it follows immediately that $c = p_0$, and consequently

$$h = a \log \frac{p_0}{p}.$$

These two formulae find frequent application. For example, if the constant $a$ is known, they enable us to find the height of a place from the barometric pressure or to find the difference of the heights of two locations by measuring the local atmospheric pressures. Again, if the atmospheric pressure and the height $h$ are known, we can determine the constant $a$, which is of great importance in **gas theory**.

**3.7.5 Progress of a Chemical Solution:** We will now consider an example from chemistry, namely the so-called **molecular reaction**. Let a substance be dissolved in a relatively large amount of solvent, say a quantity of cane sugar in water. If a chemical reaction takes place, the **chemical law of mass action** in this simple case states that the rate of reaction is proportional to the quantity of reacting substance present. If we assume that the cane-sugar is being transformed by **catalytic action** into invert sugar and if we denote by $u(x)$ the quantity of cane sugar which at time $x$ is still unchanged, the velocity of reaction will be $-du/dx$, and, in accordance with the law of mass action, there holds an equation of the form

$$\frac{du}{dx} = -ku$$

where $k$ is a constant which depends on the reacting substance. From this instantaneous law, we immediately obtain, as in 3.7.1, an integral law, which gives us the amount of cane sugar as a function of the time

$$u(x) = ae^{-kx}.$$

This formula show us clearly how the chemical reaction tends asymptotically to its final state $u = 0$, that is, the complete transformation of the reacting substance. The constant $a$ is obviously the quantity present at $x = 0$.

**3.7.6 Making and Breaking of an Electric Circuit:** As a final example, consider the growth of a (direct) electric current when a circuit is completed (or its decay, when it is broken). If $R$ is the **resistance of the circuit** and $I$ the **impressed electro-motive force** (Voltage), the current $I$ will gradually increase from its original value 0 to the steady final value $E/R$. Hence we must consider $I$ as a function of the time. The growth of the current depends on the self-induction of the circuit ; the circuit has a characteristic constant $L$ - the **coefficient of self-induction** - of such a nature that as the current increases, an electro-motive force of magnitude $L dI/dx$, opposed to the external electromotive force $E$, is developed. We obtain from **Ohm's law**, according to which the product of the resistance and the current is at each instant equal to the actual **effective voltage**, the relation

$$IR = E - L\frac{dI}{dx}.$$

Hence, we put

$$f(x) = I(x) - \frac{E}{R};$$

we find immediately that $f'(x) = -R/L \cdot f(x)$, so that, by the theorem in 3.7.1, $f(x)=f(0)e^{-Rx/L}$. Recalling that $I(0) = 0$, we see that $f(0) = -E/R$ and thus obtain the expression for the current as a function of the time

$$I = f(x) + \frac{E}{R} = \frac{E}{R}(1 - e^{-Rx/L}).$$

We see from this expression that as the circuit is closed the current tends asymptotically to its steady value $E/R$.

**Exercises 3.7:**

1. The function $f(x)$ satisfies the equation

$$f(x+y) = f(x)f(y).$$

($a$) If $f(x)$ is differentiable, either $f(x) \equiv 0$ or else $f(x) = e^{ax}$,
($b$)* If $f(x)$ is continuous, either $f(x) \equiv 0$ or else $f(x) = e^{ax}$.

2. If a differentiable function $f(x)$ satisfies the equation

$$f(xy) = f(x) + f(y),$$

then $f(x) = \alpha \log x$.

3. A quantity of radium weighs 1 g at time $t = 0$, At time $t = 10$ (years), it has diminished to 0.997g. After what time will it have diminished to 0.5 g?

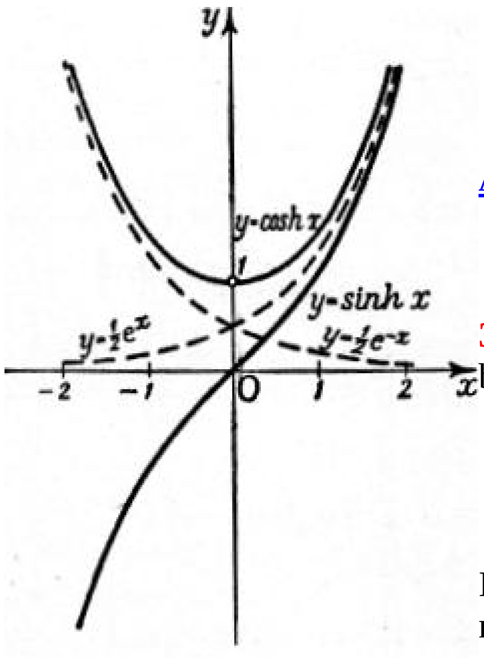



Fig. 17

4. Solve the differential equations:

$$(a)\ y' = \alpha(y - \beta). \qquad (c)\ y' - \alpha y = \beta e^{\alpha x}.$$
$$(b)\ y' - \alpha y = \beta. \qquad (d)\ y' - \alpha y = \beta e^{\gamma x}.$$

Answers and Hints

## 3.8 The Hyperbolic Functions

**3.8.1 Analytical Definition:** In many applications, the exponential function does not enter by itself, but in combinations of the form

$$\frac{1}{2}(e^x + e^{-x}) \quad \text{or} \quad \frac{1}{2}(e^x - e^{-x}).$$

It is convenient to introduce these and similar combinations as special functions with the notation:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

and call them the hyperbolic sine, hyperbolic cosine, hyperbolic tangent, and hyperbolic cotangent, respectively. The functions sinh $x$, cosh $x$ and tanh $x$ are defined for all values of $x$, while in the case of coth $x$ the point $x = 0$ must be excluded. This notation is designed to express a certain analogy with the trigonometric functions; it is this analogy, which we are about to study in detail. as it justifies special consideration of these new functions. Figs. 17,

18 and 19 show the graphs of the hyperbolic functions; the dotted lines in Fig.17 are the graphs of $y = e^x/2$ and $y = e^{-x}/2$, from which the graphs of sinh $x$ and cosh $x$ may readily be constructed.We see that cosh $x$ is an even function, i.e., a function which remains unchanged when $x$ is replaced by $-x$, while sinh $x$ is an odd function, i.e., a function which changes sign when $x$ is replaced by -ii.

The function

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

is (by its definition) positive for all values of $x$. It has its minimum when $x = 0$, while cosh 0 = 1.

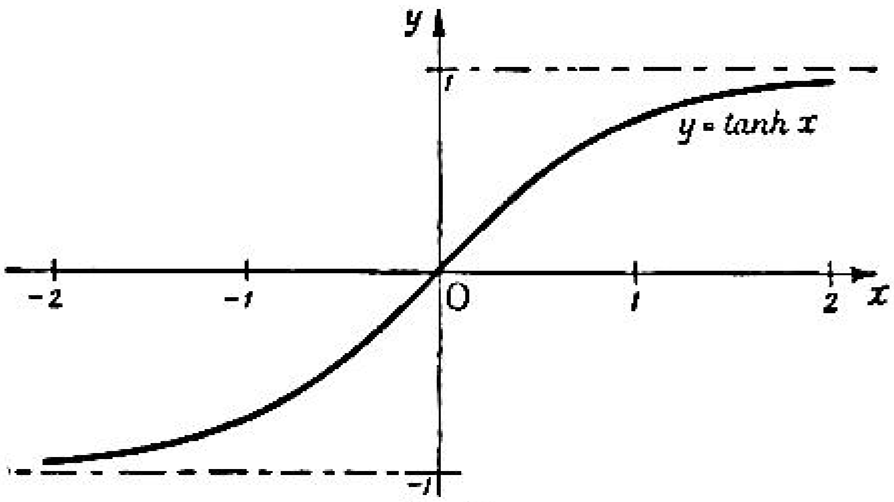


Fig. 18

There exists between cosh $x$ and sinh $x$ the fundamental relation

$$\cosh^2 x - \sinh^2 x = 1,$$

which follows immediately from the definitions of these functions.If we now denote the independent variable by $t$ instead of $x$ and write

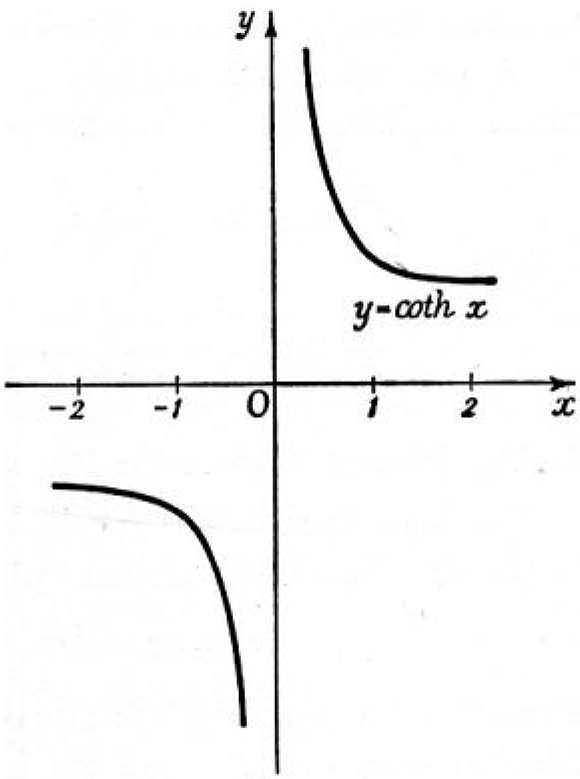


Fig. 19

$$x = \cosh t, \quad y = \sinh t,$$

we have

$$x^2 - y^2 = 1;$$

thus, the point with the coordinate $x$=cosh $t$, $y$ = sinh $t$ moves along the rectangular hyperbola $x^2 - y^2 =1$ as $t$ runs through the whole scale of values from $-\infty$ to $+\infty$. According to the defining equation, $x \geq 1$ and we may easily convince ourselves that y runs through

the whole scale of values -∞ to +∞ with $t$; in fact, if $t$ tends to infinity so does $e^t$, while $e^{-r}$ tends to zero. We may therefore state more exactly that as $t$ runs from -∞ to +∞, the equations $x = \cosh t$, $y = \sinh t$ give us one branch, namely, the right-hand one, of the rectangular hyperbola.

**3.8.2 Addition Theorems and Formulae for Differentiation:** From the definitions of our functions follow the formulae, known as addition theorems:

$$\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b,$$
$$\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b.$$

The proofs are obtained at once by writing

$$\cosh(a+b) = \frac{e^a e^b + e^{-a} e^{-b}}{2}, \quad \sinh(a+b) = \frac{e^a e^b - e^{-a} e^{-b}}{2},$$

and setting in these equations

$$e^a = \cosh a + \sinh a, \quad e^{-a} = \cosh a - \sinh a,$$
$$e^b = \cosh b + \sinh b, \quad e^{-b} = \cosh b - \sinh b.$$

The analogy between these formulae and the corresponding trigonometrical formulae is clear. The only difference in the addition theorems is one sign in the first formula.

A corresponding analogy holds for the differentiation formulae. Remembering that $d(e^x)/dx = e^x$, we readily find

$$\frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\sinh x = \cosh x,$$

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x}, \quad \frac{d}{dx}\coth x = -\frac{1}{\sinh^2 x}.$$

At times, it is convenient to introduce the functions sech $x = 1/\cosh x$, cosech $x = 1/\sinh x$.

**3.8.3 The Inverse Hyperbolic Functions:** To the hyperbolic functions $x = \cosh t$, $y = \sinh t$ correspond the inverse functions

$$t = \operatorname{arcosh} x, \quad t = \operatorname{arsinh} y.$$

The notation $\cosh^{-1} x$, etc. is also used.

Since the function $\sinh t$ is monotonic increasing throughout the interval $-\infty < t < \infty$, its inverse function is uniquely determined for all values of $y$; on the other hand, we learn from a glance at the graph (Fig. 17) that $t = \operatorname{arcosh} x$ is not uniquely determined, but has an ambiguity of sign, because there corresponds to a given value of $x$ not only the number $t$, but also the number $-t$. Since $\cosh t \geq 1$ for all values of $t$, its inverse $\operatorname{arcosh} x$ is defined only for $x \geq 1$.

We can express these inverse functions very easily in terms of the logarithm by regarding the quantity $e^t = u$ in the definitions

$$x = \frac{e^t + e^{-t}}{2}, \quad y = \frac{e^t - e^{-t}}{2}$$

as unknowns and solving these (quadratic) equations for $u$. Then

$$u = x \pm \sqrt{(x^2 - 1)}, \quad u = y + \sqrt{(y^2 + 1)};$$

since $u = e^t$ can have only positive values, the square root in the second equation must be taken with the positive sign, while in the first equation either sign is possible. In the logarithmic form

$$t = \log(x \pm \sqrt{(x^2 - 1)}) = \operatorname{arcosh} x,$$
$$t = \log(y + \sqrt{(y^2 + 1)}) = \operatorname{arsinh} y.$$

In the case of $\operatorname{arcosh} x$, the variable $x$ is restricted to the interval $x \geq 1$, while $\operatorname{arsinh} y$ is defined for all values of $y$.

The formula gives us two values, $\log\{x + \sqrt{(x^2 - 1)}\}$ and $\log\{x - \sqrt{(x^2 - 1)}\}$, for $\operatorname{arcosh} x$, corresponding to the two branches of $\operatorname{arcosh} x$. Since

$$\{x+\sqrt{(x^2-1)}\}\,\{x-\sqrt{(x^2-1)}\}=1,$$

the sum of these two values of arcosh *x* is zero, which agrees with an earlier remark.

The inverses of the hyperbolic tangent and hyperbolic cotangent can be defined analogously and can also be expressed in terms of logarithms. We denote these functions by artanh *x* and arcotanh *x*; expressing the independent variable everywhere by *x*, we readily obtain

$$\operatorname{artanh} x=\frac{1}{2}\log\frac{1+x}{1-x}\text{ in the interval }-1<x<1,$$

$$\operatorname{arcoth} x=\frac{1}{2}\log\frac{x+1}{x-1}\text{ in the intervals }x<-1,\ x>1.$$

The differentiation of these inverse functions should be carried out by the reader himself; he may use either the rule for differentiating an inverse function or the chain rule in conjunction with the above expressions for the inverse functions in terms of logarithms. If *x* is the independent variable, the results are

$$\frac{d}{dx}\operatorname{arcosh} x=\pm\frac{1}{\sqrt{(x^2-1)}},\quad \frac{d}{dx}\operatorname{arsinh} x=\frac{1}{\sqrt{(x^2+1)}},$$

$$\frac{d}{dx}\operatorname{artanh} x=\frac{1}{1-x^2},\quad \frac{d}{dx}\operatorname{arcoth} x=\frac{1}{1-x^2}.$$

The last two formulae do not contradict each other, since the first holds only for $-1 < x < 1$ and the second only for $x < -1$ and $1 < x$. The two values of
*d*ar cosh *x*/*dx*, expressed by the sign ± in the first formula, correspond to the two different branches of the curve

$$y=\operatorname{arcosh} x=\log\{x\pm\sqrt{(x^2-1)}\}.$$

**3.8.4 Further Analogies:** In the above representation of the rectangular hyperbola by the quantity *t*, we did not attempt to bring out any geometrical meaning of the parameter *t* itself. We shall now return to this matter and thus gain still more insight into the analogy between the trigonometric and hyperbolic functions. If we represent the circle by the equation $x^2 + y^2 = 1$ by means of a parameter *t* in the form $x = \cos t$, $y = \sin t$, we can interpret the quantity *t* as an angle or a length of arc measured along the

circumference; however, we may also regard *f* as twice the area of the circular sector, corresponding to that angle, the area being reckoned positive or negative a«cording *to as* the angle is positive or negative.

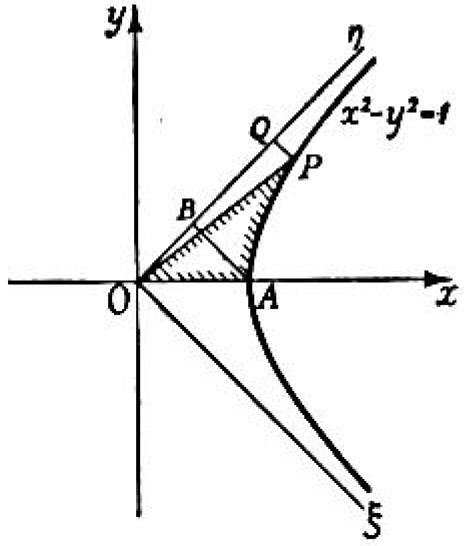



Fig. 20.—Parametric representation of the hyperbola

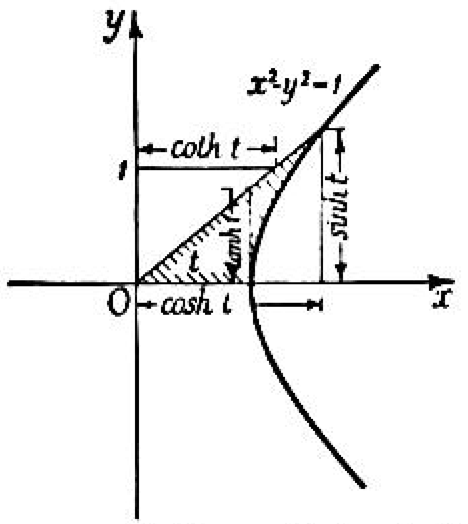



Fig. 21.—To illustrate the hyperbolic functions

We now make the analogous statement that for the hyperbolic functions the quantity *t* is twice the area of the hyperbolic sector, shown shaded in Fig. 20.

Just as the notation $t$ = arcos $x$ recalls that $t$ is an arc of a circle of reference, so $t$ = ar cosh $x$ recalls that $t$ is a certain area connected with a rectangular hyperbola.

The proof is obtained without difficulty, if we refer the hyperbola to its asymptotes as axes by means of the coordinate transformation

$$x - y = \sqrt{2}\,\xi, \quad x + y = \sqrt{2}\,\eta,$$

or

$$x = \frac{1}{\sqrt{2}}(\xi + \eta), \quad y = \frac{1}{\sqrt{2}}(\eta - \xi);$$

with these new coordinates, the equation of the hyperbola becomes $\xi\eta = ½$. We thus see immediately that the area in question is equal to the area of the figure *ABQP;* in fact, the two right-angled triangles *OPQ* and *OAB* have the same area, according to the equation of the hyperbola. ' The two points *A* and *P* obviously have the coordinates

$$\xi = \frac{1}{\sqrt{2}}, \quad \eta = \frac{1}{\sqrt{2}} \quad \text{and} \quad \xi = \frac{x-y}{\sqrt{2}}, \quad \eta = \frac{x+y}{\sqrt{2}},$$

respectively, and we thus obtain for double the area of our figure

$$2\int_{1/\sqrt{2}}^{(x+y)/\sqrt{2}} (1/2\eta)\, d\eta = \log(x+y) = \log\{x \pm \sqrt{(x^2-1)}\}.$$

A comparison of this expression with the formula in 3.8.3 for the inverse function *t* = ar cosh *x* shows us that our statement about the quantity *t* is true.

In conclusion, it may be pointed out that, as shown in Fig. 21 above, the hyperbolic functions can be diagrammatically represented on the circle.

Numerical values of the hyperbolic functions, which are useful in a variety of calculations, can be found in many tables,some of which are: J. B. Dale, Five figure Tables of Mathematical Functions (Ainold, 1918}, K. Hayashi, Fünfstellige Tafeln der Kreis- und Hyperbel-funktionen (Berlin, 1930); E. Jahnlie and P. Emde, Funktionentafeln mit Formeln und Kurven (German and English, Leipzig 1933.

## Exercises:

1. Prove the formula

$$\sinh a + \sinh b = 2\sinh\left(\frac{a+b}{2}\right)\cosh\left(\frac{a-b}{2}\right).$$

Obtain similar formulae for sinh *a* - sinh *b*, cosh *a* + cosh *b*, cosh *a* — cosh *b*.

2. Express tanh(*a* ± *b*) in terms of tanh *a* and tanh *b*, coth(*a* ± *b)* in terms of coth *a* and coth *b*, sinh *a*/2 and cosh *a*/2 in terms of cosh *a*.

3. Differentiate

$$(a)\ \cosh x + \sinh x; \quad (b)\ e^{\tanh x + \coth x}; \quad (c)\ \log \sinh (x + \cosh^2 x);$$

$$(d)\ \operatorname{ar}\cosh x + \operatorname{ar}\sinh x; \quad (e)\ \operatorname{ar}\sinh(\alpha \cosh x); \quad (f)\ \operatorname{ar}\tanh \frac{2x}{1+x^2}.$$

4. Calculate the area bounded by the catenary $y = \cosh x$, the ordinates $x = a$ and $x = b$, and the $x$.axis.

Answers and Hints

## 3.9 The Order of Magnitude of Functions

The various functions which we have met in this chapter exhibit very important differences as regards their behaviour for large values of their arguments or, as we also say, of the order of magnitude of their increase. In view of the great importance of this, we shall discuss here briefly this matter, even though it is not directly linked to the idea of the integral or of the derivative.

**3.9.1 The Concept of Order of Magnitude. The simplest Cases:** If the variable $x$ increases beyond all bounds, then, for $a > 0$, the functions $a$, $\log x$, $e^x$, $e^{ax}$ will also increase beyond all bounds. However, as regards the manner of this increase, we can immediately point out an essential difference between them. For example, the function $x^3$ will become infinite to a higher order than $x^2$; this means that, as $x$ increases, the quotient $x^3/x^2$ itself increases beyond all bounds. Similarly, we shall say that the function $x^\alpha$ becomes infinite to a higher order than $x^\beta$, if $\alpha > \beta > 0$, etc.

Quite generally, we shall say of two functions $f(x)$ and $g(x)$, the absolute values of which increase with $x$ beyond all bounds, that $f(x)$ becomes infinite of a higher order than $g(x)$, if, as $x$ increases, the quotient $|f(x)/g(x)|$ increases beyond all bounds; we shall say that **$f(x)$ becomes infinite of a lower order than $g(x)$**, if the quotient $|f(x)/g(x)|$ tends to zero as $x$ increases; and we shall say that the two functions become infinite of the same order of magnitude if as $x$ increases the quotient $|f(x)/g(x)|$ has a limit other than 0 or at least remains between two fixed positive bounds. For example, the function $ax^3+bx^2+c=f(x)$, where $a \neq 0$, will be of the same order of magnitude as the function $x^3=g(x)$; in fact, the quotient $\left|\frac{f(x)}{g(x)}\right| = \left|\frac{ax^3+bx^2+c}{x^3}\right|$ has the limit $|a|$. On the other hand, the function $x^3 + x + 1$ becomes infinite of a higher order of magnitude than the function $x^2 + x + 1$.

A sum of two functions $f(x)$ and $\phi(x)$, where $f(x)$ is of higher order of magnitude than $\phi(x)$, has the same order of magnitude as $f(x)$. In fact,

$$\left|\frac{f(x)+\phi(x)}{f(x)}\right| = \left|1+\frac{\phi(x)}{f(x)}\right|,$$

and, by hypothesis, this expression tends to 1as $x$ increases.

We might be tempted to measure the order of magnitude of functions by a scale, assigning to the quantity $x$ the order of magnitude 1 and to the power $x^{\alpha}$ ($\alpha > 0$) the order of magnitude $\alpha$. Obviously, then a polynomial of the $n$-th degree has the order of magnitude $n$; a rational function, the degree of the numerator of which is greater than that of the denominator by $h$, has then the order of magnitude $h$.

**3.9.2 The Order of Magnitude of the Exponential Function and the Logarithm:** However, it turns out that any attempt to fix the order of magnitude of an arbitrary function by the above scale must end in failure. In fact, there are functions which become infinite of higher order than the power $x^{\alpha}$ of $x$, no matter however large is the chosen $\alpha$; again, there are functions which become infinite of lower order than the power $x^{\alpha}$, no matter how small the positive number $\alpha$ is chosen. Hence, these functions will not fit in anywhere into our scale. Without entering into a detailed theory of order of magnitude we shall prove the theorem: If $a$ is an arbitrary number greater than 1, then the quotient $a^{x}$ tends to infinity as $x$ increases. In order to prove this statement, we construct the function

$$\phi(x) = \log\frac{a^{x}}{x} = x\log a - \log x;$$

obviously, it is sufficient to show that this increases beyond all bounds if $x$ tends to $+\infty$. For this purpose, we consider the derivative

$$\phi'(x) = \log a - \frac{1}{x}$$

and note that for $x \geq 0$ this is not less than the positive number 1/2 log $a$, whence, for $x \geq c$

$$\phi(x) - \phi(c) = \int_c^x \phi'(t)\,dt \geqq \int_c^x \tfrac{1}{2}\log a\,dt \geqq \tfrac{1}{2}(x-c)\log a,$$

$$\phi(x) \geqq \phi(c) + \tfrac{1}{2}(x-c)\log a,$$

and the right-hand side becomes infinite as $x$ increases.

We shall give a second proof of this important theorem. If we write $\sqrt{a} = b = 1 + h,$ we have $b > 1$ and $h > 0$. Let $n$ be an integer such that $n \le \xi < n+1$; we may take $x > 1$, so that $n \ge 1$. Applying the Lemma of I.5.3, we have

$$\sqrt{\left(\frac{a^x}{x}\right)} = \frac{b^x}{\sqrt{x}} = \frac{(1+h)^x}{\sqrt{x}} > \frac{(1+h)^n}{\sqrt{(n+1)}} > \frac{1+nh}{\sqrt{(n+1)}} > \frac{nh}{\sqrt{2n}} = \frac{h}{\sqrt{2}}\sqrt{n};$$

so that

$$\frac{a^x}{x} > \frac{h^2}{2} \cdot n,$$

whence it tends to infinity with $x$.

There follow many results from the fact just proved. For example, for every positive index $\alpha$ and every number $a > 1$, the quotient $a^x/x^\alpha$ tends to infinity as $x$ increases, that is: The exponential function becomes infinite of a higher order of magnitude than any power of $x$.

In order to see this, we need only show that the $\alpha$-th root of the expression, that is,

$$\frac{a^{x/\alpha}}{x} = \frac{1}{\alpha} \cdot \frac{a^{x/\alpha}}{x/\alpha} = \frac{1}{\alpha} \cdot \frac{a^y}{y} \qquad \left(y = \frac{x}{\alpha}\right),$$

tends to infinity. However, this follows immediately from the preceding theorem, when $x$ is replaced by $y = x/\alpha$.

We can prove in a similar fashion the theorem: For every positive value of $\alpha$, the quotient $(\log x)/x^\alpha$ tends to zero when $x$ tends to infinity, that is, the logarithm becomes infinite at a lower order of magnitude than any arbitrarily small positive power of $x$.

The proof follows immediately, if we set $\log x = y$, so that our quotient is transformed into $y/e^{\alpha y}$. We then set $e^\alpha = a$; then $a$ is a number $> 1$ and our quotient $y/a^y$ approaches 0 as $y$ increases. Since $y$ approaches infinity with $x$, our theorem is proved.

Another very simple proof maybe suggested: For $x > 1$ and $\varepsilon > 0$,

$$\log x = \int_1^x \frac{d\xi}{\xi} < \int_1^x \xi^{\epsilon - 1} d\xi = \frac{1}{\epsilon}(x^{\epsilon} - 1);$$

if we choose $\varepsilon$ smaller than $\alpha$ and divide both members of the inequality by $x^{\alpha}$, then as $x \to \infty$ it follow that $(\log x)/x^{\alpha} \to 0$.

Based on these results, we can construct functions of an order of magnitude by far higher than that of the exponential function and other functions of an order of magnitude far lower than that of the logarithm. For example, the function $e^{e^x}$ is of higher order than the exponential function, and the function log log $x$ of lower order than the logarithm; obviously, we can repeat these iteration processes as often as we like, piling up the symbols $e$ or log to any extent we please.

**3.9.3 General Remarks:** These considerations show that it is not possible to assign by systematic reasoning to all functions definite numbers as orders of magnitude in such a way that, when two functions are compared, the function of the higher order of magnitude has the higher number. For example, if the function $x$ is of the order of magnitude 1 and the function $x^{1+\varepsilon}$ of the order of magnitude $1 + \varepsilon$, then the function $x \log x$ must be of an order of magnitude which is greater than I and less than $1 + \varepsilon$ no matter how small $\varepsilon$ is chosen. But there exists no such number. Moreover, it is readily seen that functions need not possess a clearly defined order of magnitude. For example, the function $\dfrac{x^2(\sin x)^2 + x + 1}{x^2(\cos x)^2 + x}$ approaches no definite limits as $x$ increase; on the contrary, for $x = n\pi$ (where $n$ is an integer) its value is $1/n\pi$, while, for $x = (n + 1/2)\pi$, it is $(n + 1/2)\pi + 1 + 1/(n + ½)\pi$. Although both the numerator and denominator become infinite, the quotient neither remains between positive bounds, nor tends to zero, nor tends to infinity. Hence, the numerator is neither of the same order as the denominator, nor of lower order, nor of higher order. This apparently startling situation merely means that our definitions are not designed in such a way that we can compare any pair of functions. This is not a defect; we have no desire to compare the orders of such functions like the numerator and denominator above, since knowledge of the value of one of them gives us no useful information about the other one.

**3.9.4 The Order of Magnitude of a Function in the Neighbourhood of an Arbitrary Point:** Just as we can investigate the behaviour of a function when $x$ increases without limit, we may also ask ourselves whether and how functions which become infinite at the point $x = \xi$ may be distinguished with regard to their behaviour at that point. Moreover, we state that the function $f(x)= 1/|x - \xi|$ becomes infinite at the first order at the point $x = \xi$, and correspondingly that the function $1/|x - \xi|^{\alpha}$ becomes infinite at the order $\alpha$, provided that $a$ is positive.

We then recognize that the function $e^{1/|x-\xi|}$ becomes infinite at higher order, and the function log $|x - \xi|$ infinite at lower order than all these powers, i.e., that there hold the limiting relations

$$\lim_{x \to \xi}(|x-\xi|^{\alpha} \cdot e^{1/|x-\xi|}) = \infty \text{ and } \lim_{x \to \xi}(|x-\xi|^{\alpha} \cdot \log|x-\xi|) = 0$$

In order to see this, we merely set $1/|x - \xi| = y$; our statements then reduce to the known theorem, since

$$|x-\xi|^{\alpha} \cdot e^{1/|x-\xi|} = e^{y}/y^{\alpha} \text{ and } |x-\xi|^{\alpha} \cdot \log|x-\xi| = -(\log y)/y^{\alpha}$$

and $y$ increases beyond all bounds as $x$ tends to $\xi$. The method of reduction of the behaviour at a finite point to the behaviour at infinity by the substitution $1/|x-\xi| = y$ frequently proves useful.

**3.9.5 The Order of Magnitude of a Function tending to Zero:** Just as we seek to describe the approach of a function to infinity more definitely by means of the concept of order of magnitude, we may also specify the way in which a function approaches zero. We say that, as $x \to \infty$, the quantity $1/x$ vanishes to the first order, the quantity $x^{-\alpha}$, where $\alpha$ is positive, to the order $\alpha$. We find once again that the function $1/\log x$ vanishes to a lower order than an arbitrary power $x^{-\alpha}$, that is, the relation

$$\lim_{x \to 0}(x^{-\alpha} \cdot \log x) = 0$$

applies for every positive $\alpha$.

In the same way, we say that, for $x = \xi$, the quantity $x - \xi$ vanishes to the first order, the quantity $|x-\xi|^{\alpha}$ to the order $\alpha$. With the above results, it is easy to prove the relations

$$\lim_{x \to 0}(|x|^{\alpha} \cdot \log|x|) = 0, \quad \lim_{x \to 0}(|x|^{-\alpha} \cdot e^{-1/|x|}) = 0$$

which are usually expressed as follows:

The function $1/\log|x|$ vanishes as $x \to 0$ to a lower order than any power of $x$; the exponential function $e^{-1/|x|}$ vanishes to a higher order than any power $x$.

## Exercises 3.8:

1. Compare the following functions with powers of $x$ *as* regards their order of magnitude as $x\to\infty$:

(a) $e^{x^{\beta}} - 1.$

(b) $(\log x)^{\beta}.$

(c) $\sin x.$

(d) $\sinh x.$

(e) $x^{1/2}\sin x \,.\, \arctan x.$

(f) $x^{1/2}\sin x + \dfrac{x^2\cos^3 x}{x^3+1}.$

(g) $\dfrac{e^{-1/x}}{1-e^{-1/x}}.$

(h) $x^{x} - 1.$

(j) $\log(x\log x).$

2. Compare the functions of Exercise 1. with $e^{ax}$, $e^{x^{a}}$, $(\log x)^{a}$.

3. Compare the functions of Exercise 1. with powers of $x$ as $x \to 0$.

4. Does the $\lim\limits_{x\to\infty} e^{x^{n}}e^{(-e^{x})}$ exist?

6. What are the limit as $x\to\infty$ , of $e^{(-e^{x})}$ and $e^{(e^{-x})}$?

6. Let $f(x)$ be a continuous function vanishing, together with its first derivative, for $x = 0$. Show that $f(x)$ vanishes to a higher order than $x$ as $x\to 0$.

7. Show that

$$f(x) = \frac{a_0x^n + a_1x^{n-1} + \dots + a_n}{b_0x^m + b_1x^{m-1} + \dots + b_m},$$

when $a_0, b_0 \neq 0$, is of the same order of magnitude as $x^{m-n}$, when $x\to\infty$.

8.* Prove that $e^x$ is not a rational function.

9.* Prove that $e^x$ cannot satisfy an algebraic equation with polynomials in $x$ as coefficients.

Answers and Hints

# Appendix to Chapter III

## A3.1 Some Special Functions

From time to time, we have made it clear by examples that the general concept of function contains many possibilities foreign to naïve intuition. As a rule, these examples were not given in terms of single analytical expressions. Hence we must show that it is possible to represent various typical discontinuities and abnormal phenomena by means of very simple expressions, built up from the elementary functions. We begin with an example without presence of a discontinuity.

**A3.1.1 The Function $y = e^{-1/x^2}$:** The function (Fig. 22), which is defined in the first instance only for values of $x$ other than zero, obviously has the limit zero as $x \longrightarrow 0$. In fact, by the transformation $1/x^2 = \xi$, this function becomes $y = e^{-\xi}$ and

$$\lim_{\xi \to \infty} e^{-\xi} = 0.$$

Hence, in order to extend our function so that it is continuous for $x = 0$

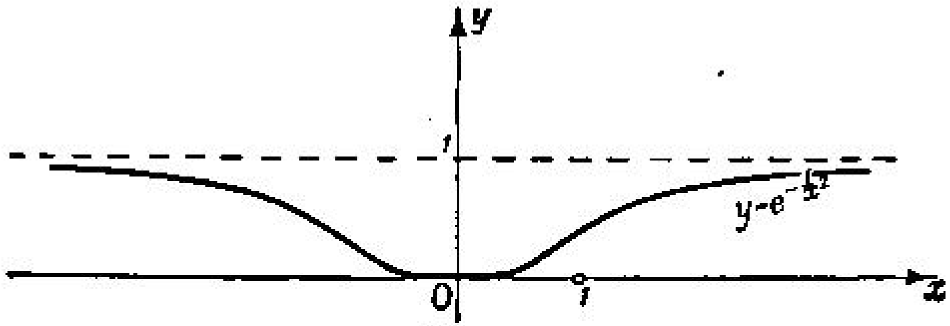


Fig. 22

we define the value of the function at the point $x = 0$ by the equation $y(0) = 0$.

By the chain rule, the derivative of our function for $x \neq 0$ is $y' = \frac{2}{x^3} e^{-1/x^2}$. If $x$ tends to 0, this derivative will also have the limit 0, as we find immediately from 3.9.5. At the point $x = 0$ itself, the derivative

$$y'(0) = \lim_{h \to 0} \frac{y(h) - y(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h}$$

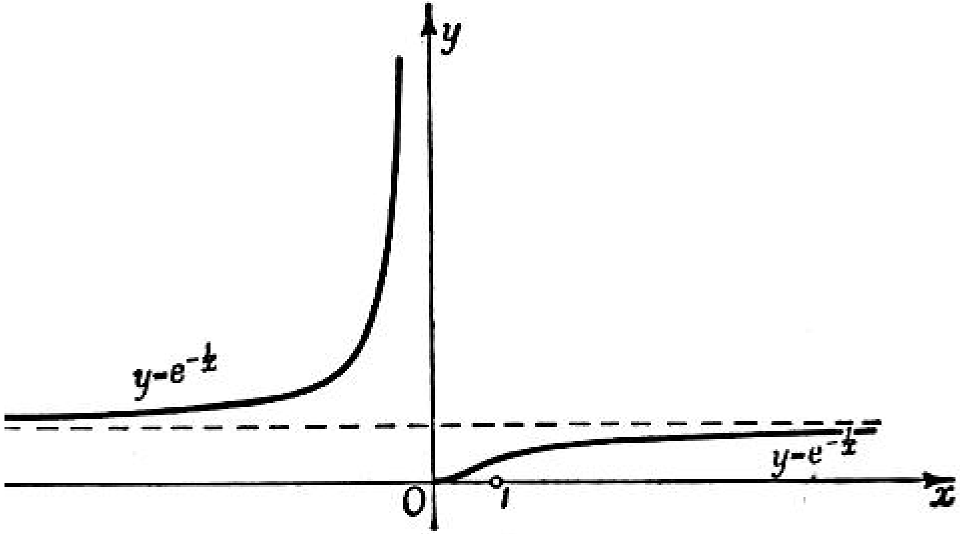



Fig. 23

is also zero.

If we form the higher derivatives for $x \neq 0$, we shall obviously always obtain the product of the function $e^{-1/x^2}$ and a polynomial in $1/x$, and the passage to the limit $x \to 0$ will always yield the limit 0. All the higher derivatives will likewise vanish, like $y'$, at the point $x = 0$.

Thus, we see that our function is continuous everywhere and differentiable as many times as we please, and yet it vanishes with all its derivatives at the point $x = 0$. We shall realize later on in A6.1how remarkable this behaviour really is.

**A3.1.2. The Function $y = e^{-1/x}$:** We may readily convince ourselves that, for positive values of $x$, this function behaves in the same way as the preceding function; if $x$ tends to 0 through positive values, the function tends to 0 and the same is true for all its derivatives. If we define the value of the function at $x = 0$ as $y(0) = 0$, all the right-hand derivatives at the point $x = 0$ will have the value 0. It is quite a different matter when $x$ tends to 0 through negative values; in fact, then the function becomes infinite and there do not exist left-hand derivatives at the point Hence, at the point $x = 0$. the function has a remarkable kind of discontinuity, quite unlike the infinite discontinuities of the rational functions, considered in 1.3.1 and 1.8.2.

**A3.1.3 The function $y = \tanh 1/x$:** We have already seen 1n 1.8.2 and A.1.5 that functions with **jump** discontinuities can be obtained from simple functions by a passage to the limit. The exponential function, defined on 3.6.4, and the principle of compounding of functions offer us another method for the construction of functions with such discontinuities from elementary functions without any further limiting process. An example of this is the function

$$y = \tanh\frac{1}{x} = \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$$

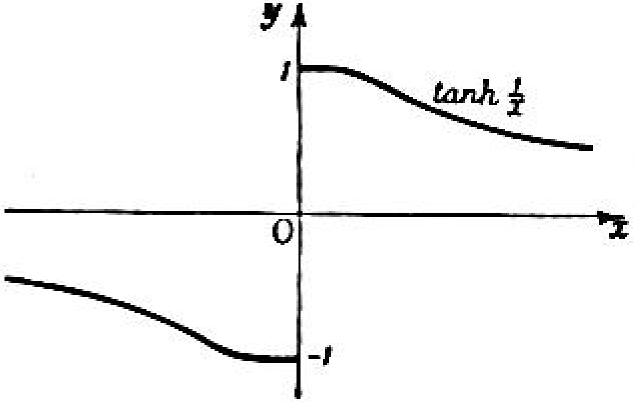


Fig. 24

with its behaviour at the point $x = 0$. In the first instance, the function is not defined at this point. If we approach the point $x = 0$ through positive values of $x$, we obviously obtain the limit 1; on the other hand, if we approach the point $x=0$ through negative values, we obtain the limit -1. The point $x = 0$ is therefore a point of discontinuity; as $x$ increases through 0, the value of the function jumps by 2 (Fig. 24). On the other hand, the derivative

$$y' = -\frac{1}{\cosh^2(1/x)}\frac{1}{x^2} = -\frac{1}{x^2}\frac{4}{(e^{1/x}+e^{-1/x})^2}$$

approaches the limit 0 from both sides, as follows readily from 3.9.5

Another example of the occurrence of a **jump** discontinuity is given by the function $y = \text{artan}1/x$ as $x \to 0$.

**A3.1.4 The Function $y = x$ tanh $1/x$:** In the case of the function

$$y = x\tanh\frac{1}{x} = x\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}},$$

the above discontinuity is removed by the factor $x$. This function has the limit 0 as $x \to 0$ from either side, so that we can again appropriately define $y(0)$ as equal to 0. Our function is then continuous at $x = 0$, its first derivative

$$y' = \tanh\frac{1}{x} - \frac{1}{x}\frac{1}{\cosh^2(1/x)}$$

has just the same kind of discontinuity as the preceding example. The graph of the function is a curve with a corner (Fig. 25); at the point $x = 0$, the function has no actual derivative, but a right-hand derivative with the value +1 and a left-hand derivative with the value -1.

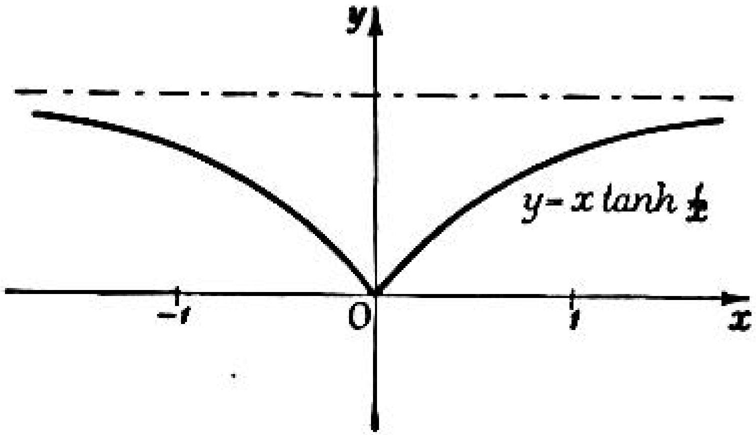



Fig. 25

**A3.1.5 The Function $y = x \sin 1/x$, $y(0) = 0$:** We have already seen that this function is not composed of a finite number of monotonic pieces - as we may say, it is not **sectionally monotonic** (cf. 10.5.1), but is nevertheless continuous (1.8.3). Indeed, its first derivative

$$y' = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}, \qquad (x \neq 0)$$

has a discontinuity at $x = 0$, because, as $x$ tends to 0, this derivative oscillates continually between bounding curves, one positive and one negative, which themselves tend to $+\infty$ and $-\infty$, respectively. At the actual point $x = 0$, the difference quotient is $[y(h) - y(0)]/h = \sin 1/h$; since, as $h \to 0$, this swings backwards and forwards between 1 and -1, an infinite number of times, the function has neither a right-hand nor a left-hand derivative.

## A3.2 Remarks on the Differentiability of Functions

The derivative of a function, which is continuous and has a derivative at every point, need not be continuous.

As the simplest example, consider the function

$$y = f(x) = x^2 \sin\frac{1}{x}.$$

This function is in the first instance not defined at $x = 0$; we shall define $f(0)$, its value there, as 0, so that the function is now defined and continuous everywhere. For all values of $x$, other than 0, the derivative is given by the expression

$$f'(x) = -x^2 \cos\frac{1}{x} \cdot \frac{1}{x^2} + 2x \sin\frac{1}{x} = -\cos\frac{1}{x} + 2x \sin\frac{1}{x}.$$

As $x$ tends to 0, $f'(x)$ has no limit. On the other hand, if we form the difference quotient

$$\frac{f(h) - f(0)}{h} = \left(h^2 \sin\frac{1}{h}\right)/h = h \sin\frac{1}{h},$$

we see at once that this tends to 0 with $h$. The derivative therefore exists for $x$=0 and has the value 0. In order to understand this intuitively - the reason for this paradoxical behaviour - we represent the function graphically (Fig. 26).

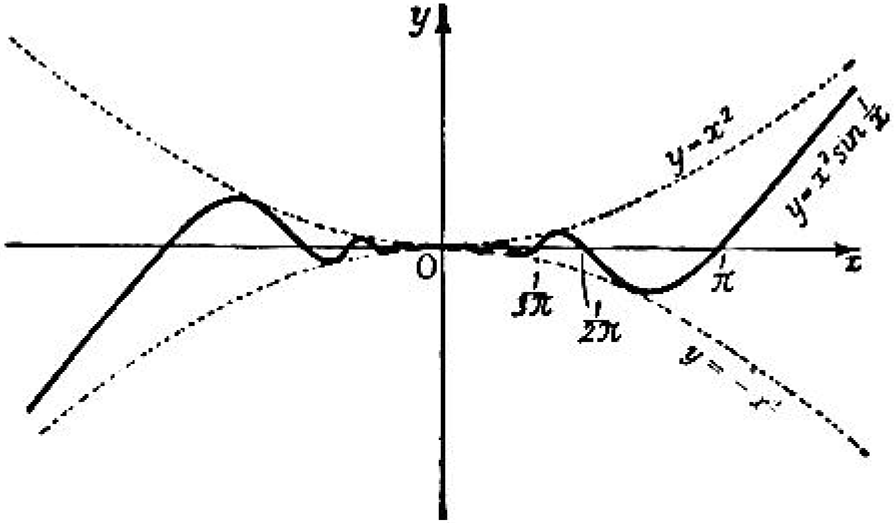



Fig. 26

It swings backwards and forwards between the curves $y = x^2$ and $y = -x^2$, which it touches in turn. Thus, the ratio of the heights of the wave-crests of our curve and their distances from the origin steadily becomes smaller. Yet, these waves do not become flatter; in fact, their slope is given by the derivative $f'(x) = 2x \sin 1/x - \cos 1/x$; at the points $x = 1/(2n\pi)$, where $\cos 1/x = 1$, this is equal to -1, at the points $x - 1/(2n + 1)\pi$, where $\cos 1/x = -1$, it is equal to +1.

In contrast to the possibility shown here - that a derivative may exist everywhere and yet not be continuous, we state the following simple theorem, which throws light on the whole series of earlier examples and discussions: If we know that, in a neighbourhood of a point $x = a$, the function $f(x)$ is continuous and has a derivative $f'(x)$ everywhere, except that we do not know whether $f'(a)$ exists, and if moreover the equation $\lim_{x \to a} f'(x) = b$ holds, then also the derivative $f'(x)$ exists at the point a and $f'(a) = b$. The proof follows immediately from the mean value theorem. In fact, we have $[f(a+h)-f(a)]/h=f'(\xi)$, where $\xi$ is a value intermediate between $a$ and $a+h$. If now $h$ tends to 0, by hypothesis, $f(\xi)$ tends to $b$, and our statement follows immediately.

A companion theorem to this theorem may be proved in a similar manner: If the function $f(x)$ is continuous in $a \leq x \leq b$ and has for $a < x < b$ a derivative, which increases beyond all bounds as $a$ tends to 0, then the right-hand difference quotient $[f(a+h)-f(a)]/h$ also increases beyond all bounds as $h$ tends to 0, so that no finite right-hand derivative exists at $x = a$. The geometrical meaning of this state of affairs is that at the point with the (finite) co-ordinates $(a, f(a))$ the curve has a vertical tangent.

## A3.3. Some Special Cases

**A3.3.1 Proof of the Binomial Theorem:** Our rules for differentiation enable us to give a simple proof of the binomial theorem; this proof will be introduced here as an example of the **method of undetermined parameters** which we shall find later on to be important. We wish to expand the quantity $(1+x)^n$ in powers of $x$ for all positive integral values of $n$. We see at once that the function $(1+x)^n$ must be a polynomial of degree $n$, i.e., it must be of the form

$$(1+x)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

and the problem is now to determine the coefficients $a_k$. If we set $x = 0$, we obtain at once $a_0 = 1$. If we differentiate both sides of the equation once, twice, three times, etc., we obtain the equations

$$\begin{gathered} n(1+x)^{n-1} = a_1 + 2a_2 x + \ldots + n a_n x^{n-1}, \\ n(n-1)(1+x)^{n-2} = 2a_2 + 3 \,.\, 2a_3 x + \ldots + n(n-1) a_n x^{n-2}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{gathered}$$

Since these equations hold for all values of $x$, we can set $x = 0$ in each of them and thus obtain for the coefficients $a_k$ the expressions

$$a_1 = n, \quad a_2 = \frac{n(n-1)}{1 \,.\, 2}, \quad a_3 = \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}, \quad \ldots$$

$$a_k = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} = \binom{n}{k}.$$

We thus obtain the binomial theorem in the form

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \ldots + \binom{n}{k}x^k + \ldots + x^n.$$

**A3.3.2 Proof of the Binomial Theorem: Leibnitz's Rule:** In connection with the above, we ask the reader to prove as an exercise that the successive differentiation of a product may be performed according to **Leibnitz's rule**:

$$\frac{d^n}{dx^n}(fg) = \frac{d^nf}{dx^n}g + \binom{n}{1}\frac{d^{n-1}f}{dx^{n-1}}\frac{dg}{dx} + \binom{n}{2}\frac{d^{n-2}f}{dx^{n-2}}\frac{d^2g}{dx^2} + \ldots$$

$$+ \binom{n}{n-1}\frac{df}{dx}\frac{d^{n-1}g}{dx^{n-1}} + f\frac{d^ng}{dx^n}.$$

However. the repeated differentiation of a compound function $y = f\{\phi(x)\}$ follows no such easily recalled law. We have from the rules for differentiation (the product and chain rules)

$$\frac{dy}{dx} = \frac{df}{d\phi}\frac{d\phi}{dx} = f'\phi',$$

$$\frac{d^2y}{dx^2} = f''\phi'^2 + f'\phi'',$$

$$\frac{d^3y}{dx^3} = f'''\phi'^3 + 3f''\phi'\phi'' + f'\phi''',$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

**A3.3.3 Further Examples of the Use of the Chain Rule. Differentiation of $f(x)^{g(x)}$. The Generalized Mean Value Theorem:** In order to form the derivative of the function $x^x$, we write $x^x = e^{x\log x}$, whence the chain rule yields

$$\frac{d}{dx}x^x = x^x(\log x + 1)$$

Similarly, we can carry out the differentiation of the more general expression

$$f(x)^{f(x)} = e^{f(x)\log f(x)}$$

by means of the chain rule as follows:

$$\frac{d}{dx}\{f(x)^{f(x)}\} = f(x)^{f(x)} \, . \, f'(x)\{\log f(x) + 1\}.$$

As a further application of the chain rule, we present here a proof of the theorem which we have already called the generalized mean value theorem of the differential calculus, the theorem being established here under less stringent conditions. Let $G(x) = u$ be a function which in the closed interval $a \leq x \leq b$ is continuous and monotonic, and which in the open interval $a < x < b$ has a derivative which is nowhere equal to 0; moreover, let $F(x)$ be a function which is also continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. Introduce by means of the inverse function $x = \Phi(u)$ of $G(x)$ the new independent variable $u$ instead of $x$ into $F(x)$, thus obtaining the compound function $f(u) = F(\Phi(u))$; by the chain rule,

$$f'(u) = F'(x)\Phi'(u) = \frac{F'(x)}{G'(x)}.$$

The ordinary mean value theorem, on application to the function $f(u)$ and to the interval between $u_1=G(a)$ and $u_2=G(b)$, shows that for an intermediate value $\omega$

$$\frac{f(u_2) - f(u_1)}{u_2 - u_1} = f'(\omega) \quad \text{or} \quad \frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(\xi)}{G'(\xi)},$$

where $\xi <= \Phi(\omega)$ is a value intermediate between $a$ and $b$.

**Exercises: 3.10**

1. Find the second derivative of $f(g(h(x)))$.

2. Differentiate the functions: ($a$) $x^{\sin x}$, ($b$) $(\cos x)^{\tan x}$,
($c$) $\log_{v(x)} u(x)$ (that is, the logarithm of $u(x)$ to the base $v(x)$, $v(x) > 0$).

3. Prove Leibnitz's rule.

4. Find the $n$-th derivative of:

| | |
|---|---|
| ($a$) $x^2 e^{ax}$. | ($d$) $\cos mx \sin kx$. |
| ($b$) $(\log x)^2$. | ($e$) $e^x \cos 2x$. |
| ($c$) $\sin x \sin 2x$. | ($f$) $(1+x)^6 e^x$ |

5.*Find the $n$-th derivative of arcsin $x$ at $x$=0, and then that of (arcsin $x$)$^2$ at $x$=0.

9. Prove that $\sum_{k=2}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$.

Answers and Hints

## Chapter IV.

# Further Development of the Integral Calculus

The rules for differentiation, formulated in the preceding chapter, have given us great power over the problem of differentiating given functions. However, almost always, its inverse problem of integration exceeds it greatly in importance, whence we must now study the art of integrating given functions. The results obtained by means of our differentiation formulae may be summed up as follows:

Every function which is formed from the elementary functions by means of a closed expression*, can be differentiated, and its derivative, if it is also a closed expression, formed from the elementary functions.

* By this we mean a function which can be built up from the elementary functions by repeated application of the rational operations and the processes of compounding and inversion. However. in this context, it should be emphasized that the distinction between **elementary functions** and other functions is itself quite arbitrary.

On the other hand, we have encountered any exactly corresponding fact relating to the integration of elementary functions. We do know that every elementary function and, in fact, every continuous function can be integrated, and we have integrated a large number of elementary functions either directly or by inversion of differentiation formulae and have found their integrals to be expressions involving only elementary functions. But we are still far from being able to find a general solution of the problem: Given a function $f(x)$, which is expressed in terms of the elementary functions by any closed expression, find an expression for its indefinite integral $F(x) = \int f(x)\,dx,$ which is itself a closed expression in terms of the elementary functions.

It is a fact that, in general, this problem is **insoluble**; it is by no means true that every elementary function has an integral which itself is an elementary function. Nevertheless, it is extremely important that we should be able to actually carry out such integrations whenever they are possible, and that we should acquire a certain amount of technical skill in the integration of given functions.

The first part of this chapter will be devoted to the development of devices which are useful for this purpose. In this connection, we expressly warn the beginner against merely memorizing the many formulae obtained by using these technical devices. Instead, students should direct their efforts towards gaining a clear understanding of the methods of integration and learning how to apply them. Moreover, they should remember that even when integration by these devices is impossible, the integral does exist (at least for all continuous functions) and can actually be computed to as high a degree of accuracy as is desired by means of the numerical methods which will be developed in Chapter VII.

In the latter part of this chapter, we shall endeavour to deepen and extend our ideas of integration and the integral, quite apart from the problem of integration techniques.

## 4.1 Elementary Integrals

First of all, we repeat that an equivalent integration formula corresponds to everyone of the earlier proved differentiation formulae. Since these elementary integrals are used over and over again as material in the art of integration, we have collected them in the table below. The right-hand column contains elementary functions, the

left-hand column the corresponding derivatives. If we read the table from the left hand side to the right hand side, we find in the right hand column an indefinite integral of the function in the left hand column.

We also remind the reader of the fundamental theorems of the differential and integral calculus, proved in 2.4, and, in particular, of the fact that the definite integral is obtained from the indefinite integral $F(x)$ by the formula

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

| $F'(x) = f(x)$ | $F(x) = \int f(x)\,dx$ |
|---|---|
| 1. $x^a \; (a \neq -1)$. | $\dfrac{x^{a+1}}{a+1}$. |
| 2. $\dfrac{1}{x}$. | $\log \lvert x \rvert$. |
| 3. $e^x$. | $e^x$. |
| 4. $a^x \; (a \neq 1)$. | $\dfrac{a^x}{\log a}$. |
| 5. $\sin x$. | $-\cos x$. |
| 6. $\cos x$. | $\sin x$. |
| 7. $\dfrac{1}{\sin^2 x} \; (\equiv \operatorname{cosec}^2 x)$. | $-\cot x$. |
| 8. $\dfrac{1}{\cos^2 x} \; (\equiv \sec^2 x)$. | $\tan x$. |
| 9. $\sinh x$. | $\cosh x$. |
| 10. $\cosh x$. | $\sinh x$. |
| 11. $\dfrac{1}{\sinh^2 x} \; (\equiv \operatorname{cosech}^2 x)$. | $-\coth x$. |
| 12. $\dfrac{1}{\cosh^2 x} \; (\equiv \operatorname{sech}^2 x)$. | $\tanh x$. |
| 13. $\dfrac{1}{\sqrt{(1-x^2)}} \; (\lvert x \rvert < 1)$. | $\begin{cases} \arcsin x. \\ -\arccos x. \end{cases}$ |
| 14. $\dfrac{1}{1+x^2}$. | $\begin{cases} \arctan x. \\ -\operatorname{arc\,cot} x. \end{cases}$ |
| 15. $\dfrac{1}{\sqrt{(1+x^2)}}$. | $\operatorname{ar\,sinh} x \equiv \log\{x + \sqrt{(1+x^2)}\}$. |
| 16. $\dfrac{1}{\pm\sqrt{(x^2-1)}} \; (\lvert x \rvert > 1)$. | $\operatorname{ar\,cosh} x \equiv \log\{x \pm \sqrt{(x^2-1)}\}$. |

Finally, for the technique of integration, the reader should have the elementary rules of integration 2.1.3 at his finger-tips.

In the following sections, we shall attempt to reduce the calculation of integrals of given functions in some way or other to the elementary integrals collected in the table above. Apart from devices which the beginner certainly could not acquire systematically, but which, on the contrary, occur only to those with long experience, this reduction is based essentially on two useful methods. Each of these methods enables us to transform a given integral in many ways; the objective of such transformations is to reduce the given integral, in one step or in a sequence of steps, to one or more of the above elementary integration formulae.

## 4.2 The Method of Substitution

The first of these useful methods for attacking integration problems is the introduction of a new variable (i.e. the **method of substitution or transformation**). The corresponding integral formula is just the chain rule of the differential calculus, expressed in the integral form.

**4.2.1 The Substitution Formula:** We assume that a new variable $u$ is introduced into a function $F(x)$ by means of the equation $x = \phi(u)$, so that $F(x)$ becomes a function of $u$:

$$F(x) = F(\phi(u)) = G(u).$$

By the chain rule of the differential calculus,

$$\frac{dG}{du} = \frac{dF}{dx}\,\phi'(u).$$

If we now write

$$F'(x) = f(x) \quad \text{and} \quad G'(u) = g(u),$$

or the equivalent expressions

$$F(x) = \int f(x)\,dx \quad \text{and} \quad G(u) = \int g(u)\,du,$$

then, on the one hand, the chain rule takes the form

$$g(u) = f(x)\phi'(u)$$

and, on the other hand, $G(u) = F(x)$ by definition, that is,

$$\int g(u)\,du = \int f(x)\,dx,$$

and we obtain the integral formula, equivalent to the chain rule,

$$\int f(\phi(u))\,\phi'(u)\,du = \int f(x)\,dx, \qquad \{x = \phi(u)\}.$$

This is the basic formula for the substitution of a new variable in an integral. It means that, if we wish to find an indefinite integral of a function of $u$, which is given in the special form $f(\phi(u))\phi'(u)$, we can instead find the indefinite integral of the function $f(x)$ as a function of $x$ and after integration return to the variable $u$ by setting $x = \phi(u)$.

For example, if we apply the formula to the integrand $\phi'(u)/\phi(u)$, we obtain

$$\int \frac{\phi'(u)}{\phi(u)}\,du = \int \frac{dx}{x} = \log|x| = \log|\phi(u)|$$

or, replacing $u$ by $x$,

$$\int \frac{\phi'(x)}{\phi(x)}\,dx = \log|\phi(x)|.$$

If we substitute in this important formula particular functions such as $\phi(x) = \log x$ or $\phi(x) = \sin x$, we obtain*

$$\int \frac{dx}{x \log x} = \log | \log x |,$$

$$\int \cot x \, dx = \log | \sin x |, \qquad \int \tan x \, dx = -\log | \cos x |.$$

A further example is

$$\int \varphi(u) \varphi'(u) \, du = \int x \, dx = \frac{1}{2} x^2 = \frac{1}{2} [\varphi(u)]^2,$$

where $f(x) = x$. This yields for $\phi(u) = \log u$

$$\int \frac{\log u}{u} \, du = \frac{1}{2} (\log u)^2.$$

*These and the following formulae are verified by showing that differentiation of the result returns the integrand. Moreover, the formulae are, of course, only asserted as true in as far as the expressions occurring in them have a meaning.

We finally consider the example

$$\int \sin^n u \cos u \, du.$$

Here $x = \sin u = \phi(u)$, whence

$$\int \sin^n u \cos u \, du = \int x^n \, dx = \frac{x^{n+1}}{n+1} = \frac{\sin^{n+1} u}{n+1}.$$

However, in many cases, we shall use the above formula in the reverse direction, starting with the right hand side - the integral $\int f(x) \, dx.$ We now have to evaluate or simplify a prescribed indefinite integral $F(x) = \int f(x) \, dx$ by introduction of the new integration variable $u$ by means of the transformation formula $x = \phi(u)$, then working out the indefinite integral

$$G(u) = \int f(\phi(u))\phi'(u)\,du,$$

and finally replacing the variable *u* in this integral by *x*. In order to carry out this last step, we must be certain that a definite value *u* actually corresponds to the value *x*, i.e., that the function $x = \phi(u)$ has an inverse. Accordingly, we now make the following assumption, in which we regard *x* as the primary variable: In the interval under consideration, $u = \psi(x)$ is a monotonic, differentiable function, the derivative $f'(x)$ of which does not vanish anywhere in the interval. We denote the inverse function, which under these conditions is definite and single-valued, by $x = \phi(u)$; its derivative is then given by $\phi'(u) == 1/\psi'(x)$. As the **basic formula** for the substitution of a new variable *u* in an integral, we obtain

$$\int f(x)\,dx = \int f(\phi(u))\,\phi'(u)\,du \qquad (u = \psi(x)).$$

The indefinite integral $\int f(x)\,dx$ can be obtained by calculation of the indefinite integral $\int f(\phi(u))\phi'(u)\,du$ and finally introducing *x* instead of *u* for the independent variable by means of the equation $u = \psi(x)$.

Hence it is not sufficient merely to express the old variable *x* in terms of the new variable *u* and then to integrate with respect to this new variable; before integrating, we must multiply by this derivative of the original variable *x* with respect to the new variable *u*.

The corresponding formula for definite integration between two limits is

$$\int_a^b f(x)\,dx = \int_{\psi(a)}^{\psi(b)} f(\phi(u))\phi'(u)\,du.$$

In the new integral, we have to choose those limits of integration which are obtained by subjecting the old integration limits to the transformation $x=\phi(u)$, $u=\psi(x)$.

In most applications, the integrand $f(x)$ will appear at the outset as a function of a function, say, $f(x)=h(u)$, where $u=\psi(x)$. It is then more convenient to write our integral formula in a slightly different form by identifying the expression $f\{\phi(u)\}$ with the expression $h(h)$. If we make for *u* the substitution $u=\psi(x)$, $x=\phi(u)$, then our transformation formula is simply

$$\int h\{\psi(x)\}dx = \int h(u)\frac{dx}{du}du.$$

As a first example, consider the integration of the function $f(x) = \sin 2x$ and introduce $u=\psi(x) = 2x$ and $h(u) = \sin u$. We have

$$\frac{du}{dx} = \psi'(x) = 2.$$

If we now substitute $u = 2x$ into the integral as the new variable, then it is not transformed into $\int \sin u\, du$, but into

$$\frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u = -\frac{1}{2}\cos 2x;$$

of course, this may of be verified at once by differentiating the right hand side.

If we integrate for $x$ between the limits 0 and $\pi/4$, the corresponding limits for $u$ are 0 and $\pi/2$, and we obtain

$$\int_0^{\pi/4}\sin 2x\, dx = \frac{1}{2}\int_0^{\pi/2}\sin u\, du = -\frac{1}{2}\cos u\Big|_0^{\pi/2} = \frac{1}{2}.$$

Another simple example is the integral $\int_1^4 \frac{dx}{\sqrt{x}}$. Here we take $u = \psi(x) = \sqrt{x}$ whence $x = \phi(u) = u^2$. Since $\phi'(u) = 2u$, we find

$$\int_1^4 \frac{dx}{\sqrt{x}} = 2\int_1^2 \frac{u\, du}{u} = 2\int_1^2 du = 2.$$

**4.2.2 Another Proof of the Substitution Formula:** Our integration formula can also be explained in another and more direct manner by aiming at the formula for definite integration and basing the proof on the meaning of the definite integral as a limit of a sum. In order to calculate the integral

$$\int_a^b h(\psi(x))\, dx$$

(for the case $a < b$), we begin with an arbitrary subdivision of the interval $a \leq x \leq b$, and then make the sub-division finer and finer. We choose these sub-divisions in the following way. If the function $u = \psi(x)$ is assumed to be monotonic increasing, there is a (1,1) correspondence between the interval $a \leq x \leq b$ on the $x$-axis and an interval $\alpha \leq u \leq \beta$ of the values of u = $\psi(z)$, where $\alpha = \psi(a)$, $\beta = \psi(b)$. We subdivide this interval into $n$ parts of length $\Delta u$; there is a corresponding subdivision of the $x$-interval into subintervals which, in general, are not all of the same length. (The assumption that these subintervals are all equal it by no means essential for the proof!) We denote the points of division of the $x$-interval by $x_0 = a$, $x_1$, $x_2$, $\cdots$, $x_n = b$ and the lengths of the corresponding sub-intervals by

$$\Delta x_1, \Delta x_2, \ldots, \Delta x_n.$$

The integral under consideration is then the limit of the sum

$$\sum_{\nu=1}^{n} h\{\psi(\xi_\nu)\}\Delta x_\nu,$$

where the value $\xi_\nu$ is arbitrarily selected from the $\nu$-th subinterval of the $x$-subdivision. This limit exists (for $\Delta u \to 0$) and is the integral, since on account of the uniform continuity of $x = \phi(u)$ the greatest of the lengths $\Delta x$ tends to 0 with $\Delta u$.

We now write this sum in the form $\sum_{\nu=1}^{n} h(u_\nu)\dfrac{\Delta x_\nu}{\Delta u}\,\Delta u,$ where $u_\nu = \psi(\xi_\nu)$. By the [mean value theorem of the differential calculus](), $\dfrac{\Delta x_\nu}{\Delta u} = \phi'(\eta_\nu),$ where $\eta_\nu$ is a suitably chosen intermediate value of the variable $u$ in the $\nu$-th sub-interval of the $u$-sub-division and $x = \phi(u)$ denotes the inverse function of $u = \psi(x)$. If we now select the value $\xi_\nu$ in such a way that $\xi$ and $\eta$ coincide, i.e., $\xi_\nu = \phi(\eta_\nu)$, $\eta_\nu = \psi(\xi_\nu)$, then our sum takes the form

$$\sum_{\nu=1}^{n} h(\eta_\nu)\phi'(\eta_\nu)\Delta u.$$

If we make here the passage to the limit, we immediately obtain the expression

$$\int_\alpha^\beta h(u)\frac{dx}{du}\,du$$

as the limiting value, that is, as the value of the integral under consideration in agreement with the formula given above.

Hence we have proved the theorem:

Let $h(u)$ be a continuous function of $u$ in the interval $\alpha \le u \le \beta$. If the function $u = \psi(x)$ is continuous and monotonic, has a continuous, non-vanishing derivative $du/dx$ in $a \le x \le b$, and $\psi(a) = \alpha, \psi(b) = \beta$, then

$$\int_a^b h\{\psi(x)\}dx = \int_a^b h(u)\,dx = \int_\alpha^\beta h(u)\frac{dx}{du}\,du.$$

This formula exhibits the advantage of Leibnitz's notation. In order to carry out the substitution $u=\psi(x)$, we only need write $(dx/du)du$ in place of $dx$, changing the limits from the original values of $x$ to the corresponding values of $u$.

**4.2.3 Examples. Integration formulae:** With the help of the substitution rule, we canevaluate a given integral $\int f(x)\,dx$ in many cases by reducing it by means of a suitable substitution $x = \phi(u)$ to one of the elementary integrations in our table. Whether such substitutions exist and how to find them are questions which have no general answer; this is rather a matter in which practice and ingenuity, in contrast to a systematic method, come into their own.

As an example, we shall work out the integral $\int \frac{dx}{\sqrt{(a^2 - x^2)}}$ using the substitution $x = \phi(u) = au$, $u=\psi(x)=x/a$, $dx=adu$, by which, with No. 13 in the table above, we obtain

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \int \frac{a\,du}{a\sqrt{(1-u^2)}} = \arcsin u = \arcsin\frac{x}{a}, \text{for } |x| < |a|.$$

For the sake of brevity, we have written the symbols $dx$ and $du$ separately, i.e., $dx = \phi'(u)du$ instead of $dx/du = \phi'(u)$. (cf. 2.3.9).

The same substitution yields also

$$\int \frac{dx}{a^2+x^2} = \int \frac{a\,du}{a^2(1+u^2)} = \frac{1}{a}\operatorname{arc}\tan u = \frac{1}{a}\operatorname{arc}\tan\frac{x}{a},$$

$$\int \frac{dx}{\sqrt{(a^2+x^2)}} = \operatorname{ar}\sinh\frac{x}{a},$$

$$\int \frac{dx}{\sqrt{(x^2-a^2)}} = \operatorname{ar}\cosh\frac{x}{a}, \text{ for } |x| > |a|,$$

$$\int \frac{dx}{a^2-x^2} = \begin{cases} \frac{1}{a}\operatorname{ar}\tanh\frac{x}{a} \text{ for } |x| < |a| \\ \frac{1}{a}\operatorname{ar}\coth\frac{x}{a} \text{ for } |x| > |a| \end{cases},$$

formulae which occur very frequently and which are easily verified by differentiating the right hand side.

In conclusion, we again emphasize the point: In our substitution process, we have made the assumption that the substitution has a unique inverse $x = \phi(u)$, and indeed that $\psi'(x)$ is nowhere equal to zero in the interval under consideration. If our assumption is not fulfilled, application of the substitution formula may easily lead to wrong conclusions. If $\psi'(x) = 0$ at isolated points of the interval of integration only, we can avoid these difficulties by subdividing this interval in such a way that $\psi'(x)$ vanishes only at the ends of a sub-interval; we can then apply the substitution to each sub-interval separately.

An application of this method leads at once to the following result, which applies to many special cases: If the derivative $\psi'(x)$ vanishes at a finite number of points, but the function $\psi(x)$remains monotonic, then the substitution formula remains valid.

## 4.3 Further Examples of the Substitution Method

We collect here several examples which the reader should study carefully by way of practice!

By the substitution $u = 1 \pm x^2$, $du = \pm 2x dx$, we find

$$\int \frac{x\,dx}{\sqrt{(1 \pm x^2)}} = \pm \sqrt{(1 \pm x^2)},$$

$$\int \frac{x\,dx}{1 \pm x^2} = \pm \tfrac{1}{2} \log |1 \pm x^2|.$$

In these formulae, we must take either the sign + or the sign - in all three places.

By the substitution $u = ax + b$, $du = adx$ ($a \neq 0$), we find

$$\int \frac{dx}{ax + b} = \frac{1}{a} \log |ax + b|,$$

$$\int (ax + b)^{\alpha}\,dx = \frac{1}{a(\alpha + 1)} (ax + b)^{\alpha+1} \qquad (\alpha \neq -1),$$

$$\int \sin(ax + b)\,dx = -\frac{1}{a} \cos(ax + b);$$

in a similar manner, using the substitution $u = \cos x$, $du = -\sin x\,dx$, we obtain

$$\int \tan x\,dx = -\log |\cos x|,$$

and by means of the substitution $u = \sin x$, $du = \cos x\,dx$

$$\int \cot x\,dx = \log |\sin x|$$

(cf. 4.2.1). Using the analogous substitutions $u=\cosh x$, $du=\sinh x\,dx$ and $u=\sinh x$, $du=\cosh x\,dx$, we obtain

$$\int \tanh x\,dx = \log |\cosh x|,$$

$$\int \coth x\,dx = \log |\sinh x|.$$

By the substitution $u = (a/b) \tan x$, $du = (a/b) \sec^2 x\,dx$, we find the two formulae

$$\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{1}{b^2} \int \frac{1}{\frac{a^2}{b^2} \tan^2 x + 1} \cdot \frac{dx}{\cos^2 x} = \frac{1}{ab} \operatorname{arc} \tan \left(\frac{a}{b} \tan x\right)$$

and

$$s_m(\alpha) = \frac{1}{m+1} \left[ \frac{\sin \frac{(m+1)\alpha}{2}}{\sin \frac{\alpha}{2}} \right]^2.$$

We evaluate the integral

$$\int \frac{dx}{\sin x}$$

by writing sin $x$ = 2 sin $x$/2·cos $x$/2 = 2 tan $x$/2 · cos² $x$/2 and setting $u$ = tan $x$/2 so that $du$ ½ sec²$x$/2$dx$; the integral then becomes

$$\int \frac{dx}{\sin x} = \int \frac{du}{u} = \log \left| \tan \frac{x}{2} \right|.$$

If we replace $x$ by $x + \pi/2$, this formula becomes

$$\int \frac{dx}{\cos x} = \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{4}\right) \right|.$$

If we also apply the known trigonometrical formulae 2 cos²$x$ = 1 + cos 2$x$ and 2 sin² $x$ = 1 - cos 2$x$, the substitution $u$ = 2$x$ yields

$$\int \cos^2 x \, dx = \tfrac{1}{2} (x + \sin x \cos x) \quad \text{and} \int \sin^2 x \, dx = \tfrac{1}{2} (x - \sin x \cos x).$$

By the substitution $x$ = cos $u$, equivalent to $u$ = arcos $x$ or, more generally,
$x = a$ cos $u$ ($a \neq 0$), we can reduce

$$\int \sqrt{(1-x^2)}\,dx \quad \text{and} \quad \int \sqrt{(a^2-x^2)}\,dx,$$

respectively, to these formulae. We thus obtain

$$\int \sqrt{(a^2-x^2)}\,dx = -\frac{a^2}{2}\,\text{arc cos}\,\frac{x}{a} + \frac{x}{2}\sqrt{(a^2-x^2)}.$$

Similarly, the substitution $x = a \cosh u$ yields

$$\int \sqrt{(x^2-a^2)}\,dx = -\frac{a^2}{2}\,\text{ar cosh}\,\frac{x}{a} + \frac{x}{2}\sqrt{(x^2-a^2)}$$

and the substitution $x = a \sinh u$

$$\int \sqrt{(a^2+x^2)}\,dx = \frac{a^2}{2}\,\text{ar sinh}\,\frac{x}{a} + \frac{x}{2}\sqrt{(a^2+x^2)}.$$

The substitution $u = a/x$, $dx = -(a/u^2)\,du$ leads to

$$\int \frac{dx}{x\sqrt{(x^2-a^2)}} = -\frac{1}{a}\,\text{arc sin}\,\frac{a}{x},$$

$$\int \frac{dx}{x\sqrt{(x^2+a^2)}} = -\frac{1}{a}\,\text{ar sinh}\,\frac{a}{x},$$

$$\int \frac{dx}{x\sqrt{(a^2-x^2)}} = -\frac{1}{a}\,\text{ar cosh}\,\frac{a}{x}.$$

Finally, we consider the three integrals

$$\int \sin mx \sin nx\,dx,\ \int \sin mx \cos nx\,dx,\ \int \cos mx \cos nx\,dx,$$

where *m* and *n* are positive integers. By the well-known trigonometrical formulae

$$\sin mx \sin nx = \frac{1}{2}\{\cos(m-n)x - \cos(m+n)x\},$$

$$\sin mx \cos nx = \frac{1}{2}\{\sin(m+n)x + \sin(m-n)x\},$$

$$\cos mx \cos nx = \frac{1}{2}\{\cos(m+n)x + \cos(m-n)x\}.$$

we can divide each of these integrals into two parts. If we now use the substitutions $u= (m + n)\, x$ and $u= (m - n)\, x$, respectively, we obtain directly the system of formulae:

$$\int \sin mx \sin nx\, dx = \begin{cases} \frac{1}{2}\left\{\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right\} \text{ if } m \neq n, \\ \frac{1}{2}\left(x - \frac{\sin 2mx}{2m}\right) \text{ if } m = n; \end{cases}$$

$$\int \sin mx \cos nx\, dx = \begin{cases} -\frac{1}{2}\left\{\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right\} \text{ if } m \neq n, \\ -\frac{1}{2}\left(\frac{\cos 2mx}{2m}\right) \text{ if } m = n; \end{cases}$$

$$\int \cos mx \cos nx\, dx = \begin{cases} \frac{1}{2}\left\{\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right\} \text{ if } m \neq n, \\ \frac{1}{2}\left(\frac{\sin 2mx}{2m} + x\right) \text{ if } m = n. \end{cases}$$

In particular, if we now integrate from $-\pi$ to $+\pi$, we obtain from these formulae the extremely important relations

$$\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = \begin{cases} 0 \text{ if } m \neq n, \\ \pi \text{ if } m = n, \end{cases}$$

$$\int_{-\pi}^{+\pi} \sin mx \cos nx \, dx = 0,$$

$$\int_{-\pi}^{+\pi} \cos mx \cos nx \, dx = \begin{cases} 0 \text{ if } m \neq n, \\ \pi \text{ if } m = n. \end{cases}$$

These are the **orthogonality relations** of the trigonometric functions which we shall meet again in Chap. IX .

**Exercises 4.1:**

Evaluate the following integrals and verify the results by differentiation:

1. $\int xe^{x^2}\,dx.$

2. $\int x^3e^{-x^4}\,dx.$

3. $\int x^2\sqrt{1+x^3}\,dx.$

4. $\int \frac{\log x}{x}\,dx.$

5. $\int \frac{dx}{x(\log x)^n}.$

6. $\int \frac{3\,dx}{9x^2-6x+2}.$

7. $\int \frac{dx}{\sqrt{(x^2-2x+5)}}.$

8. $\int \frac{6x}{2+3x}\,dx.$

9. $\int \frac{x+1}{\sqrt{(1-x^2)}}\,dx.$

10. $\int \frac{dx}{\sqrt{(5+2x+x^2)}}.$

11. $\int \frac{dx}{\sqrt{(3-2x-x^2)}}.$

12. $\int \frac{x\,dx}{x^2-x+1}.$

13. $\int \frac{x\,dx}{\sqrt{(x^2-4x+1)}}.$

14. $\int \frac{(x+1)\,dx}{\sqrt{(2+2x-3x^2)}}.$

15. $\int \frac{dx}{x^2+x+1}.$

16. $\int \frac{dx}{x^2-x+1}.$

17. $\int \frac{dx}{x^2+2ax+b}.$

18. $\int \frac{x^4}{1-x}\,dx.$

19. $\int \sin^3 x\cos^4 x\,dx.$

20. $\int \sin^2 x\cos^5 x\,dx.$

21. $\int x^3(\sqrt{1-x^2})^5\,dx.$

22. $\int \frac{x^3}{\sqrt{(1-x^2)}}\,dx.$

23. $\int_0^1 \frac{\arc\tan x}{1+x^2}\,dx.$

24. $\int_0^\pi \cos^n x\sin x\,dx.$

25. $\int_0^4 \frac{x\,dx}{\sqrt{(1+3x^2)}}.$

26. $\int_a^b \frac{x}{(1+x^2)^2}\,dx.$

27. $\int_a^b \frac{x^3}{1-x}\,dx\ (1<a<b).$

28. $\int_0^{\pi/2} x\sin 2x^2\,dx.$

29. Evaluate $\int_0^1 (1-x)^n\,dx$ (where $n$ is a positive integer) by substitution.

Answers and Hints

## 4.4 Integration by parts

The second useful method for dealing with integration problems is given by the differentiation formula for a product:

$$(fg)' = f'g + fg'.$$

### 4.4.1 General Remarks:

If we write this formula as an integral formula, we obtain (cf. 3.2.1)

$$f(x)g(x) = \int g(x)f'(x)\,dx + \int f(x)g'(x)\,dx$$

or

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

This formula is referred to as the formula for **integration by parts**. The calculation of one integral is thereby reduced to the calculation of another integral. In fact, if we split up the integrand of an integral $\int \omega(x)\,dx$ into a product $\omega(x) = f(x)\cdot\phi(x)$ and find the indefinite integral

$$g(x) = \int \phi(x)\,dx$$

of the one factor $\phi(x)$, so that $\phi(x) = g'(x)$, then our formula reduces the integral $\int \omega(x)\,dx = \int f(x)\phi(x)\,dx = \int f(x)g'(x)\,dx$ to the integral $\int g(x)f'(x)\,dx$, which in some cases is more readily evaluated than the original form. Since a given function $\omega(x)$, which occurs as an integrand, can be

regarded as a product $f(x)\phi(x) = f(x)g'(x)$ in a great many different ways, this formula provides us with a very effective tool for the transformation of integrals.

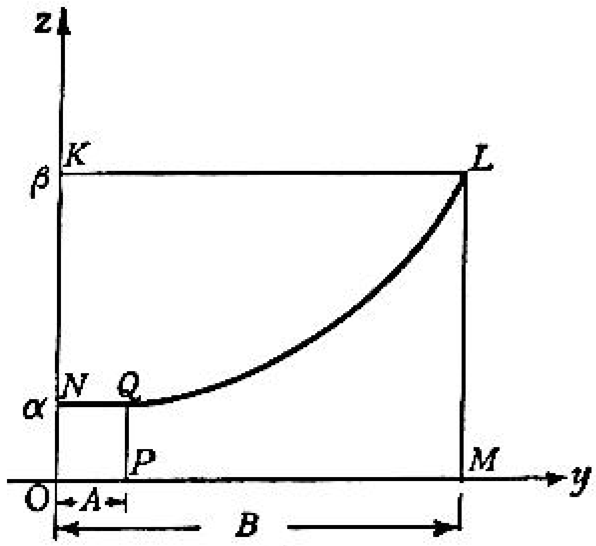


Fig. 1

Written as a formula for **definite integration**, the formula for **integration by parts** is

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$
$$= f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\,dx.$$

In fact, in order to obtain the formula for definite integration from the formula for indefinite integration (2.4.4), we only need replace, first of all, the variable appearing on both sides in the formula for the indefinite integral by the value $x=b$, then by the value $x = a$ and write down the difference of these two expressions.

A simple interpretation of this formula, at least with suitable restrictions on the functions involved, can be given. Suppose that $y = f(x)$ and $z = g(x)$ are monotonic, and that $f(x)=A$, $f(b)=B$, $g(a)=\alpha$, $g(b)=\beta$; we can then form the inverse of the first function and substitute it into the equation, thus obtaining $z$ as a function of $y$. Let this function be monotonic increasing. Since $dy = f'(x)dx$ and $dz = g'(x)dx$, the formula for integration by parts can be rewritten

$$\int_A^B z\,dy + \int_\alpha^\beta y\,dz = B\beta - A\alpha,$$

in agreement with the relation made clear by Fig. 1,

$$\text{area } NQLK + \text{area } PMLQ = \text{area } OMLK - \text{area } OPQN.$$

The following example may serve as a first illustration:

$$\int \log x\,dx = \int \log x \,.\, 1 \,.\, dx.$$

We write the integrand in this way, in order to indicate that we intend to set $f(x) = \log x$ and $g'(x)=1$, so that we have $f'(x) = 1/x$ and $g(x) = x$. Our formula then becomes

$$\int \log x \, dx = x \log x - \int \frac{x}{x} \, dx = x \log x - x.$$

This last expression is therefore the integral of the logarithm, as may be verified at once by differentiation.

### 4.4.2 Examples:

The following additional examples may help the reader to grasp this method.

If we set $f(x) = x$, $g'(x) = e^x$, we have $f'(x) = 1$, $g(x) = e^x$, and

$$\int x e^x \, dx = e^x (x - 1).$$

In a similar way, we obtain

$$\int x \sin x \, dx = -x \cos x + \sin x \quad \text{and} \quad \int x \cos x \, dx = x \sin x + \cos x.$$

For $f(x) = \log x$, $g'(x) = x^a$, we have the relation

$$\int x^a \log x \, dx = \frac{x^{a+1}}{a+1} \left( \log x - \frac{1}{a+1} \right).$$

We must assume here that $a \neq 1$. For $a = -1$, we obtain (cf. 4.1.1)

$$\int \frac{1}{x} \log x \, dx = (\log x)^2 - \int \log x \, . \, \frac{dx}{x};$$

transferring the integral from the right-hand side to the left hand side, we have

$$\int \frac{1}{x} \log x \, dx = \frac{1}{2} (\log x)^2.$$

We calculate the integral $\int \operatorname{arc}\sin x\,dx$ by taking $f(x) = \operatorname{arsin} x$, $g'(x) = 1$, whence

$$\int \operatorname{arc}\sin x\,dx = x \operatorname{arc}\sin x - \int \frac{x\,dx}{\sqrt{(1-x^2)}}.$$

The integration on the right-hand side can be performed as in 4.2.4; we thus find that

$$\int \operatorname{arc}\sin x\,dx = x \operatorname{arc}\sin x + \sqrt{(1-x^2)}.$$

In the same way, we calculate the integral

$$\int \operatorname{arc}\tan x\,dx = x \operatorname{arc}\tan x - \frac{1}{2}\log(1+x^2)$$

and many other integrals of a similar type.

The following examples are somewhat different; a repeated application of the method of integration by parts brings us back to the original integral, for which we thus obtain an equation.

Integrating by parts twice, we obtain

$$\begin{aligned}\int e^{ax}\sin bx\,dx &= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx\\ &= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx,\end{aligned}$$

and solving this equation for the integral $\int e^{ax}\sin bx\,dx,$

$$\int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx).$$

A similar procedure yields

$$\int e^{ax} \cos bx \, dx = \frac{1}{a^2 + b^2} e^{ax} (a \cos bx + b \sin bx).$$

**4.4.3 Recurrence Formulae:** In many cases, the integrand is not only a function of the independent variable, but also of an integral index *n*; on integrating by parts, instead of the value of the integral, we obtain another similar expression in which the index *n* has a smaller value. We thus arrive, after a number of steps, at an integral, which we can process by means of our table of integrals. Such a procedure is called a **recurrence process**. The following examples illustrate this method: By repeated integration by parts, we can calculate the trigonometrical integrals

$$\int \cos^n x \, dx, \quad \int \sin^n x \, dx, \quad \int \sin^m x \cos^n x \, dx,$$

provided that *m* and *n* are integers. In fact, we find that

$$\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx;$$

here we can rewrite the right-hand side in the form

$$\cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx,$$

and thus obtain the recurrence relation

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

This formula enables us to keep on lowering the index in the integrand until we finally arrive at the integral

$$\int \cos x \, dx = \sin x \quad \text{or} \quad \int dx = x,$$

according to whether *n* is odd or even. In a similar way, we obtain the analogous recurrence formulae

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

and

$$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx.$$

In particular, these formulae enable us to compute the integrals

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x\cos x)$$

and

$$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x\cos x),$$

which we have already obtained by the method of substitution (cf. 4.2.3).

It need hardly be mentioned that the corresponding integrals for the hyperbolic functions can be calculated in exactly the same manner.

Further recurrence relations are obtained by the transformations:

$$\int (\log x)^m\,dx = x(\log x)^m - m\int (\log x)^{m-1}\,dx,$$

$$\int x^m e^x\,dx = x^m e^x - m\int x^{m-1}e^x\,dx,$$

$$\int x^m \sin x\,dx = -x^m\cos x + m\int x^{m-1}\cos x\,dx,$$

$$\int x^m \cos x\,dx = x^m\sin x - m\int x^{m-1}\sin x\,dx,$$

$$\int x^a(\log x)^m\,dx = \frac{x^{a+1}(\log x)^m}{a+1} - \frac{m}{a+1}\int x^a(\log x)^{m-1}\,dx \quad (a \neq -1).$$

**4.4.4 Wallis' Product:** The recurrence formula for the integral $\int \sin^n x\,dx$ leads on an elementary way to a most remarkable expression for the number $\pi$ as an infinite product. We suppose that $n > 1$ and insertthe limits 0 and $\pi/2$ in the formula

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

obtaining

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx \text{ for } n > 1.$$

Repeated application of the recurrence formula to the right-hand side yields for the cases $n = 2m$ and $n = 2m + 1$

$$\int_0^{\pi/2} \sin^{2m} x\,dx = \frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\int_0^{\pi/2} dx,$$

$$\int_0^{\pi/2} \sin^{2m+1} x\,dx = \frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\frac{2}{3}\cdot\int_0^{\pi/2} \sin x\,dx,$$

whence

$$\int_0^{\pi/2} \sin^{2m} x\,dx = \frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2},$$

$$\int_0^{\pi/2} \sin^{2m+1} x\,dx = \frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\frac{2}{3}.$$

Division now yields

$$\frac{\pi}{2} = \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdots \frac{2m.2m}{(2m-1).(2m+1)} \; \frac{\int_0^{\pi/2} \sin^{2m} x\, dx}{\int_0^{\pi/2} \sin^{2m+1} x\, dx}.$$

The quotient of the two integrals on the right hand side converges to 1 as *m* increases, as we recognize from the following considerations. In the interval $0<x<\pi/2$, we find

$$0 < \sin^{2m+1} x \leqq \sin^{2m} x \leqq \sin^{2m-1} x,$$

whence

$$0 < \int_0^{\pi/2} \sin^{2m+1} x\, dx \leqq \int_0^{\pi/2} \sin^{2m} x\, dx \leqq \int_0^{\pi/2} \sin^{2m-1} x\, dx.$$

If we here divide each term by $\int_0^{\pi/2} \sin^{2m+1} x\; dx$ and note that by the first formula, proved above,

$$\frac{\int_0^{\pi/2} \sin^{2m-1} x\, dx}{\int_0^{\pi/2} \sin^{2m+1} x\, dx} = \frac{2m+1}{2m} = 1 + \frac{1}{2m},$$

we find that

$$1 \leqq \frac{\int_0^{\pi/2} \sin^{2m} x\, dx}{\int_0^{\pi/2} \sin^{2m+1} x\, dx} \leqq 1 + \frac{1}{2m},$$

which yields the statement above, whence there applies the relation

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\cdots\frac{2m}{2m-1}\,\frac{2m}{2m+1}.$$

This product formula (due to Wallis), with its simple law of formation, presents a remarkable relation between the number $\pi$ and the integers. If we observe that

$$\lim_{m\to\infty} \frac{2m}{2m+1} = 1,$$

we can write

and, if we take the square root and then multiply numerator and denominator by 2 · 4 ··· ,··· (2m — 2), we find

$$\sqrt{\frac{\pi}{2}} = \lim_{m\to\infty} \frac{2.4\ldots(2m-2)}{3.5\ldots(2m-1)}\sqrt{2m} = \lim_{m\to\infty} \frac{2^2.4^2\ldots(2m-2)^2}{(2m-1)!}\sqrt{2m}$$

$$= \lim_{m\to\infty} \frac{2^2.4^2\ldots(2m)^2}{(2m)!}\,\frac{\sqrt{2m}}{2m}.$$

Hence, we finally obtain

$$\lim_{m\to\infty} \frac{(m!)^2\,2^{2m}}{(2m)!\,\sqrt{m}} = \sqrt{\pi},$$

a form of Wallis' product which will be of use in the Appendix to Chapter VII.

**Exercises 4.2:** Evaluate the integrals 1 - 14:

15. **Prove the formula**

$$\int e^x p(x)\,dx = e^x\{p(x) - p'(x) + p''(x) - + \cdots\},$$

where $p(x)$ is any polynomial.

1. $\displaystyle\int \frac{x\cos x}{\sin^2 x}\,dx.$ 2. $\displaystyle\int \frac{x^7}{(1-x^4)^2}\,dx.$ 3. $\displaystyle\int x^2\cos x\,dx.$

4. $\displaystyle\int x^3 e^{-x^2}\,dx.$ 5. $\displaystyle\int_{-\pi}^{\pi} x^2\cos nx\,dx$ ($n$ a positive integer).

6. $\displaystyle\int_{-\pi}^{\pi} x^2\sin nx\,dx$ ($n$ a positive integer). 7. $\displaystyle\int x^3\cos x^2\,dx.$

8. $\displaystyle\int \sin^4 x\,dx.$ 9. $\displaystyle\int \cos^6 x\,dx.$ 10. $\displaystyle\int x^4\sqrt{1-x^2}\,dx.$

11. $\displaystyle\int x^2 e^x\,dx.$ 12. $\displaystyle\int \frac{\log x}{x^n}\,dx$ $(n \neq 1).$

13. $\displaystyle\int x^m\log x\,dx$ $(m \neq 1).$ 14. $\displaystyle\int x^2(\log x)^2\,dx.$

16. Show that for all odd positive values of $n$ the integral $\int e^{-x^2}x^n\,dx$ can be evaluated in terms of elementary functions.

17. Show that, if $n$ is even, the integral $\int e^{-x^2}x^n\,dx$ can be evaluated in terms of elementary functions and the integral $\int e^{-x^2}dx$ (for which tables have been computer).

18. Prove that

$$\int_0^x\left(\int_0^u f(t)\,dt\right)du = \int_0^x f(u)(x-u)\,du.$$

19.* Exercise 18. yields a formula for the second iterated integral. Prove that the $n$-th iterated integral of $f(x)$ is given by

$$\frac{1}{(n-1)!}\int_0^x f(u)(x-u)^{n-1}\,du.$$

Answers and Hints

# 4.5 Integration of Rational Functions

The most important general class of functions, which are integrable in terms of elementary functions, are the rational functions

$$R(x) = \frac{f(x)}{g(x)},$$

where $f(x)$ and $g(x)$ are the polynomials:

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_0,$$
$$g(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0 \; (b_n \neq 0).$$

We recall that every polynomial can be integrated at once and that the integral is itself a polynomial, whence we need only consider those rational functions for which the denominator is not a constant. Moreover, we can always assume that the degree of the numerator is less than the degree ($n$) of the denominator, because otherwise we can divide the polynomial $f(x)$ by the polynomial $g(x)$ and obtain a remainder of degree less than $n$; in other words, we can write $f(x)=q(x)g(x)+r(x)$, where $q(x)$ and $r(x)$ are also polynomials and $r(x)$ is of degree less than $n$. The integration of $f(x)/g(x)$ is then reduced to the integration of the polynomial $q(x)$ and of the **proper fraction** $r(x)/g(x)$. Moreover, we note that the $f(x)/g(x)$ can be represented as the sum of the functions $a_\nu x^\nu/g(x)$, so that we need only consider integrands of the form $x^\nu/g(x)$.

**4.5.1 The Fundamental Types:** We shall not at once proceed to the integration of the most general rational function of the above type, but instead shall study only those functions in which the denominator $g(x)$ is of a particularly simple type, namely, $g(x) = (\alpha x + \beta)^n$ - a power of a linear $\alpha x+\beta$ $(\alpha \neq 0)$ - or $g(x) = (ax^2+2bx+c)^n$ - a power of a definite * quadratic expression. In the first case, we introduce a new variable $\xi = \alpha x + \beta$. Then $d\xi/dx = \alpha$ and $x = (\xi - \beta)/\alpha$ is also a linear function of $\xi$. Each numerator $f(x)$ becomes a polynomial $\phi(\xi)$ of the same degree, whence

$$\int \frac{f(x)}{(\alpha x+\beta)^n}\,dx = \frac{1}{\alpha}\int \frac{\phi(\xi)}{\xi^n}\,d\xi.$$

* A **quadratic expression** $Q(x) = \alpha x^2 + 2bx + c$ is said to be **definite**, if it takes for for all real values of $x$ values with one the same sign, i.e., if the equation $Q(x) = 0$ has no real roots. For this, it is necessary and sufficient that $ac\text{-}b^2$ is positive.

In the second case, we write

$$ax^2 + 2bx + c = \frac{1}{a}(ax+b)^2 + \frac{d^2}{a} \qquad (d^2 = ac - b^2,\ d > 0),$$

observing that, since we have assumed our expression to be definite, $ac - b^2$ must be positive and $a \neq 0$. After introducing the new variable

$$\xi = \frac{ax+b}{d}$$

we obtain an integral with the denominator

$$\left[\frac{d^2}{a}(1+\xi^2)\right]^n.$$

Hence, in order to integrate rational functions the denominators of which are powers of a linear expression or of a definite quadratic expression, it is sufficient to be able to integrate the functional types

$$\frac{1}{x^n},\quad \frac{x^{2\nu}}{(x^2+1)^n},\quad \frac{x^{2\nu+1}}{(x^2+1)^n}.$$

In fact, we shall see that, in general, even these types need not be treated, for we can reduce the integration of every rational function to the integration of the very special forms of these three functions, obtained by taking $\nu = 0$. Hence, we will now consider the integration of the three expressions

$$\frac{1}{x^n}, \quad \frac{1}{(x^2+1)^n}, \quad \frac{x}{(x^2+1)^n}.$$

**4.5.2 Integration of the Fundamental Types:** The integration of the first type of function $1/x^n$ yields immediately log $|x|$, if $n = 1$ and $-1/\{(n - 1)x^{n-1}\}$, if $n > 1$, so that in both cases the integral is again an elementary function. Functions of the third type can be integrated immediately after introduction of the new variable $\xi = x^2 + 1$, whence we obtain $2xdx = d\xi$ and

$$\int \frac{x}{(x^2+1)^n}\,dx = \frac{1}{2}\int \frac{d\xi}{\xi^n} = \begin{cases} \frac{1}{2}\log(x^2+1) & \text{if } n = 1, \\ -\dfrac{1}{2(n-1)(x^2+1)^{n-1}} & \text{if } n > 1. \end{cases}$$

Finally, in order to calculate the integral

$$I_n = \int \frac{dx}{(x^2+1)^n},$$

where $n$ has any value exceeding 1, we use the **recurrence method**. In fact, if we set

$$\frac{1}{(x^2+1)^n} = \frac{1}{(x^2+1)^{n-1}} - \frac{x^2}{(x^2+1)^n},$$

we can transform the right-hand side by integration by parts, using formulae in 4.3 with

$$f(x) = x, \quad g'(x) = \frac{x}{(x^2+1)^n}.$$

Then, as we have just discovered above,

$$g(x) = -\frac{1}{2}\,\frac{1}{(n-1)(x^2+1)^{n-1}},$$

whence we obtain

$$I_n = \int \frac{dx}{(x^2+1)^n} = \frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{2n-3}{2(n-1)}\int \frac{dx}{(x^2+1)^{n-1}}.$$

The calculation of the integral $I_n$ is thus reduced to that of the integral $I_{n-1}$. If $n - 1 > 1$, we apply the same process to the latter integral and continue the process until we finally arrive at the expression

$$\int \frac{dx}{(x^2+1)} = \text{arc tan } x.$$

We thus see that the integral * $I_n$ can be explicitly expressed in terms of rational functions and the function artan $x$.

* The integral of the function $1/(x^2 - 1)^n$ can be calculated in the same way; we reduce it by the corresponding recurrence method to the integral

$$\int \frac{dx}{1-x^2} = \text{artanh}\, x \quad (\text{or arcoth}\, x).$$

Incidentally, we could also have integrated the function $1/(x^2 + 1)^n$ directly using the substitution $x$=tanh $t$; we should then have obtained $dx = \sec^2 t\, dt$ and $1/(1 + x^2) = \cos^2 t$, so that

$$\int \frac{dx}{(x^2+1)^n} = \int \cos^{2n-2} t\, dt,$$

and we have already learnt in 4.4.3 how to evaluate this integral.

**4.5.3 Partial fractions:** We are now in a position to integrate the most general rational functions by virtue of the fact that every such function can be represented as the sum of so-called **partial fractions**, i.e., as the sum of a polynomial and a finite number of rational functions, each one of which has either a power of a linear expression as its denominator and a constant for its numerator or a power of a definite quadratic expression as its denominator and a linear function as its numerator. If the degree of the numerator $f(x)$ is less than that of the denominator $g(x)$, the polynomial does not occur. We can now integrate each partial fraction, since, by 4.5, the denominator can be reduced to one of the special form $x^n$ and $(x^2 + 1)^n$, and the fraction is then a combination of the fundamental types integrated in 4.5.2.

We shall not give the general proof of the possibility of this resolution into partial fractions. On the contrary, we shall confine ourselves to stating the theorem in an form intelligible for the reader and to showing by examples how the resolution into partial fractions can be carried out in typical cases. In practice, one deals only with comparatively simple functions, since otherwise the computations become far too complicated.

As we know from elementary algebra, every polynomial $g(x)$ can be written in the form

$$g(x)=a(x-a_1)^{l_1}(x-a_2)^{l_2}\dots(x^2+2b_1x+c_1)^{r_1}(x^2+2b_2x+c_2)^{r_2}\dots$$

Here, the numbers $a_1$, $a_2$, ··· are the real and distinct roots of the equation $g(x)=0$ and the positive integers $l_1$, $l_2$, ··· indicate how often they are repeated; the factors $x^2 + 2b_\nu x + c_\nu$ indicate definite quadratic expressions, of which no two are the same, with conjugate complex roots and the positive integers $r_1, r_2, \cdots$ give the numbers of times that these roots are repeated.

We assume that the denominator is either given to us in this form or else that we have brought it to this form by calculating the real and imaginary roots. Moreover, let us assume that the numerator $f(x)$ has a lower degree than the denominator (cf. 4.5). Then the theorem on the resolution into partial fractions is:

For each factor $(x - \alpha)^l$, where $\alpha$ is any one of the real roots and $l$ is the number of times it is repeated, we can determine an expression of the form

$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\dots+\frac{A_l}{(x-a)^l},$$

and for each quadratic factor $Q(x) = x^2 + 2bx + c$ in our product, which is raised to the power $r$, we can determine an expression of the form

$$\frac{B_1+C_1x}{Q}+\frac{B_2+C_2x}{Q^2}+\dots+\frac{B_r+C_rx}{Q^r},$$

in such a way that the function $f(x)/g(x)$ is the sum of all these expressions. In other words, the quotient $f(x)/g(x)$ can be represented by a sum of fractions, each of which belongs to one or another of the types integrated in 4.5.2.

We present here a brief outline of the method by which the possibility of this decomposition into partial fractions is proved. If $g(x) = (x - a)^k h(x)$ and $h(a)\neq 0$, then obviously there vanishes on the right hand side of the equation

$$\frac{f(x)}{g(x)}-\frac{f(a)}{h(a)(x-a)^k}=\frac{1}{h(a)}\,\frac{f(x)h(a)-f(a)h(x)}{(x-a)^k\,h(x)}$$

the numerator for $x = -a$, whence it has the form $h(a)(x - a)^m f_1(x)$, where $f_1(x)$ is also a polynomial, the integer $m \geq 1$ and $f_1(a) \neq 0$. Writing $f(a)/h(a) = \beta$, we obtain

$$\frac{f(x)}{g(x)} - \frac{\beta}{(x-a)^k} = \frac{f_1(x)}{(x-a)^{k-m} h(x)}.$$

Continuing the process, we can keep on reducing the degree of the power of ($x$-$a$) in the denominator until finally no such factor is left. We repeat for the remaining fraction the process for some other roots of $g(x)$ and do so as many times as $g$(») has distinct factors. This is done not only for the real, but also for the complex roots until we arrive at the complete decomposition into partial fractions.

In particular cases, the splitting into partial fractions can be done easily by inspection. If, for example, $g(x) = x^2 - 1$, we see at once that

$$\frac{1}{x^2-1} = \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1},$$

whence

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|.$$

More generally, if $g(x) = (x - \alpha)(x - \beta)$, that is, if $g(x)$ is a non-definite quadratic expression with two real series $\alpha$ and $\beta$, we have

$$\frac{1}{(x-\alpha)(x-\beta)} = \frac{1}{\alpha-\beta}\frac{1}{x-\alpha} - \frac{1}{\alpha-\beta}\frac{1}{x-\beta},$$

whence

$$\int \frac{dx}{(x-\alpha)(x-\beta)} = \frac{1}{\alpha-\beta}\log\left|\frac{x-\alpha}{x-\beta}\right|.$$

**4.5.4 Example. The Bi-molecular Reaction:** A simple example of the application of this easy reduction to partial fractions is given by the so-called bimolecular reaction. Let there be given two reagents the original concentrations of which in mols per until volume are $a$ and $b$, where we assume that $a < b$; we will assume that there is formed at time $t$ in the unit volume a quantity $x$ (mols) of the reaction product. Then, according to the law of mass action, in the simplest case - reaction between one molecule of each of the reagents - the rate of increase of the quantity $x$ is

given by the equation $dx/dt = k(a - x)\{b - x)$. The problem is to determine the function $x(t)$. Conversely, if we think of the time $t$ as a function of «, we have

$$\frac{dt}{dx} = \frac{1}{k(a-x)(b-x)} = \frac{1}{k(b-a)}\left(\frac{1}{a-x} - \frac{1}{b-x}\right);$$

integration now yields

$$kt = \frac{1}{a-b}\log\frac{a-x}{b-x} + c, \quad \text{for } x < a < b.$$

We determine the constant of integration $c$ by the condition that at time $t = 0$ there has not yet formed a product of reaction, so that

$$\frac{1}{a-b}\log\frac{a}{b} + c = 0.$$

Thus, we finally obtain

$$kt = \frac{1}{a-b}\log\frac{1-\frac{x}{a}}{1-\frac{x}{b}},$$

and, by solving for $x$, the required function

$$x = \frac{ab(1-e^{(a-b)kt})}{b-ae^{(a-b)kt}}.$$

**4.5.5 Further Examples of Resolution into Partial Fractions. The Method of Undetermined Coefficients:** If $g(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$, where $\alpha_i \neq \alpha_k$, $i \neq k$, i.e., if the equation $g(x) = 0$ has only single real roots, the expression in terms of partial fractions has the simple form

$$\frac{1}{g(x)} = \frac{a_1}{x-a_1} + \frac{a_2}{x-a_2} + \dots + \frac{a_n}{x-a_n}.$$

We obtain explicit expressions for the coefficients $a_1$, $a_2$, ⋯, if we multiply both sides of this equation by $(x - a_1)$, cancel the common factor $(x - a_1)$ in the numerator and denominator on the left hand side and in the first term on the right hand side, and then put $x = a_1$. This yields

$$a_1 = \frac{1}{(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n)}.$$

The reader will observe that the denominator on the right hand side is $g'(a_1)$, i.e., the derivative of the function $p(x)$ at the point $x = \alpha_1$.

As a typical example of a denominator $g(x)$ with multiple roots, consider the function $1/\{x^2(x-1)\}$. The preliminary statement

$$\frac{1}{x^2(x-1)} = \frac{a}{x-1} + \frac{b}{x} + \frac{c}{x^2}$$

as shown in 4.5.3 leads us to the required result. If we multiply both sides of this equation by $x^2(x-1)$, we obtain

$$1 = (a+b)x^2 - (b-c)x - c,$$

which is true for all values of $x$, from which we have to determine the coefficients $a$, $b$, $c$. This condition cannot hold unless all the coefficients of the polynomial $(a+b)x^2 - (b—c)x - c - 1$ are zero, i.e., we must have

$$a + b = b - c = c + 1 = 0 \text{ or } c = -1,\ b = -1,\ a = 1.$$

Thus, we obtain the resolution

$$\frac{1}{x^2(x-1)} = \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2},$$

and consequently

$$\int \frac{dx}{x^2(x-1)} = \log|x-1| - \log|x| + \frac{1}{x}.$$

We shall now split up the function $1/[x(x^2 + 1)]$ - an example of the case when the zeroes of the denominator are complex - in accordance with the equation

$$\frac{1}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1}.$$

We obtain for the coefficients $a + b = c = a - 1$, so that

$$\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1},$$

and consequently

$$\int \frac{dx}{x(x^2+1)} = \log|x| - \frac{1}{2}\log(x^2+1).$$

We consider as a third example the function $1/(x^4 + 1)$. Even Leibnitz found this to be a troublesome integration. We can represent the denominator as the product of two quadratic factors:

$$x^4 + 1 = (x^2+1)^2 - 2x^2 = (x^2 + 1 + \sqrt{2}x)(x^2 + 1 - \sqrt{2}x).$$

We now know that the resolution into partial fractions will have the form

$$\frac{1}{x^4+1} = \frac{ax+b}{x^2+\sqrt{2}x+1} + \frac{cx+d}{x^2-\sqrt{2}x+1}.$$

In order to determine the coefficients $a$, $b$, $c$, $d$, we have the equation

$$(a+c)x^3 + (b + d - a\sqrt{2} + c\sqrt{2})x^2 + (a + c - b\sqrt{2} + d\sqrt{2})x + (b + d - 1) = 0,$$

which is satisfied by the values

$$a = \frac{1}{2\sqrt{2}}, \quad b = \frac{1}{2}, \quad c = -\frac{1}{2\sqrt{2}}, \quad d = \frac{1}{2},$$

whence

$$\frac{1}{x^4+1} = \frac{1}{2\sqrt{2}} \cdot \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{1}{2\sqrt{2}} \cdot \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1},$$

and, applying the method in 4.5.1, we obtain

$$\int \frac{dx}{x^4+1} = \frac{1}{4\sqrt{2}} \log|x^2+\sqrt{2}x+1| - \frac{1}{4\sqrt{2}} \log|x^2-\sqrt{2}x+1|$$
$$+ \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}x+1) + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}x-1),$$

which is readily verified by differentiation.

**Exercises 4.3:**

Evaluate the integrals

1. $\int \frac{dx}{2x-3x^2}.$
2. $\int \frac{dx}{x^3-x}.$
3. $\int \frac{3\,dx}{x(x+1)^3}.$
4. $\int \frac{x^2+x+1}{3x^2-2x-5}\,dx.$
5. $\int \frac{dx}{(x-1)^2(x^2+1)}.$
6. $\int \frac{x^2\,dx}{(x-1)^2(x^2+1)}.$
7. $\int \frac{dx}{1-x^3}.$
8. $\int \frac{dx}{1+x^3}.$
9. $\int \frac{(x-4)}{(x^2+1)(x-2)}\,dx.$
10. $\int \frac{x+4}{(x^2-1)(x+2)}\,dx.$
11. $\int \frac{x^6}{1-x^4}\,dx.$
12.* $\int \frac{dx}{x^6+1}.$
13. $\int \frac{x^2}{x^4+x^2-2}\,dx.$
14. $\int \frac{dx}{x^2(x^2+1)^2}.$

# 4.6 Integration of certain other Classes of Functions

## 4.6.1 Preliminary Remarks on the Rational Representation of the Trigonometric and Hyperbolic Functions:

The integration of certain other general classes of functions can be reduced to the integratioa of rational functions. We shall better understand this reduction, if we begin by stating certain elementary facts about the trigonometric and hyperbolic functions. If we set $t = \tan x/2$, elementary trigonometry yields the simple formulae

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2};$$

in fact,

$$\frac{1}{1+t^2} = \cos^2\frac{x}{2} \quad \text{and} \quad \frac{t^2}{1+t^2} = \sin^2\frac{x}{2},$$

whence we obtain the above equations from the elementary formulae

$$\sin x = 2\cos^2\frac{x}{2}\tan\frac{x}{2} \quad \text{and} \quad \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}.$$

These equations show that sin $x$ and cos $x$ can be expressed rationally in terms of the quamtity $t=\tan x/2$. Differentiation now yields

$$\frac{dt}{dx} = \frac{1}{2\cos^2 x/2} = \frac{1+t^2}{2},$$

whence

$$\frac{dx}{dt} = \frac{2}{1+t^2},$$

whence the derivative $dx/dt$ is also a rational expression in $t$.

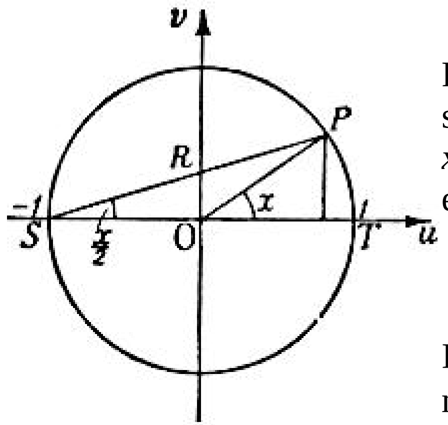



**Fig. 2.—Parametric representation of the trigonometric functions**

Fig. 2 shows the geometrical representation and the geometrical meaning of our formulae. It shows the circle $u^2 + v^2 = 1$ in a *uv*-plane. If $x$ denotes the angle *POT*, then $u = \cos x$ and $v = \sin x$. By a theorem in elementaly geometry, the angle *OSP* with its vertex at the point $u$=-1, $v$=0 is equal to $x/2$, and we can deduce from the figure the geometrical meaning of the parameter:

$$t = \tan x/2 = OR.$$

If the point $P$ starts from $S$ and moves once around the circle in the positive direction, i.e., if $x$ moves through the interval from $-\pi$ to $+\pi$, the quantity $t$ will move exactly once throngh the entire range of the values from $-\infty$ to $+\infty$.

We may correspondingly express the hyperbolic functions $\cosh x = (e^x+e^{-x})/2$ and $\sinh x=(e^x-e^{-x})/2$ as rational functions of a third quantity. The most obvious way is to set $e^x = \tau$, so that we have for these hyperbolic functions the rational expressions

$$\cosh x = \frac{1}{2}\left(\tau + \frac{1}{\tau}\right), \quad \sinh x = \frac{1}{2}\left(\tau - \frac{1}{\tau}\right).$$

Once again, $dx/dt = 1/\tau$. However, we obtain a closer analogy with the trigonometric functions by introduction of the quantity $t = \tanh x/2$; we then arrive at the formulae

$$\sinh x = \frac{2t}{1-t^2}, \quad \cosh x = \frac{1+t^2}{1-t^2}.$$

Differentiation of $t = \tanh x/2$ yields this time the rational expression

$$\frac{dx}{dt} = \frac{2}{1-t^2}$$

for the derivative $dx/dt$. Once again, the quantity $t$ has a geometrical meaning similar to that which it has in the case of the trigonometric functions, as we see at once from Fig. 3.

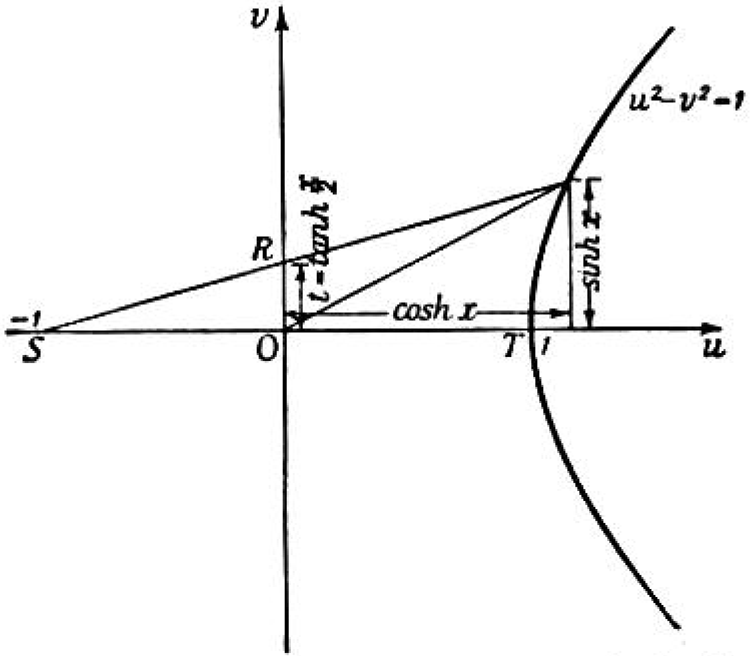



Fig. 3.—Parametric representation of the hyperbolic functions

However, while in the case of the trigonometrio functions, *t* must run through the entire range of values from - ∞ to +∞, in order to yield all pairs of values of cos *x* and sin *x*, in the case of the hyperbolic functions, *t* is limited to the interval -1 < t < l.

After these preliminary remarks, we turn to our integration problem.

**4.6.2 Integration of *R*(cos *x*, sin *x*):** Let *R*(cos *x*, sin *x*) denote an expression which ia rational in the two functions sin *x* and cos *x*, i.e., an expression which is formed rationally from these two functions and given constants such as

$$\frac{3\sin^2 x + \cos x}{3\cos^2 x + \sin x}.$$

If we apply the substitution $t = \tan x/2$, the integral

$$\int R(\cos x, \sin x)\, dx$$

becomes the integral

$$\int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \frac{2}{1+t^2}\, dt,$$

and we have now under the integral sign a rational function of *t*. Thus, we have obtained theoretically the integral of our expression, since we can now perform the integration by the methods of the preceding section.

**4.6.3 Integration of *R*(cosh *x*, sinh *x*):** In the same way, if *R*(cosh *x*, sinh *x*) is an expression which is rational in terms of the hyperbolic functions cosh *x* and sinh *x*, we can integrate it by means of the substitution $t = \tanh x/2$. Recalling that

$$\frac{dx}{dt} = \frac{2}{1-t^2}$$

we find

$$\int R(\cosh x, \sinh x)\, dx = \int R\left(\frac{1+t^2}{1-t^2}, \frac{2t}{1-t^2}\right) \frac{2}{1-t^2}\, dt.$$

(According to a previous remark, we could also have introduced $\tau = e^x$ as a new variable and expressed cosh *x* and sinh *x* in terms of *t*.) The integration is once again reduced to that of a rational function.

**4.6.4 Integration of $R\{x, \sqrt{(1-x^2)}\}$ :** The integral $R\{x, \sqrt{(1-x^2)}\}$ can be reduced to the type in 4.6.2 by using the substitution

$$x = \cos u,\ \sqrt{(1-x^2)} = \sin u, \quad dx = -\sin u\, du;$$

the transformation $t = \tan x/2$ leads to the integration of a rational function. By the way, we could have carried out the reduction in a single step by the substitution

$$t = \sqrt{\left(\frac{1-x}{1+x}\right)};\ x = \frac{1-t^2}{1+t^2};\ \sqrt{(1-x^2)} = \frac{2t}{1+t^2};\ \frac{dx}{dt} = \frac{-4t}{(1+t^2)^2},$$

that is, we could have obtained directly a rational function by introduction of a new variable $t$=tan$u$/2.

**4.6.5 Integration of** $R\{x, \sqrt{(x^2-1)}\}$**:** The integral of this function is transformed by the substitution $x$ = cosh $u$ into the type 4.6.3, when we again introduce

$$t=\sqrt{\left(\frac{x-1}{x+1}\right)}=\tanh\frac{u}{2}.$$

**4.6.6 Integration of** $R\{x, \sqrt{(x^2+1)}\}$ **:** The integral of this function is transformed by the substitution $x$ = sinh $u$ into the type 4.6.3, whence it can be integrated in terms of elementary functions. Instead of the further reduction to an integral of a rational function by the substitution $e^u=\tau$ or tanh $u$/2 = $t$, we could have reached the integral of a rational function by a single step by either of the substitutions

$$\tau=x+\sqrt{(x^2+1)},\quad t=\frac{-1+\sqrt{(x^2+1)}}{x}.$$

**4.6.7 Integration of** $R\{x, \sqrt{(ax^2+2bx+c)}\}$ **:**The integral of this function is rational in terms of $x$ and the square root of an arbitrry polynomial of the second degree in $x$; it can be reduced immediately to one of the types above. As in 4.5.1,we write

$$ax^2+2bx+c=\frac{1}{a}(ax+b)^2+\frac{ac-b^2}{a}.$$

If $ac - b^2 > 0$, we introduce a new variable $\xi$= by means of the transformation $\xi=\frac{ax+b}{\sqrt{(ac-b^2)}}$, whence the surd takes the form $\sqrt{\left\{\frac{ac-b^2}{a}(\xi^2+1)\right\}}$. Expressed in terms of $\xi$, we have now the type of 4.6.6. The constant $a$ must be positive here, in order that the square root may have real values.

If $ac - b^2 = 0$, $a > 0$, then we see from

$$\sqrt{(ax^2+2bx+c)} = \sqrt{a}\left(x+\frac{b}{a}\right)$$

that the integrand was rational from the start.

Finally, if $ac - b^2 < 0$, we set $\xi = \frac{ax+b}{\sqrt{(b^2-ac)}}$ and obtain for the surd the expression $\sqrt{\left\{\frac{b^2-ac}{a}(\xi^2-1)\right\}}$. If $a$ is positive, our integral is thus reduced to the type 4.6.5; on the other hand, if $a$ is negative, we write the surd in the form $\sqrt{\left\{\frac{b^2-ac}{a}(\xi^2-1)\right\}}$ and see that the integral is thus reduced to the type of 4.6.4

**4.6.8 Further Examples of Reduction to Integrals of Rational Functions:** We shall briefly mention two other types of functions which can be integrated by reduction to rational functions:

(1) Rational expressions involving two different surds of linear expressions

$$R\{x, \sqrt{(ax+b)}, \sqrt{(\alpha x+\beta)}\};$$

(2) Expressions of the form

$$R\left\{x, \sqrt[n]{\left(\frac{ax+b}{\alpha x+\beta}\right)}\right\},$$

where $a$, $b$, $\alpha$, $\beta$ are constants. In (1), we introduce the new variable $\xi = \sqrt{(\alpha x+\beta)}$, so that $\alpha x+\beta=\xi^2$, whence

$$x = \frac{\xi^2-\beta}{\alpha} \quad \text{and} \quad \frac{dx}{d\xi} = \frac{2\xi}{\alpha};$$

then

$$\int R\{x, \sqrt{(ax+b)}, \sqrt{(\alpha x+\beta)}\}\,dx$$
$$= \int R\left\{\frac{\xi^2-\beta}{\alpha}, \sqrt{\frac{1}{\alpha}\{a\xi^2-(a\beta-b\alpha)\}}, \xi\right\}\frac{2\xi}{\alpha}\,d\xi,$$

which is of the type 4.6.7.

If we introduce in the second case the new variable

$$\xi = \sqrt[n]{\left(\frac{ax+b}{\alpha x+\beta}\right)},$$

we have

$$\xi^n = \frac{ax+b}{\alpha x+\beta}, \quad x = \frac{-\beta\xi^n+b}{\alpha\xi^n-a}, \quad \frac{dx}{d\xi} = \frac{a\beta-b\alpha}{(\alpha\xi^n-a)^2}\,.\,n\xi^{n-1},$$

and immediately arrive at

$$\int R\left(x, \sqrt[n]{\left(\frac{ax+b}{\alpha x+\beta}\right)}\right)dx$$
$$= \int R\left(\frac{-\beta\xi^n+b}{\alpha\xi^n-a}, \quad \xi\right)\frac{a\beta-b\alpha}{(\alpha\xi^n-a)^2}\,n\xi^{n-1}\,d\xi,$$

which is an integral of a rational function.

**4.6.9 Comments on Exercises:** The preceding discussion is mainly of theoretical interest. In the case of complicated expressions, the actual calculations would be far too involved. It is therefore expedient to employ, whenever it is possible, a special form of the integrand, in order to simplify the work. For example, in order to integrate the expression

$$1/(a^2\sin^2 x + b^2\cos^2 x),$$

it is better to use the substitution $t = \tan x$ instead of that given in 4.6.2, since $\sin^2 x$ and $\cos^2 x$ can be expressed rationally in terms of $\tan x$ and it is therefore unnecessary to go back to $t = \tan x/2$. The same is true for every expression formed rationally from $\sin^2 x$, $\cos^2 x$ and $\sin x \cos x$*, because $\sin x \cos x = \tan x \cos^2 x$ can, of course, be expanded in terms of $\tan x$. Moreover, for the calculation of many integrals, a trigonometrical form is to be preferred to a rational one, provided that the trigonometrical form can be evaluated by some simple recurrence method.

For example, although the integrand in $\int x^n \{\sqrt{(1-x^2)}\}^m \, dx$ can be reduced to a rational form, it is simpler to write $x = \sin u$ and convert it into $\int \sin^n u \cos^{m+1} u \, du,$ since this can be treated easily by the recurrence method in 4.4.3 (or by using the addition theorems to reduce the powers of the sine and cosine to sines and cosines of multiple angles).

For the evaluation of the integral

$$\int \frac{dx}{a \cos x + b \sin x} \quad (a^2 + b^2 > 0),$$

we determine, instead of referring to the general theory, a number $A$ and an angle $\theta$ in such a way that

$$a = A \sin \theta, \quad b = A \cos \theta,$$

that is, we write

$$A = \sqrt{a^2 + b^2}, \quad \sin \theta = \frac{a}{A}, \quad \cos \theta = \frac{b}{A}.$$

The integral then becomes

$$\frac{1}{A} \int \frac{dx}{\sin (x + \theta)},$$

and, on introducing the new variable $x+\theta$, (4.2.3) yields the value of the integral

$$\frac{1}{4}\log\left|\tan\frac{x+\theta}{2}\right|.$$

## Exercises 4.4:

Evaluate the integrals

1. $\int \frac{dx}{1+\sin x}$.

2. $\int \frac{dx}{1+\cos x}$.

3. $\int \frac{dx}{2+\sin x}$.

4. $\int \frac{dx}{\sin^3 x}$.

5. $\int \frac{dx}{\cos x}$.

6. $\int_0^{\pi/2} \frac{dx}{3+\cos x}$.

7. $\int \frac{dx}{1+\cos^2 x}$.

8. $\int \frac{dx}{3+\sin^2 x}$.

9. $\int \tan^3 x \, dx$.

10. $\int \frac{dx}{\sin x+\cos x}$.

11. $\int \frac{\sin^2 x+\cos^3 x}{3\cos^2 x+\sin^4 x}\sin x \, dx$.

12. $\int \sqrt{(x^2-4)}\, dx$.

13. $\int \sqrt{(4+9x^2)}\, dx$.

14. $\int \frac{dx}{(x-2)\sqrt{(x^2-4x+3)}}$.

15. $\int x\sqrt{(x^2+4x)}\, dx$.

16. $\int \frac{dx}{\sqrt{x}+\sqrt{(1-x)}}$.

17. $\int \frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}\, dx$.

18. $\int \frac{\sqrt{(x-a)}}{1+\sqrt{(x-a+1)}}\, dx$.

19. $\int \frac{dx}{\sqrt{(x-a)}+\sqrt{(x-b)}}$.

Answers and Hints

### 4.7 Remarks on Functions which are not Integrable in Terms of Elementary Functions

**4.7.1 Definition of Functions by means of Integrals. Elliptic Integrals:** The above examples of types of functions which can be integrated by reduction to rational functions practically exhaust the list of such functions. Attempts to express general integrals such as

$$\int \frac{dx}{\sqrt{(a_0 + a_1 x + \ldots + a_n x^n)}}, \quad \int \sqrt{(a_0 + a_1 x + \ldots + a_n x^n)}\, dx$$

or $\int \frac{e^x}{x}\, dx$ in terms of elementary functions have always ended up in failure; and finally, in the Nineteenth Century, it has been proved that it is actually impossible to cany out integrations in terms of elementary functions. Hence, if the objective of the integral calculus were integration of functions in terms of elementary functions, we should have come to a definite halt. But such a restricted objective has no intrinsic justiflcation; indeed, it is of a somewhat artificial nature. We know that the integral of every continuous function exists and is itself a continuona function of the upper limit, and this fact has nothing to do with the question whether an integral can be expressed in terms of elementary functions or not. The distinguishing features of the elementary functions are based on the fact that their properties are easily recognized, that their application to numerical problems is often facilitated by convenient tables or, as in the case of the rational functions, that they are readily calculated with as great a degree of accuracy as we please.

Whenever an integral of a function cannot be expressed in terms of functions with which we are already acquainted, there is nothing to stop us from introducing such an integral as a new **higher function** into analysis, which really means no more than giving it a name. Whether the introduction of such a new function is convenient or not depends on the its properties, the frequency at which it occurs and the ease with which it can be manipulated in theory and in practice. Hence, in this sense, the process of integration forms a base for the generation of new functions.

After all, we are already acquainted with this principle from our dealings with the elementary functions. Thus, we found ourselves obliged in 3.6 to introduce the previously unknown integral of $1/x$ as a new function, which we called the logarithm and the properties of which we could easily determine. We could have introduced the trigonometric functions in a similar way, only making use of the rational functions, the process of integration, and the process of inversion. For this purpose, we only need introduce one or the other of the equations

$$\operatorname{arc}\tan x = \int_0^x \frac{dt}{1+t^2} \quad \text{or} \quad \operatorname{arc}\sin x = \int_0^x \frac{dt}{\sqrt{(1-t^2)}}$$

as the definitions of the function artan *x* or arsin *x*, respectively, in order to arrive at the trigonometric functions by inversion. By this process, the definition of these functions becomes separated from geometry, but we are naturally left with the task of also developing their properties independently of geometry.

We shall not go here into the development of these ideas. The essential step is to prove the addition theorem for the inverse functions, i,e. for sin and tan.

The first and most important example, which leads us beyond the region of elementary functions, is are the **elliptic integrals**. These are integrals in which the integrand is formed in a rational way from the variable of integration and the square root of an expression of the third or fourth degree. Among these integrals, the function

$$u(s) = \int_0^s \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

turns out to be of special importance. Its inverse function *s*(*u*) has a correspondingly important role. In particular, for $k = 0$, we obtain $u(s) = \text{arsin } x$ and $s(u) = \sin u$, respectively. The function *s*(*u*) has been examined as thoroughly and tabulated as the elementary functions. However, this leads us away from the line of the present discussioa and into the realm of the so-called **elliptic functions**, which occupy a central position in the theory of functions of a complex variable. We shall merely note here that the name **elliptic integral** arises from the fact that such integrals enter into the problem of the determination of the length of an arc of an ellipse.(Chapter V)

Moreover, we may note that integrals which at first sight have quite a different appearance turn out after a simple substitution to be elliptic integrals. For example, the integral

$$\int \frac{dx}{\sqrt{(\cos\alpha - \cos x)}}$$

is transformed by the substitution $u = \cos x/2$ into the integral

$$-k\sqrt{2}\int \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \quad k = \frac{1}{\cos\alpha/2},$$

the integral

$$\int \frac{dx}{\sqrt{(\cos 2x)}}$$

by the substitution u = sin *x* into the integral

$$\int \frac{du}{\sqrt{(1-u^2)(1-2u^2)}},$$

and, finally, the integral

$$\int \frac{dx}{\sqrt{(1-k^2\sin^2 x)}}$$

by the substitution $u$ = sin $x$ into the integral

$$\int \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}.$$

**4.7.2 On Differentiation and Integration:** We insert here another remark on the relationship between differentiation and integration. Differentiation may be viewed to be a more elementary process than integration, since it does not lead us away from the domain of known functions. On the other hand, we must remember that the differentiability of an arbitrary, continuous function is by no means a foregone conclusion, but a very stringent additional assumption. In fact, we have seen that there are continuous functions which are not differentiable at isolated points and we may mention without proof that, since the time of Weierstrass, many examples have been constructed of continuous functions which do not possess a derivative anywhere (cf. Titchmarsh, The Theory of Functions (Oxford, 1932), §§ 11.21 - 11.23, pp/ 350- 354.) Hence, there is much less in the mathematical definition of continuity than simple intuition would lead us to suspect. In contrast, even though integration in terms of elementary functions is not always possible, we are under all circumstances at least certain that the integral of a continuous function exists.

Altogether, we see that integration and differentiation cannot be simply classified as more elementary and less elementary, but that from some points of view the one and from other points of view the other should be considered to be more elementary.

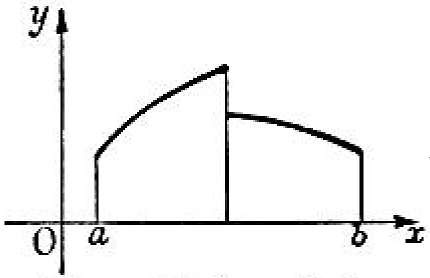


Fig. 4.—The integral of a discontinuous function

In as far as the concept of integral is concerned, we shall see in the next section that it is not closely tied to the assumption that the integrand is continuous, but that it may be extended to wider classes of functions with discontinuities.

## 4.8 Extension of the Concept of Integral. Improper Integrals

**4.8.1 Functions with Jump Discontinuities**. To start with, we see that there arises no difficulty with an extension of the concept of integral to the case where the integrand has jump discontinuities at one or more points in the interval of integration. In fact, we need only take the integral of the function as the sum of the integrals over the separate subintervals in which the function is continuous. The integral then retains its intuitive meaning as an area (Fig. 4).

We should really note that, in our previous definition of integral, we assumed that the interval is closed and tbe function is continuous in the closed interval. This presents no difficulties, since we can extend in each closed subinterval the function so that it is continuous by taking for the value of the function at the end-point the limit of the function as $x$ approaches the end-point from inside the interval.

**4.8.2 Functions with Infinite Discontinuities:** It is quite a different matter when a function has an infinite discontinuity inside the interval or at one of its ends. In order to gain even a notion of an integral in this case, we must introduce a further limiting process. Before stating the general definition, we shall illustrate some of the possibilities by means of examples.

We begin with the integral

$$\int \frac{dx}{x^\alpha},$$

where $\alpha$ is a positive number. The integrand $1/x^\alpha$ becomes infinite as $x \to 0$, whence we cannot extend the integral to the lower limit 0. However, we can try to find what happens as we take the integral from the positive limit $\varepsilon$ to the limit 1, say, and finally let $\varepsilon \to 0$. According to the elementary rules of integration, we obtain, provided $\alpha \neq 1$,

$$\int_\varepsilon^1 \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}(1 - \varepsilon^{1-\alpha}).$$

We recognize immediately that the following possibilities arise: (1) $\alpha > 1$, when, as $\varepsilon \to 0$, the right hand side tends to $\infty$; (2) $\alpha < 1$, when the right hand side tends to the limit $1/(1 - \alpha)$. hence, in the second case, we shall simply take this limiting value as the integral between the limits 0 and 1. In the first case, we shall say that the integral from 0 to 1 does not exist. (3) In the third case, where $\alpha = 1$, the integral will be equal to $-\log \varepsilon$ and therefore, as $\varepsilon \to 0$, it approaches no limit, but tends to $\infty$, that is, the integral from 0 to 1 does not exist.

Another example of the extension of the integral of a function up to an infinite discontinuity is given by the integrand $\frac{1}{\sqrt{(1-x^2)}}$. We find that

$$\int_0^{1-\varepsilon} \frac{dx}{\sqrt{(1-x^2)}} = \arcsin(1-\varepsilon).$$

If we let $\varepsilon$ tend to 0, the right hand side converges to the definite limit $\pi/2$; we therefore call this the value of the integral $\int_0^1 \frac{dx}{\sqrt{(1-x^2)}}$, even though the integrand becomes infinite at the point $x = 1$.

In order to extract a perfectly general concept from these examples, we note, in the first place, that it clearly makes no essential difference whether the discontinuity of the integrand lies at the upper or the lower end of the interval of integration. We now make the following statement:

If in an interval $a \leq x \leq b$ the function $f(x)$ is continuous with the single exception of the end-point $b$, we define $\int_a^b f(x)\,dx$ as the limit

$$\lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx$$

when the point $b - \varepsilon$ approaches the end-point $b$ from inside the interval provided such a limit exists.

In this case, we say that the **improper integral** $\int_a^b f(x)\,dx$ converges. However, if no such limit exists, we say that the integral $\int_a^b f(x)\,dx$ does not exist or does not converge or that it **diverges**.

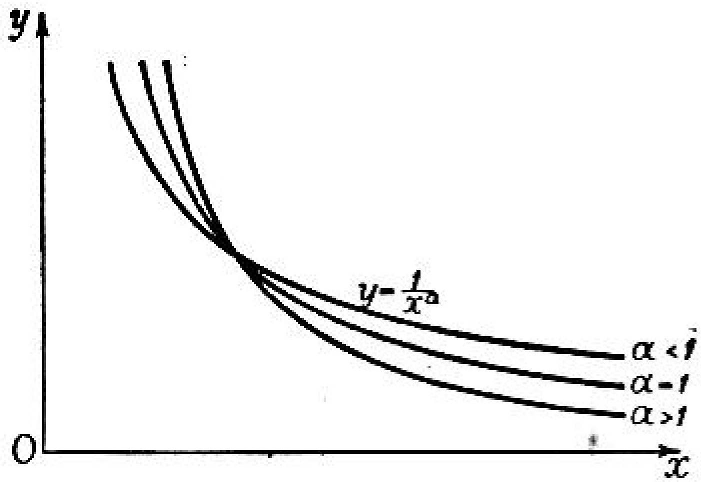



Fig. 5.—To illustrate the convergence or divergence of improper integrals

An analogous definition holds for the case where the lower limit of the interval of integration instead of the upper one is the exceptional point.

Even improper integrals can be intepreted as areas. In the first instance, of course, there is no sense in speaking of the area of a region which estends to infinity; yet, one may attempt to define such an area by means of a passage to the limit from a bounded region with a finite area. For example, the above results for the function $1/x^{\alpha}$ imply that the area, bounded by $x$ axis, the line $x=\varepsilon$ and the curve $y = 1/x^{\alpha}$ tends to a finite limit as $\varepsilon \to 0$, provided that $\alpha < 1$, and that it tends to infinity if $\alpha \geq 1$. This fact may be simply expressed as follows: The area between the $x$-axis, the $y$-axis, the curve and the line $x = 1$ is finite or infinite according to whether $\alpha < 1$ or $\alpha \geq 1$.

Naturally, intuition can give us no precise information about the finitenees or infiniteness of the area of a region stretching to infinity. We can only say about such a region that the more closely its sides approach one another, the more likely it is to have a finite area. In this sense, Fig. 6 illustrates the fact that for $\alpha < 1$, the area under our curve remains finite, while it is infinite for $\alpha \geq 1$.

In order to discover whether a function $f(x)$, which has an infinite discontinuity at the point $x = b$, can be integrated up to $b$, we can often save ourselves a special investigation by means of the criterion:

Let the function $f(x)$ be positive (A8.3); we will show that this restriction of sign can be removed) in the interval $a \le x \le b$ and let $\lim_{x \to b} f(x) = \infty$. Then the integral $\int_a^b f(x)\,dx$ converges, if there exist both a positive number $\mu$ less than 1 and a fixed number $M$ independent of $x$ such that everywhere in the interval $a \le x \le b$ the inequality $f(x) \le M/(b - x)^{\mu}$ is true, in other words, if at the point $x = b$, the function $f(x)$ becomes infinite at a lower order at least than the first. On the other hand, the integral diverges, if there exist both a number $\nu \ge 1$ and a fixed number $N$ such that everywhere in the interval $a \le x \le b$ the inequality $f(x) \ge N/(b - x)^{\nu}$ is true, in other words, if at the point $x = b$ the function $f(x)$ becomes infinite to the first order at least.

The proof follows almost immediately from a comparison with the very simple special case discussed above. In order to prove the first part of the theorem, we observe that for $0 < \varepsilon < b - a$, we have

$$0 \leqq \int_a^{b-\varepsilon} f(x)\,dx \leqq \int_a^{b-\varepsilon} \frac{M}{(b-x)^{\mu}}\,dx.$$

As $\varepsilon \to 0$, the integral on the right hand side, which is obtained from the integral $\int \frac{dx}{x^{a}}$ (cf. 2.7.2), by a simple change of notation, has a limit and therefore remains bounded. Moreover, the values of $\int_a^{b-\varepsilon} f(x)\,dx$ increase monotonically as $\varepsilon \to 0$; since they are also bounded, they must possess a limit and therefore the integral $\int_a^b f(x)\,dx$ converges.

The parallel proof of the second part of the theorem is left as an exercise for the reader.

We likewise see at once that exactly analogous theorems hold when the lower limit of the integral is a point of infinite discontinuity. If a point of infinite discontinuity lies in the interior of the interval of integration, we merely use this point to subdivide the interval into two parts and then apply the above considerations separately to each of these.

As a further example, we consider the elliptic integral

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad (k^2 < 1).$$

We conclud at oncee from the identity 1 - $x^2$ = (1 - $x$)(l + $x$) that, as $x \to 1$, the integrand becomes only infinite at order ½, whence the improper integral exists.

**4.8.3 Infinite Interval of Integration:** Another important extension of the concept of integral occurs when one of the limits of integration as infinite. In order to make this extension precise, we introduce the notation: If the integral

$$\int_a^A f(x)\,dx,$$

where $a$ is fixed, tends to a definite limit when $A$ increases positively beyond all bounds, we denote the limit by

$$\int_a^\infty f(x)\,dx,$$

and call it the integral of the function $f(x)$ from $a$ to $\infty$ . Of course, such an integral does not necessarily exist or, as we often will say, converge.

Simple examples of the various possibilities are again given by the function $f(x)=1/x^\alpha$ :

$$\int_1^A \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}(A^{1-\alpha} - 1).$$

We see here that, if we again exclude the case $\alpha = 1$, the integral to infinity exists for the case $\alpha > 1$, and, in fact,

$$\int_1^\infty \frac{dx}{x^\alpha} = \frac{1}{\alpha - 1};$$

in contrast, when $\alpha < 1$, the integral no longer exists. In the case $\alpha = 1$, the integral again clearly fails to exist, since log $x$ tends to infinity with $x$. Hence we see that, as regards integration over an infinite interval, the functions $1/x^\alpha$ do not behave in the same way as for integration up to the origin. This statement also is made plausible by a glance

at Fig. 5. In , we see that, the larger is $\alpha$, the more slowly do the curves draw in towards the $x$-axis when $x$ is large, so that we can readily assume that the area under consideration tends to a definite limit for sufficiently large values of $\alpha$.

The following criterion for the existence of an integral with an infinite limit is often useful. We again assume that for sufficiently large valued of $x$, say, for $x \geq \alpha$, the integrand has always the same sign, which, without loss of generality, we can choose to be positive. (As we shall see in Chapter 8, this restriction of the sign is easily removed.) Then we have the statement:

The integral $\int_a^\infty f(x)\,dx$ converges if the function $f(x)$ vanishes at infinity to a higher order than the first, that is, if there is a number $\nu > 1$ such that for all values of $x$, no matter how large, the relation $0 < f(x) \leq M/x^\nu$ is true, where $M$ is a fixed number independent of $x$. Again, the integral diverges, if the function remains positive and vanishes at infinity to an order not higher than the first, that is, if there is a fixed number $N > 0$ such that $xf(x) \geq N$.

The proof of these criteria, which run exactly parallel to the previous argument, is left to the reader.

A very simple example is the integral $\int_a^\infty \frac{1}{x^2}\,dx\,(a > 0).$ The integrand vanishes at infinity to the second order. As a matter of fact, we see at once that the integral does converge, because $\int_a^A \frac{1}{x^2}\,dx = \frac{1}{a} - \frac{1}{A}$ whence

$$\int_a^\infty \frac{1}{x^2}\,dx = \frac{1}{a}.$$

Another equally simple example is

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{A\to\infty} (\arctan A - \arctan 0) = \frac{\pi}{2}.$$

**4.8.4 The Gamma Function:** Another example of particular importance in analysis is offered by the socalled **Γ–function**

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \qquad (n > 0).$$

Here too the criterion of convergence is satisfied; for example, if we choose $\nu$=2, we have $\lim_{x\to\infty} x^\nu . e^{-x} x^{n-1} = 0,$
since the exponential function $e^x$ tends to zero to a higher order than any power $1/x^m$ ($m > 0$). This gamma function, which we can think of as a function of the number $n$ (not necessarily an integer), satisfies a remarkable relation, which we can arrive at in the following way by integration by parts: To begin with, we have

$$\int e^{-x} x^{n-1} dx = -e^{-x} x^{n-1} + (n-1) \int e^{-x} x^{n-2} dx.$$

If we take this formula between the limits 0 and $A$ and then let $A$ increase beyond all bounds, we immediately obtain

$$\Gamma(n) = (n-1) \int_0^\infty e^{-x} x^{n-2} dx = (n-1)\Gamma(n-1),$$

and by this recurrence formula, provided $\mu$ is an integer and $0 < \mu < n$,

$$\Gamma(n) = (n-1)(n-2) \ldots (n-\mu) \int_0^\infty e^{-x} x^{n-\mu-1} dx.$$

In particular, if $n$ is a positive integer, we have

$$\Gamma(n) = (n-1)(n-2) \ldots 3 . 2 . 1 \int_0^\infty e^{-x} dx,$$

and, since

$$\int_0^\infty e^{-x} dx = 1,$$

it follows finally that

$$\Gamma(n) = (n-1)(n-2)\ldots 2\,.\,1 = (n-1)!$$

This expression for a factorial by an integral is of importanoe in many applications.

The integrals

$$\int_0^\infty e^{-x^2}\,dx, \quad \int_0^\infty x^n e^{-x^2}\,dx$$

also converge, as we may easily verify by means of our criterion.

**4.8.5 The Dirichlet Integral:** A convergent integral, important in many applications, the convergence of which does not follow directly from our criterion and which is a simple case of a type investigated by Dirichlet, is

$$I = \int_0^\infty \frac{\sin x}{x}\,dx.$$

This integral is easily seen to be convergent, if the upper limit is finite, because sin $x/x \to 1$ as $x \to 0$. Its convergence in the infinite interval is due to the periodic change of sign of the integrand, which causes the contributions to the integral from neighbouring intervals of length $\pi$ almost to cancel each another. In order to utilize this fact, we write the expression

$$D_{AB} = \int_A^B \frac{\sin x}{x}\,dx$$

in the form

$$D_{AB} = \int_A^{A+\pi} \frac{\sin x}{x}\,dx - \int_B^{B+\pi} \frac{\sin x}{x}\,dx + \int_{A+\pi}^{B+\pi} \frac{\sin t}{t}\,dt,$$

introduce in the last of the three integrals the new variable $x = t - \pi$, whence sin $t$ = -sin x, and obtain

$$D_{AB} = \int_A^{A+\pi} \frac{\sin x}{x}\,dx - \int_B^{B+\pi} \frac{\sin x}{x}\,dx - \int_A^B \frac{\sin x}{x+\pi}\,dx.$$

Addition of this expression to the original one for $D_{AB}$ yields

$$2D_{AB} = \int_A^{A+\pi} \frac{\sin x}{x}\,dx - \int_B^{B+\pi} \frac{\sin x}{x}\,dx + \pi\int_A^B \frac{\sin x}{x(x+\pi)}\,dx.$$

Hence, if we assume that $B > A > 0$, it follows that

$$|\, 2D_{AB} \,| < \frac{2\pi}{A} + \pi\int_A^B \frac{dx}{x^2};$$

in fact, we may use the method of 2.2.6, while observing that

$$-\frac{1}{x} \leqq \frac{\sin x}{x} \leqq \frac{1}{x}$$

and

$$-\frac{1}{x^2} \leqq \frac{\sin x}{x\,(x+\pi)} \leqq \frac{1}{x^2}$$

for positive values of $x$. The integral on the right hand side converges, by our criterion and our formula shows that $|D_{AB}| \to 0$ as both $A$ and $B$ tend to infinity. Now

$$|\, D_{0B} - D_{0A} \,| = |\, D_{AB} \,|,$$

and it follows from Cauchy's convergence test that $D_{0B}$ tends to a definite limit as $B \to \infty$. In other words, the integral $I$ exists. Another proof of this result is given in the Chapter VIII and, moreover, in Chapter IX we shall show that $I$ has the value $\pi/2$.

**4.8.6 Substitution:** It is obvioua that all rules for the substitution of new variables, etc. remains valid for **convergent improper integrals**. As an example, in order to calculate $\int_0^\infty xe^{-x^2}dx$, we introduce the new variable $u = x^2$ and obtain

$$\int_0^\infty xe^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-u}\,du = \lim_{A\to\infty}\frac{1}{2}(1-e^{-A}) = \frac{1}{2}.$$

Other examples of the use of substitution in the investigation of improper integrals are the **Fresnel integrals**, which occur in the theory of **diffraction of light**:

$$F_1 = \int_0^\infty \sin(x^2)\,dx, \quad F_2 = \int_0^\infty \cos(x^2)\,dx.$$

The substitution $x^2 = u$ yields

$$F_1 = \frac{1}{2}\int_0^\infty \frac{\sin u}{\sqrt{u}}\,du, \quad F_2 = \frac{1}{2}\int_0^\infty \frac{\cos u}{\sqrt{u}}\,du.$$

Integrating by parts, we have

$$\int_A^B \frac{\sin u}{\sqrt{u}}\,du = \frac{\cos A}{\sqrt{A}} - \frac{\cos B}{\sqrt{B}} - \frac{1}{2}\int_A^B \frac{\cos u}{u^{3/2}}\,du.$$

As $A$ and $B$ tend to $\infty$, the first two terms on the right hand side tend to 0 and, by the criterion of 4.8.4, the integral also tends to 0. Hence, by the same argument as for the Dirichlet integral, we see that the integral $F_1$ converges. The convergence of the integral $F_2$ is proved in exactly the same way.

These Fresnel integrals show that an improper integral may exist even although the integrand does not tend to zero as $x \to \infty$. In fact, an improper integral can exist even when the integrand is unbounded, as is shown by the example

$$\int_0^\infty 2u \cos(u^4)\, du.$$

When $u^4 = n\pi$, i.e. when $u = \sqrt[4]{n\pi}, n = 0, 1, 2, \cdots$, the integrand becomes $2\sqrt[4]{n\pi} \cos n\pi = \pm 2\sqrt[4]{n\pi}$, so that the integrand is unbounded. However, the substitution $u^2 = x$ reduces the integral to

$$\int_0^\infty \cos(x^2)\, dx,$$

which we have just shown to be convergent.

By means of a substitution, an improper integral may often be transformed into a proper one. For example, the transformation $x = \sin u$ yields

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} = \int_0^{\pi/2} du = \frac{\pi}{2}.$$

On the other hand, integrals of continuous functions may be transformed into improper integrals; this oocurs if the transformation $u = \phi(x)$ is such that at the end of the interval of integration the derivative $\phi'(x)$ vanishes, so that $dx/du$ is infinite.

**Exercises 4.5:**

Test the convergence of the improper integals 1-11:

1. $\int_{-3}^{3} \frac{dx}{x^2}$. 2. $\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}}$. 3. $\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$.

4. $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$. 5. $\int_0^{\pi} \frac{dx}{1-\cos x}$.

6. $\int_A^B \frac{dx}{\sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}}$, where $a_1$, $a_2$, $a_3$, $a_4$ are all different and lie between $A$ and $B$.

7. $\int_0^{\infty} \frac{\arctan x}{1+x^2}\, dx$. 8. $\int_0^{\infty} \frac{\arctan x}{1-x^3}\, dx$.

9. $\int_1^{\infty} \frac{x}{1-e^x}\, dx$. 10. $\int_0^{\infty} \frac{x}{e^x-1}\, dx$. 11. $\int_0^{\pi/2} \log \tan x\, dx$.

12.* Prove that $\int_0^{\infty} \sin^2\left[\pi\left(x+\frac{1}{x}\right)\right] dx$ does not exist.

13.* Prove that $\lim_{k\to\infty} \int_0^{\infty} \frac{dx}{1+kx^{10}} = 0$.

14. For what value of $s$ converges ($a$) $\int_0^{\infty} \frac{x^{s-1}}{1+x}\, dx$, ($b$) $\int_0^{\infty} \frac{\sin x}{x^s}\, dx$.

15.* Does $\int_0^{\infty} \frac{\sin t}{1+t}\, dt$ converge?

16.* (a) If $\alpha$ is a fixed positive number, prove that

$$\lim_{h\to 0} \int_{-a}^{a} \frac{h}{h^2+x^2}\, dx = \pi.$$

($b$) If $f(x)$ is continuous in the interval $-1 \le x \le 1$, prove that

$$\lim_{h \to 0} \int_{-1}^{1} \frac{h}{h^2 + x^2} f(x)\, dx = \pi f(0).$$

[Answers and Hints]

## Chapter IV, Mixed Exercises:

Evaluate the integrals in 1 - 10:

1. $\int e^{\arcsin x}\, dx.$

2. $\int \sin^8 x \cos^6 x\, dx.$ (By a shorter method than that of the text, using trigonometrical identities.)

3. $\int (\log x)^2\, dx.$ 4. $\int \frac{\sin x\, dx}{3 + \sin^2 x}.$ 5. $\int \sqrt{1 - e^{-2x}}\, dx.$

6. $\int_{-1}^{+1} x e^{-x^2 \tan^2 x}\, dx.$ 7. $\int_{\frac{1}{2}}^{2} \frac{1}{x} \sin\left(x - \frac{1}{x}\right) dx.$

8.* Prove that $\lim_{x \to \infty} e^{-x^2} \int_0^x e^{t^2} dt = 0.$

9. Assuming that $|\alpha| \neq |\beta|$, prove that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \sin \alpha x \sin \beta x\, dx = 0.$$

10. Evaluate $\int_{-1}^{1} x^3 e^{-x^4} \cos 2x\, dx.$

11.* Prove that the substitution $x = (\alpha t + \beta)/(\gamma t + \delta)$, where $\alpha\delta - \gamma\beta \neq 0$, transforms the integral

$$\int \frac{dx}{\sqrt{ax^4 + bx^3 + cx^2 + dx + e}}$$

into an integral of a similar type, and that, if the biquadratic

$$ax^4 + bx^3 + cx^2 + dx + e$$

has no repeated factors, neither has the new biquadratic in $t$ which takes its place.

Prove that the same statements are true for

$$\int R(x, \sqrt{ax^4 + bx^3 + cx^2 + dx + e})\,dx,$$

where $R$ is a rational function.

12. Find the limit as $n \to \infty$ of $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$.

13.* Find the limit of

$$b_n = \frac{1}{\sqrt{n^2 - 0}} + \frac{1}{\sqrt{n^2 - 1}} + \frac{1}{\sqrt{n^2 - 4}} + \dots + \frac{1}{\sqrt{n^2 - (n-1)^2}}.$$

14.* Prove that $\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e}$.

15.* If $\alpha$ is any real number greater than $-1$, evaluate

$$\lim_{n\to\infty} \frac{1^\alpha + 2^\alpha + 3^\alpha + \dots + n^\alpha}{n^{\alpha+1}}.$$

Answers and Hints

# Appendix to Chapter IV

## The Second Mean Value Theorem of the Integral Calculus

The method of integration by parts yields us an easy way for proving an important theorem on the estimation of integrals, usually called the **second mean value theorem of the integral calculus**.

Let the function $\phi(x)$ be monotonic and continuous in the interval $a \leq x \leq b$, the derivative $\phi'(x)$ be continuous and moreover $f(x)$ be an arbitrary function which is continuous in the same interval. Then one has the second mean value theorem of the integral calculus:

There exists a number $\xi$ such that $a \leq \xi \leq b$ for which

$$\int_a^b f(x)\,\phi(x)\,dx = \phi(a)\int_a^\xi f(x)\,dx + \phi(b)\int_\xi^b f(x)\,dx.$$

We prove this by noting first that we can assume that $\phi(b) = 0$; in fact, replacement of $\phi(x)$ by $\phi(x) - \phi(b)$ changes both sides of the equation by the same amount and yields a function which vanishes at $x = b$. Moreover, we can assume that $\phi(a) > 0$; in fact, if $\phi(a) < 0$, we need only replace $\phi(x)$ by $-\phi(x)$, which changes the sign of both sides of the equation. (The case $\phi(a) = 0$ is trivial; in fact, if both $\phi(a)$ and $\phi(b)$ vanish, $\phi(x)$ must be identically zero and our equation becomes $0 = 0$.) Hence, we must only prove that, if $\phi(x)$ is continuous and monotonic decreasing and $\phi(b) = 0$, then

$$\int_a^b f(x)\,\phi(x)\,dx = \phi(a)\int_a^\xi f(x)\,dx.$$

We now set $F(x) = \int_a^x f(x)\,dx$ and apply the formula for integration by parts to the left hand side of the last equation; we then have

$$\int_a^b f(x)\phi(x)\,dx = F(x)\phi(x)\Big|_a^b + \int_a^b F(x)\{-\phi'(x)\}\,dx.$$

The integrated part vanishes, since $F(a)$ and $\phi(b)$ are zero. The expression $-\phi'(a)$ is everywhere positive, so that we can apply the first mean value theorem of the integral calculus. We thus find that the integral on the right hand side has the value

$$F(\xi)\int_a^b \{-\phi'(x)\}\,dx, \quad a \leqq \xi \leqq b.$$

However,

$$F(\xi) = \int_a^\xi f(x)\,dx \quad \text{and} \quad \int_a^b \{-\phi'(x)\}\,dx = \phi(a) - \phi(b) = \phi(a)$$

and our theorem is established.

This theorem can be extended (although we shall not carry out the proof) to more general classes of functions. The theorem remains true for all continuous monotonic functions $\phi(x)$ whether they have derivatives or not. In fact, it is true for any discontinuous monotonic function for which we are in a position to integrate $f(x)\phi(x)$.

## Chapter V.

# Applications

After dealing with a few preliminaries, we shall show in this chapter how what we can apply what we have learnt so far in many ways in geometry and physics .

### 5.1 Representation of Curves

**5.1.1 Parametric representation:** As we have seen in 1.2.3, when we represent a curve by means of an equation $y = f(x)$, we must always restrict ourselves to a single-valued branch. Hence it is often more convenient - when dealing, in particular, with a closed curve - to introduce other analytical methods of representation. The most general and at the same time the most useful representation of a curve is parametric representation. Instead of considering one of the rectangular co-ordinates as a function of the other, we think of both the co-ordinates $x$ and $y$ as functions of a third independent variable $t$, the socalled **parameter**; the point with the co-ordinates $x$ and $y$ then describes the curve as $t$ traverses a definite interval. Such parametric representations have already been encountered. For example, for the circle $x^2+y^2 = a^2$, we obtain a parametric representation in the form

$$x = a\cos t,\ y = a\sin t.$$

Here, as we know already, $t$ has the geometrical meaning of an angle at the centre of the circle. For the ellipse $x^2/a^2 + y^2/b^2 = 1$ we likewise have the parametric representation $x = a\cos t$, $y = b\sin t$, where $t$ is the solaced **eccentric angle**, that is, the angle at the centre corresponding to the point of the circumscribed circle lying vertically above or

below the point $P$ ($a\cos t$, $b\sin t$) of the ellipse (Fig. 1). In both these cases, the point with the co-ordinates $x$, $y$ describes the complete circle or ellipse as the parameter $t$ traverses the interval from 0 to $2\pi$.

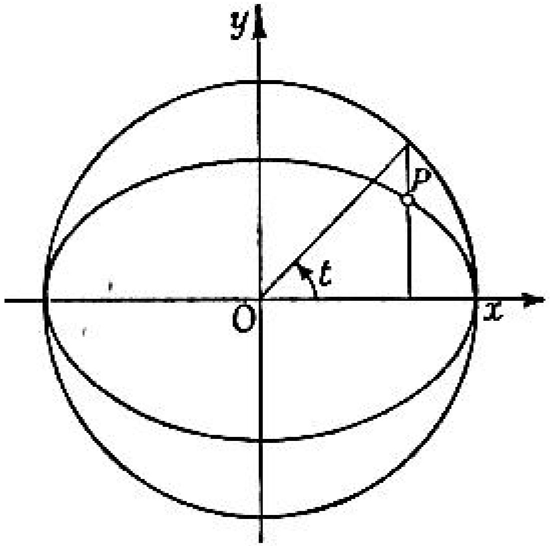



Fig. 1

In general, we can seek to represent a curve parametrically by taking

$$x = \phi(t) = x(t), \quad y = \psi(t) = y(t),$$

that is, by considering two functions of a parameter $t$; the shorter notation $x(t)$ and $y(t)$ will henceforth be used whenever there is no danger of confusion. For a given curve, these two functions $\phi(t)$ and $\psi(t)$ must be determined in such a way that the totality of pairs of functional values $x(t)$ and $y(t)$ corresponding to a given interval of values of $t$ yields all the points on the curve and no points which are not on the curve. If a curve, in the first instance, is given in the form $y=f(x)$, we can arrive at a representation of this kind by first writing $x=\phi(t)$, where $\phi(t)$ is any continuous monotonic function which in a definite interval passes exactly once through each of the values of $x$ in question; it then follows that $y = f\{\phi(t)\}$, that is, the second function $\psi(t)$ is determined by compounding $f$ and $\phi$. We thus see that, owing to the arbitrariness in the choice of the function $\phi$, we have a great deal of freedom in representing a given curve parametrically; in particular, we may actually take $t=x$ and thus think of the original representation $y = f(x)$ as a parametric representation with the parameter $t = x$.

The advantage of the parametric representation is that this arbitrariness may be utilized for purposes of simplification. For example, we represent the curve $y = \sqrt[3]{x^2}$ by taking $x = t^3$, $y = t^2$, so that $\phi(t) = t^3$, $\psi(t) = t^2$. The point with the co-ordinates $x$,y will then describe the entire curve (**semi-cubical parabola**) as $t$ varies from $-\infty$ to $+\infty$.

On the other hand, if a curve is originally given in parametric representation $x=\phi(t)$, $y = \psi(t)$ and we wish to obtain the equation of the curve in non-parametric form, that is, in the form $y = f(x)$, we have only to eliminate the parameter $t$ from the two equations. In the case of the above parametric representations of the circle and ellipse, we can do this at once by squaring and using the equation $\sin^2 t + \cos^2 t = 1$. (Another example is given below.) In general, we should have to find an expression for $t$ from the equation $x = \phi(t)$ by means of the inverse function $t = \Phi(x)$ and substitute this into $y=\psi(t)$, in order to obtain the representation $y = \psi\{\Phi(x)\} = f(x)$. Naturally, during such an elimination, we must ordinarily restrict ourselves to a portion of the curve; in fact, to a portion which is not intersected twice by any line parallel to the $y$-axis.

However, it may happen that the equation $y = f(x)$ obtained in this way represents more than the original parametric representation. For example, the equations $x = a \sin t$, $y = b \sin t$ represent only the finite portion of the line $y = bx/a$ between the points $x = -a$, $y = -b$ and $x = a$, $y = b$, whereas the equation $y = bx/a$ represents the entire line.

The parametric representation has associated with it a definite sense in which a curve is described, corresponding to the direction in which the values of the parameter increase; we shall call this direction the **positive sense**. For example, if the point $x = x(t)$, $y = y(t)$ describes a curve $C$ as $t$ traverses an interval $t_0 \leq t \leq t_1$ and the end-points $P_0$ and $P_1$ of the curve correspond to $t_0$ and $t_1$, respectively, then the curve is traversed positively in the direction from $P_0$ to $P_1$. If we introduce $\tau = -t$ as a new parameter, the curve $C$ will correspond to the values $-t_1 \leq \tau \leq -t_0$ of the variable $\tau$, and the points $P_0$ and $P_1$ will correspond to $\tau = -t_0$ and $\tau = -t_1$, respectively. If we now traverse the curve from $P_0$ to $P_1$, we proceed in the direction in which the values of the parameter $\tau$ decrease, that is, in the **negative sense**. In general, a change of parameter $t = t(\tau)$ preserves the sense in which a curve is described, if the function $t(\tau)$ is monotonic increasing, but reverses it, if the function $t(\tau)$ is monotonic decreasing.

**5.1.2 Interpretation of the Parameter. Change of Parameter:** In many cases, we can give an immediate physical interpretation to the parameter $t$, that is **time**. Any motion of a point in the plane may be expressed mathematically by the fact that the co-ordinates $x$ and $y$ appear as functions of the time. Hence, these two functions determine the motion along a path or trajectory in parametric form.

An example of this are the **cycloids** which arise when a circle rolls along a straight line on another circle. We limit ourselves here to the simplest case, in which a circle of radius $a$ rolls along the $x$-axis and we consider a point on its circumference. This point then describes a **common cycloid**. If we choose the origin of the co-ordinate system and the initial time in such a way that for time $t = 0$ the corresponding point of the curve coincides with the origin, we obtain (Fig. 2) the parametric representation for the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t),$$

where $t$ denotes the angle through which the circle has turned from its original position; in the case when the velocity of rolling is uniform, it is proportional to the time.

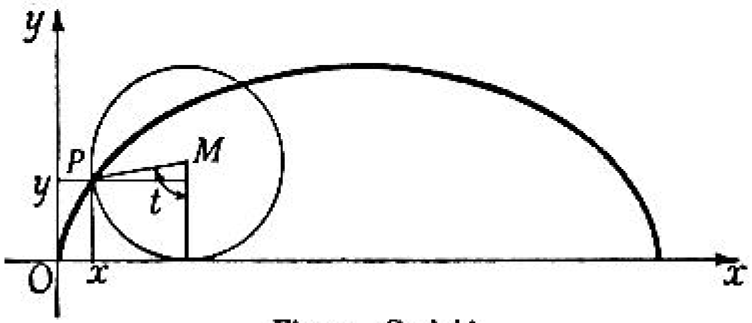



Fig. 2.—Cycloid

By elimination of the parameter $t$, we can obtain the equation of the curve in non-parametric form, however, at the cost of the neatness of the expression. We have

$$\cos t = \frac{a-y}{a}, \quad t = \arccos\frac{a-y}{a}, \quad \sin t = \pm\sqrt{\left\{1 - \frac{(a-y)^2}{a^2}\right\}},$$

whence

$$x = a \arccos\frac{a-y}{a} \mp \sqrt{\{(2a-y)y\}},$$

and we obtain $x$ as a function of $y$.

In the parametric representation of a given curve, we have a great deal of freedom in the choice of the parameter (5.1.1). For example, we could take instead of the time $t$ the quantity $\tau = t^2$ as parameter or, indeed, any arbitrary quantity $\tau$ which is related to the original parameter $t$ by an arbitrary equation of the form $\tau = \omega(t)$, where we assume that for the entire interval of values of $t$ under consideration this function has a unique inverse $t = \kappa(\tau)$. If increasing values of $\tau$ correspond to increasing values of $t$, the positive sense of description remains the same; otherwise it is reversed.

Naturally, parametric representation is not limited to rectangular co-ordinates; for example, it can just as well be used with the polar co-ordinates $r$ and $\theta$, which are linked to the rectangular co-ordinates by the well-known equations

$$x = r\cos\theta,\ r\sin\theta \text{ or } r = \sqrt{(x^2+y^2)},\ \sin\theta = y/r,\ \cos\theta = x/r;$$

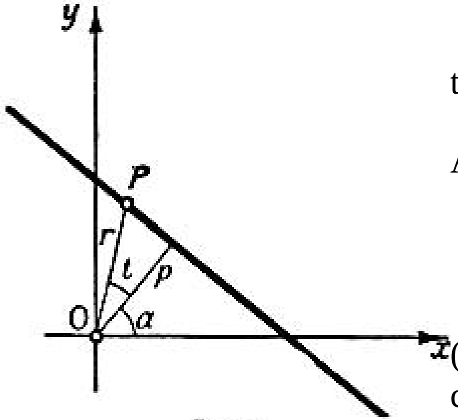


Fig. 3

the equations of the curve would then be $r = r(t)$, $\theta = \theta(t)$.

As an example, the straight line may be represented parametrically (Fig. 3) by

$$r = \frac{p}{\cos t}, \quad \theta = \alpha + t$$

($p$ and $\alpha$ being constants), from which we immediately obtain the equation of the line in polar co-ordinates

$$r = \frac{p}{\cos(\theta - \alpha)},$$

by eliminating the parameter $t$.

**5.1.3 The Derivatives for a Parametrically Represented Curve:** If, on the one hand, a curve is given by an equation $y = f(x)$ and, on the other hand, parametrically by $x = x(t)$, $y = y(t)$, then we must have $y = f\{x(t)\}$. By the chain rule for differentiation, it follows that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

or

$$y' = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}},$$

where we use as an abbreviation for differentiation with respect to the parameter $t$ a dot · over the variable (**Newton's notation**) instead of the dash ' ; we shall reserve the latter for differentiation with respect to $x$.

For example, for the cycloid, we have

$$\dot{x} = a(1 - \cos t) = 2a \sin^2 \frac{t}{2},$$

$$\dot{y} = a \sin t = 2a \sin \frac{t}{2} \cos \frac{t}{2}.$$

These formulae show that the cycloid has a cusp with a vertical tangent at the points *i* = 0, ±2$\pi$, ±4$\pi$, ··· , at which it meets the *x*-axis, because, on approaching these points, the derivative $y' = \dot{y}/\dot{x} = \cot(t/2)$ becomes infinite. At these points, *y* is equal to 0, while everywhere else y > 0.

The equation of the tangent to the curve is

$$(\xi - x)\dot{y} - (\eta - y)\dot{x} = 0,$$

where $\xi$ and $\eta$ are the **current co-ordinates**, that is, the variable co-ordinates corresponding to an arbitrary point on the tangent. For the equation of the normal, i.e., the straight line through a point of the curve, perpendicular to the tangent at that point, we likewise obtain

$$(\xi - x)\dot{x} + (\eta - y)\dot{y} = 0.$$

The **direction cosines of the tangent**, that is, the cosines of the angles $\alpha$, $\beta$ which the tangent makes with the *x* and *y* axes, respectively, are given by

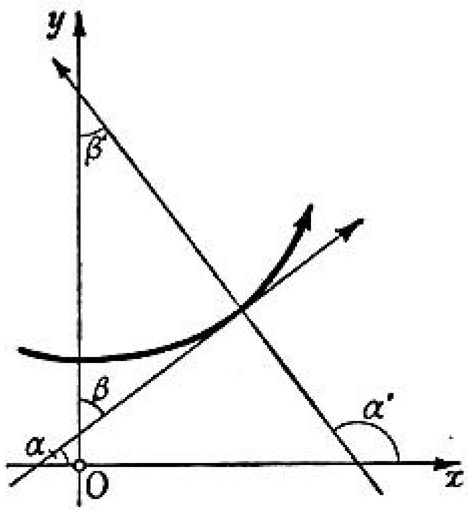


**Fig. 4.—Direction cosines of the tangent and the normal**

$$\cos \alpha = \frac{\dot{x}}{\pm\sqrt{(\dot{x}^2 + \dot{y}^2)}}, \quad \cos \beta = \frac{\dot{y}}{\pm\sqrt{(\dot{x}^2 + \dot{y}^2)}},$$

as we may verify by elementary methods. The corresponding **direction cosines of the normal** (Fig. 4) are given by

$$\cos \alpha' = \frac{-\dot{y}}{\pm\sqrt{(\dot{x}^2 + \dot{y}^2)}}, \quad \cos \beta' = \frac{\dot{x}}{\pm\sqrt{(\dot{x}^2 + \dot{y}^2)}}.$$

These formulae show us that at every point, at which $\dot{x}$ and $\dot{y}$ are continuous and $\dot{x}^2 + \dot{y}^2 \neq 0$, the direction of the tangent varies continuously with *t*. This is the most

important case for us; however, it is interesting to illustrate by examples the various possibilities which arise when our assumptions are not fulfilled and we cannot state directly that the tangent keeps on turning continuously. At a point, at which $\dot{x} = \dot{y} = 0$, the tangent may or may not turn continuously. As one example, we have the curve $x = t^3$, $y = t^2$, discussed in 2.3.5, which has a cusp at the origin even though $\dot{x}$ and $\dot{y}$ are continuous everywhere. Consider as another example the curve $x = t^3$, $y = t^3$, which is the straight line $y = x$. This curve has the same tangent direction everywhere; the latter is therefore continuous, although the derivatives $\dot{x}$ and $\dot{y}$ both vanish for $t = 0$. Moreover, at a point at which $\dot{x}$ and $\dot{y}$ are discontinuous, the direction of the tangent may or may not be continuous. In fact, let $\phi(t)$ be any continuous monotonic increasing function, defined for $t_1 \le t \le t_2$, which has a sharp corner at $t = t_3$, $t_1 \le t \le t_2$. Then the curve $x = t$, $y = \phi(t)$, which is the same curve as $y == \phi(x)$, has a sharp corner at $x = t_3$; while the curve $x = \phi(t)$, $y = \phi(t)$, which is a segment of the straight line $y = x$, has a constant tangent direction even though the derivatives $\dot{x}$ and $\dot{y}$ do not exist at $t = t_3$. This indicates that, if we wish to investigate the behaviour of the tangent at a point where our theorem does not apply, we should first use the formulae to find $\cos\alpha$ or $\cos\beta$ as functions of $t$ and then investigate these direction cosines themselves.

From a well-known formula in trigonometry or analytical geometry, we find that the angle between the two curves represented parametrically by $x=x_1(t)$, $y=y_1(t)$ and $x=x_2(t)$, $y=y_2(t)$, respectively, (that is, the angle between their tangents or normals) is given by the expression

$$\cos\delta = \frac{\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2}{\pm\sqrt{(\dot{x}_1^2 + \dot{y}_1^2)}\sqrt{(\dot{x}_2^2 + \dot{y}_2^2)}}.$$

The indeterminacy of the signs of the square roots in the last few formulae suggests that the angles are not completely determined, since we can still specify either sense of direction on the tangent or normal as **positive**. Taking the square root as positive, as it is usually done, corresponds to choosing for the positive direction on the tangent the direction in which the parameter increases, and for the positive direction on the normal the direction obtained by rotating the tangent through an angle $\pi/2$ in the positive, i.e., the counter-clockwise sense.,

The second derivative $y'' = d^2y/dx^2$ is obtained in the following way by means of the chain rule and the rule for differentiating a quotient:

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dt}\,\frac{dt}{dx} = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)\frac{1}{\dot{x}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}\,\frac{1}{\dot{x}},$$

whence

$$y'' = \frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}.$$

**5.1.4 Change of Axes for Parametrically represented Curves:** If we rotate the axes through an angle $\alpha$ in the positive direction, the new rectangular co-ordinates $\xi, \eta$ and the old ones $x, y$ are interrelated by the equations

$$\begin{aligned} x &= \xi \cos\alpha - \eta \sin\alpha, & \xi &= x \cos\alpha + y \sin\alpha, \\ y &= \xi \sin\alpha + \eta \cos\alpha, & \eta &= -x \sin\alpha + y \cos\alpha. \end{aligned}$$

Thus, the new co-ordinates $\xi$ and $\eta$ are specified along with $x$ and $y$ as functions of the parameter $t$. We obtain at once by differentiation

$$\begin{aligned} \dot{x} &= \dot{\xi} \cos\alpha - \dot{\eta} \sin\alpha, & \dot{\xi} &= \dot{x} \cos\alpha + \dot{y} \sin\alpha, \\ \dot{y} &= \dot{\xi} \sin\alpha + \dot{\eta} \cos\alpha, & \dot{\eta} &= -\dot{x} \sin\alpha + \dot{y} \cos\alpha. \end{aligned}$$

Let the curve be given in polar co-ordinates and both polar and rectangular co-ordinates be given as functions of a parameter $t$. Then, by differentiation with respect to $t$, we obtain from the equations $x = r \cos\theta$, $y = r \sin\theta$ the formulae

$$\left.\begin{aligned} \dot{x} &= \dot{r} \cos\theta - r \sin\theta \,.\, \dot{\theta}, \\ \dot{y} &= \dot{r} \sin\theta + r \cos\theta \,.\, \dot{\theta}, \end{aligned}\right\} \quad . \quad . \quad . \quad (a)$$

which are frequently used in passing from rectangular to polar co-ordinates. As an example, consider the polar equation of a curve, $r = f(\theta)$ which might arise from a parametric representation $r=r(t)$, $\theta=\theta(t)$ by elimination of the parameter $t$. The angle $\psi$ between the radius vector to a point on the curve and the tangent to the curve at that point is then given by

$$\tan\psi = \frac{f(\theta)}{f'(\theta)}.$$

We can convince ourselves of this in the following way. If we think of the curve as being given by an equation $y = F(x)$ and use $\theta$ as a parameter, so that $\dot{\theta} = 1$ and $\dot{r} = f'(\theta)$, we have

$$\tan\alpha = y' = \frac{\dot{y}}{\dot{x}} = \frac{\dot{r}\tan\theta + r}{\dot{r} - r\tan\theta}$$

(cf. Fig. 5 and equations ($a$) above). In addition, $\psi = \alpha - \theta$, whence

$$\tan\psi = \frac{y' - \tan\theta}{1 + y'\tan\theta} = \frac{r + r\tan^2\theta}{\dot{r} + \dot{r}\tan^2\theta} = \frac{r}{\dot{r}}.$$

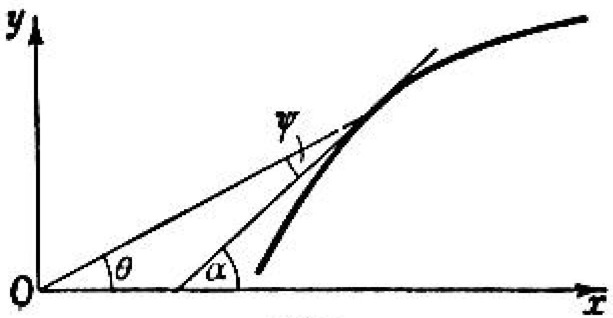



Fig. 5

This formula can also be established by geometrical methods.

**5.1.5 General Remarks:**. In discussing given curves, we sometimes consider properties which do not assert anything about the form of a curve itself, but merely something about the position of the curve with respect to the co-ordinate system; for example, the occurrence of a horizontal tangent, expressed by the equation $\dot{y} = 0$, or the occurrence of a vertical tangent, expressed by $\dot{x} = 0$. Such properties do not persist when the axes are rotated.

In contrast to this, a **point of inflection** will still be a point of inflection after the axes have been rotated. According to 5.1.4 , the condition for a point of inflection is

$$\dot{x}\ddot{y} - \ddot{x}\dot{y} = 0.$$

If we replace on the left hand side the expressions $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}$ by their values in terms of the new co-ordinates $\xi, \eta$, we readily obtain

$$\dot{x}\ddot{y} - \ddot{x}\dot{y} = \dot{\xi}\ddot{\eta} - \ddot{\xi}\dot{\eta}.$$

Hence it follows from the equation $\dot{x}\ddot{y} - \ddot{x}\dot{y} = 0$ that $\dot{\xi}\ddot{\eta} - \ddot{\xi}\dot{\eta} = 0,$ so that our equation expresses a property of the point of the Curve which is independent of the co-ordinate system.

We shall often see later on that properties which are truly geometrical are expressed by formulae the form of which is not altered by rotation of the axes.

**Exercises 5.1:**

1. Find in parametric form the equation of the curve

$$x = a \cos 2\theta \cos \theta$$
$$y = a \cos 2\theta \sin \theta.$$

2. A circle *c* of radius *r* rolls on the outside of a fixed circle *0* of radius *R*. The point *P* on the circumference of *c* moves with *o* and describes a curve called the **epicycloid**. Find the parametric representation of the epicycloid (consider *c* to rotate with constant velocity and measure time so that at $t = 0$ the point *P* is in contact with the circle *C*)

3. Sketch the epicycloid for the special case $r = R$ and find its parametric equations, (This particular epicycloid is called the **cardioid**.)

4. If in 2. the radius *r* is less than *R* and *c* rolls inside *C*, the point *P* describes a **hypercycloid**. Find its parametric equations.

5. Sketch the hypercycloid (1) for $R = 4r$, (2) for *R* as $3r$.

6. Sketch the hypercyloid for $R = 4r$ (the **astroid**) and find its non-parametric equation.

7. Find the parametric equations for the curve $x^3 + y^3 = 3axy$ (the **folium of Descartes**), choosing as parameter *t* the tangent of the angle between the *x*-axis and the radius vector from the origin to the point $(x, y)$.

8. Find the formula for the angle $\alpha$ between two curves $r = f(\theta)$ and $r = g(\theta)$ in polar co-ordinates.

9. Find the equation of the curves which everywhere intersect the straight lines through the origin at the same angle $\alpha$.

10. Let $c$ be a fixed curve and $P$ a fixed point with co-ordinates $x_0,y_0$. The **pedal curve** of $C$ with respect to $P$ is defined to be the locus of the foot of the perpendicular from $P$ on the tangent to $C$. Find the parametric representation of the pedal of $C$, if $C$ is itself given parametrically by $x = f(t)$, $y = g(t)$.

11. Find the pedal curve of the circle $C$, ($a$) with respect to its centre $M$, ($b$) with respect to a point $P$ on its circumference.

15. Find the pedal curve of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ with respect to the origin.

Answers and Hints

## 5.2 Applications to the Theory of Plane Curves

We shall consider two different kinds of geometrical properties or quantities associated with curves. The first type consists of properties or quantities which depend only on the **behaviour of the curve in the small**, i.e. in the immediate neighbourhood of a point, and which can be expressed analytically by means of the local derivative. Properties of the second type depend on the entire course or a portion of the curve and are expressed analytically by means of the concept of integral. We shall begin by considering properties of the second type.

**5.2.1 Orientation of Area:** The idea of area was our starting point for the definition of the integral; but the connection between the definite integral and area is still somewhat incomplete. The areas, with which we are concerned in geometry, are bounded by given closed curves; on the other hand, the area measured by the integral $\int_{x_0}^{x_1} f(x)\,dx$ is bounded only partly by the given curve $y = f(x)$, the rest of the boundary consisting of lines which depend on the choice of the co-ordinate system. If we wish to determine the area interior to a closed curve, such as a circle or an ellipse, by means of integrals of this type, we have to use some such device as breaking up the area into several parts, each of which is bounded by a single-valued branch of the curve as well as by the $x$-axis and the corresponding ordinates.

It is convenient for the discussion of this general case to make first some remarks on the determination of the sign of an area under consideration. For any figure, bounded by an arbitrary closed curve which does not intersect itself, we can relate the sign of its area to the purely geometrical idea of the sense in which the curve is described,

following the convention: We say that the boundary of a region is described in the positive **sense**, if we go around the boundary in such a direction that the interior of the region is on the left; the opposite sense we call **negative**. If we then consider a region, the boundary of which is traversed in an assigned sense - a solaced **oriented region**, we consider the area to be positive if this sense is positive, and negative if this sense is negative (Fig. 6).

If we wish to avoid the words right and left in such a context, we say that the triangle, the ordered vertices of which are the origin, the point $x = 1, y = 0$ and the point $x = 0, y = 1$, is described in the positive sense, if the vertices are passed in the order mentioned. For every other region, we say that the boundary is positively described. if it is described in the same sense as this triangle, otherwise it is described negatively

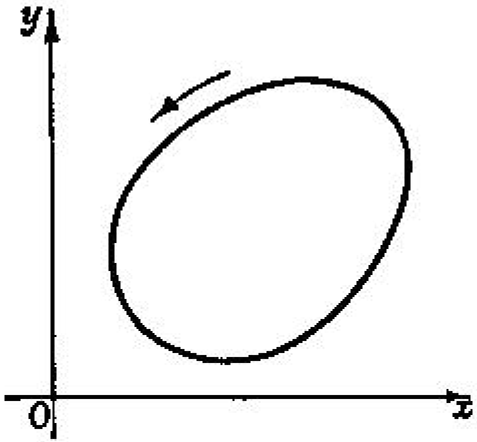


Fig. 6.—A positive area

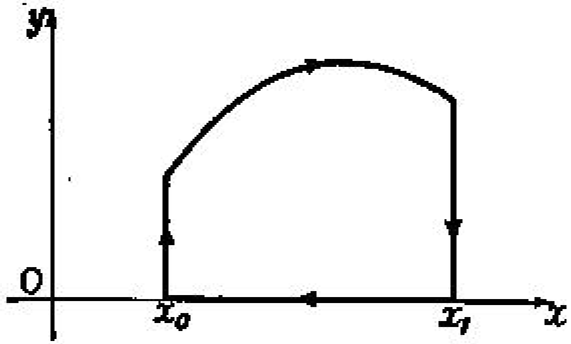


Fig. 7

In particular, let in the interval $a \leq x \leq b$ the function $f(x)$ be everywhere positive. We consider the closed curve obtained by starting at the point $x = b = x_1$, $y = 0$, traversing the $x$-axis back to the point $x = a = x_0$, $y = 0$, then proceeding along the ordinate to the curve $y = f(x)$, then along the curve to the ordinate $x = b$, and finally along the ordinate to the $x$-axis (Fig. 7). The absolute value of the area interior to this curve - the number of square units contained in it - is, as we know, $\int_a^b f(x)\,dx.$ . Hence, denoting by $A_{01}$ the area with its sign as determined above, the integral yields the value $A_{01}$ except for its sign. In order to determine the sign, we need only observe that the boundary of the region is traversed in the negative sense, so that $A_{01}$ is negative; hence

$$A_{01} = -\int_a^b f(x)\,dx.$$

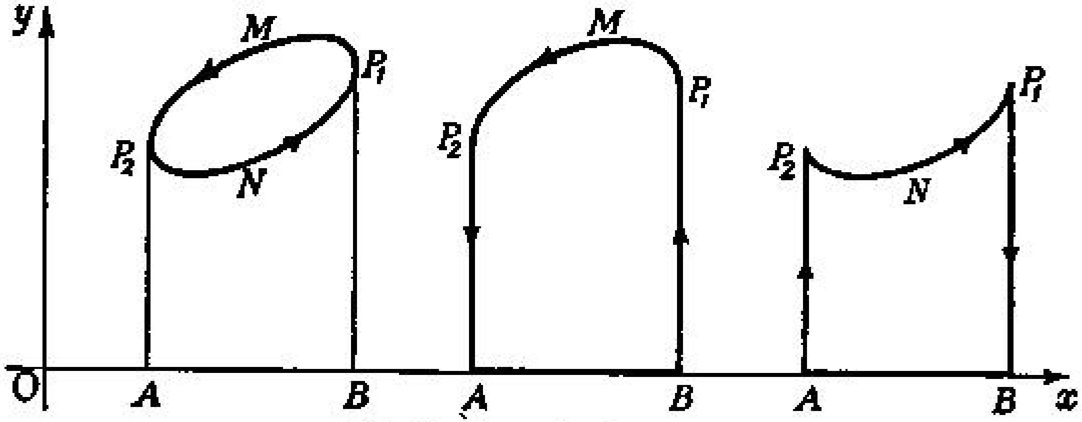



Fig. 8.—Area of a closed curve

Similarly, if $a > b$, we find that, according to our convention, $A_{01}$ is positive, while the integral

$$\int_a^b f(x)\,dx.$$

is negative, whence in either case $A_{01}$ is given by the above equation.

**5.2.2 The General formula for the Area as an Integral:**. After these preliminaries, the difficulties mentioned at the beginning can now be avoided in a simple way by representing our curve parametrically. If we introduce formally $t$ into the above integral as a new independent variable, writing $x = x(t)$, $y = y(t)$, we have

$$A_{01} = -\int_{t_0}^{t_1} y(t)\,\dot{x}(t)\,dt,$$

where $t_0$ and $t_1$ are the values of the parameter corresponding to the abscissae $x_0 = a$ and $x_1 = b$, respectively. We assume here that the considered branch of the curve $y=f(x)$ is related to an interval $t_0 \leq t \leq t_1$ by a (1,1) correspondence, that $f(x)$* is everywhere positive and that ) never vanishes in this interval. As we have seen, our expression then yields the area of the region bounded by the curve, the lines $x = a$ and $x = b$, and the $x$-axis. It is, of course, still subject to the disadvantages mentioned above. We shall now show that, if the curve $x=x(t)$, $y=y(t)$, $t_0 \leq t \leq t_1$ is a closed curve bounding a region of area $A_{01}$, this area is given by an integral which in form is exactly the same as the preceding one.

* i.e., is such that everyone of its points corresponds to a single value of $t$ in the interval $t_0 \leq t \leq t_1$ and conversely.

$$s_m(\alpha) = \frac{1}{m+1}\left[\frac{\sin\frac{(m+1)\alpha}{2}}{\sin\frac{\alpha}{2}}\right]^2.$$

Consider now a closed curve which is represented parametrically by the equations $x=x(t)$, $y = y(t)$, the curve being described just once as $t$ describes the interval $t_0 \leq t \leq t_1$. In order that the curve may be closed, it is essential that $x(t_0)= x(t_1)$ and $y(t_0)=y(t_1)$. We shall assume that the derivatives are continuous except at most for a finite number of

jump-discontinuities, and that $-\int_{t_0}^{t_1} y\dot{x}\,dt,$ differs from zero except perhaps at a finite number of points which may be corners of the curve.

A continuous curve $x = x(t)$, $y = y(t)$ is said to have a **corner** at $t = t_0$, if the positive direction of the tangent approaches a limit as $(t - t_0) \to 0$ through positive values and approaches a limit as $(t - t_0) \to 0$ through negative, but the two limits are not the same.

We shall first consider a closed curve which has no corners and is convex and of such a type that no straight line intersects it at more than two points. We denote by $P_1$ and $P_2$ the points at which the curve has a vertical tangent; these tangents are said to be **lines of support** at $P_1$ and $P_2$, respectively, because the points of the curve in the neighbourhood of $P_1$ and $P_2$ lie entirely on one side of the line. We can then (Fig. 8 above) regard the area to be bounded by the curve as the sum of the area $A_{12}$, bounded by the closed curve $P_1MP_2ABP_1$, formed as in the preceding section, and the area $A_{21}$, bounded by the closed curve $P_2NP_1BAP_2$. We assume here that the curve is described in the positive sense, as in the figure; by our sign convention, $A_{12}$ is then positive and $A_{21}$ negative. Let the point $x(t)$, $y(t)$ describe the upper part of the curve from $P_1$ to $P_2$ as $t$ moves from $t_0$ to $\tau$, and the lower part from $P_2$ to $P_1$ as $t$ moves from $\tau$ to $t_1$. We then obtain immediately

$$A_{12} = -\int_{t_0}^{\tau} y(t)\dot{x}(t)\,dt\ ,\ A_{21} = -\int_{\tau}^{t_1} y(t)\dot{x}(t)\,dt,$$

whence the total area bounded by the convex curve is

$$A = -\int_{t_0}^{t_1} y(t)\dot{x}(t)\,dt.$$

If we denote by the **absolute area** of a region the number of square units contained in it, - which is, of course, never negative - then the above expression always yields the absolute area bounded by the curve except perhaps for the sign. In order to see what happens when we reverse the sense in which the curve is described, we simply take the same integral from $t_1$ to $t_0$ instead of from $t_0$ to $t_1$; then our integral becomes

$$-\int_{t_1}^{t_0} y\dot{x}\,d\tau,$$

which is equal to $-A$. We thus recognize the truth of the statement:

The area represented by our formula is positive or negative according to the sense in which the boundary is described.

In drawing the figure, we have assumed that $y > 0$ for all points of the curve. This really does not restrict the generality of the result. In fact, if we displace the curve by a distance $a$ parallel to the $y$-axis, without rotating it, in other words, replace $y$ by $y+a$, the area is unchanged; the value of the integral is likewise unaltered, for the above integral is replaced by

$$-\int_{t_0}^{t_1}(y+a)\dot{x}(t)\,dt,$$

and, since the curve is closed,

$$\int_{t_0}^{t_1} a\dot{x}\,dt = a\{x(t_1) - x(t_0)\} = 0.$$

Two simple observations enable us to extend our results. Firstly, our formula remains valid for closed curves which do not intersect themselves, even if they are not convex, but have a more general form as is shown in Fig. 9. Secondly, the derivatives may have jump discontinuities or may both vanish at a finite number of points, which may represent corners; according to 4.8, the function $y\dot{x}$ remains integrable. (The ordinate to a corner-point is considered to be a line of support, if the curve lies in the neighbourhood of the point entirely to one side of the ordinate). We assume that the curve has only a finite number of lines of support, corresponding to the points $P_1$, $P_2$, ⋯ , $P_n$ and subdivide the

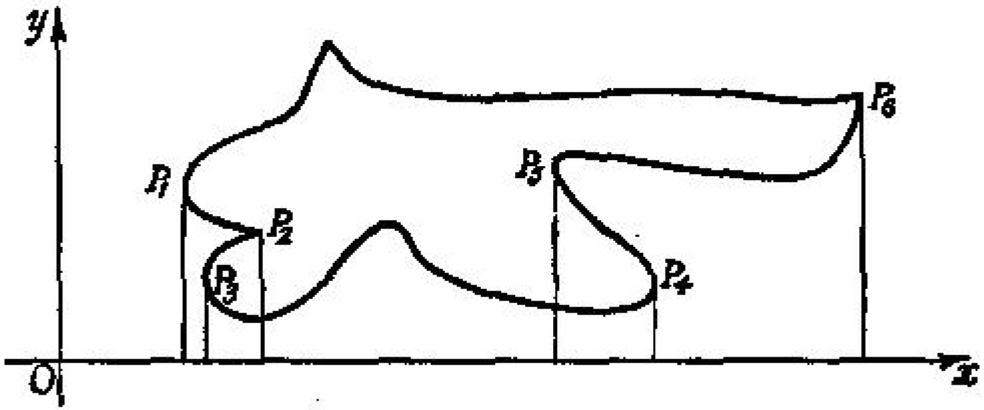



Fig. 9

curve into the single-valued branches $P_1P_2 \cdots P_{n-1}P_nP_1$. Then, as in Fig. 9, we obtain the area bounded by the curve in the form $A = A_{12}+A_{23}+\cdots+A_{n-1,n}+A_{n1}$. (cf. Fig. 9, which illustrates this for the case $n = 6$.) If we express each of these portions of area parametrically and combine the expressions into a single integral, we find that the area bounded by the curve is given by

$$-\int_{t_0}^{t_1} y\dot{x}\,dt,$$

which, as before, has the same sign as the sense in which the boundary curve is traversed.

In a certain sense, our formula even gives us the area in the case where the curve intersects itself. But we shall not enter into such a discussion here; readers who wish to do so may turn to A5.2,

We can express our formula for the area in a more elegant symmetrical form if we first apply to the integral integration by parts:

$$\int_{t_0}^{t_1} y\dot{x}\,dt = -\int_{t_0}^{t_1} x\dot{y}\,dt + xy\Big|_{t_0}^{t_1}.$$

Since the curve is closed,

$$x(t_0) = x(t_1), \quad y(t_0) = y(t_1),$$

whence

$$A = -\int_{t_0}^{t_1} y\dot{x}\,dt = \int_{t_0}^{t_1} x\dot{y}\,dt.$$

If we form the arithmetic mean of the two expressions, we obtain the symmetrical form

$$A = -\frac{1}{2}\int_{t_0}^{t_1} (y\dot{x} - x\dot{y})\,dt.$$

Instead of finding for the area the second expression above by integration by parts, we could have derived it by using the fact that, as regards the definition of area, the $x$-axis and the $y$-axis are interchangeable, except that the sense of rotation which brings the $x$-axis into the $y$-axis along the shortest way is opposite to the sense which brings the $y$-axis into the $x$-axis along the shortest way.

5.2.3 Remarks and an Example: In connection with these expressions, we must make a remark of a fundamental nature. Both the proof and the statement of the formulae depend on a particular system of rectangular co-ordinates. But the value of the area - a purely geometrical quantity - cannot depend on the chosen co-ordinate system. It is therefore important to show that a change of co-ordinates does not affect our integrals.

If the axes are merely displaced without rotation, obviously the integrals are unaltered (cf. above). Now let us assume that the axes are rotated through an angle $\alpha$; instead of $x$ and $y$, we now have new variables $\xi$ and $\eta$, defined by the equations

$$x = \xi\cos\alpha - \eta\sin\alpha, \quad y = \xi\sin\alpha + \eta\cos\alpha,$$

the new variables being also functions of the parameter $t$. If we recall that

$$\dot{x} = \dot{\xi}\cos\alpha - \dot{\eta}\sin\alpha, \quad \dot{y} = \dot{\xi}\sin\alpha + \eta\cos\alpha,$$

a short calculation yields

$$y\dot{x} - x\dot{y} = \eta\dot{\xi} - \xi\dot{\eta},$$

whence

$$A = -\frac{1}{2}\int_{t_0}^{t_1}(y\dot{x} - x\dot{y})\,dt = -\frac{1}{2}\int_{t_0}^{t_1}(\eta\dot{\xi} - \xi\dot{\eta})\,dt.$$

This equation expresses the fact that the area does not depend on the co-ordinate system.

Our integral expression for the area also does not depend on the choice of parameter. In fact, let us introduce a new parameter $\tau$ by the equation $\tau = \tau(t)$; we have

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt}, \quad \frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt},$$

so that

$$\begin{aligned} -\int_{t_0}^{t_1}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)dt &= -\int_{t_0}^{t_1}\left(y\frac{dx}{d\tau} - x\frac{dy}{d\tau}\right)\frac{d\tau}{dt}\,dt \\ &= -\int_{\tau_0}^{\tau_1}\left(y\frac{dx}{d\tau} - x\frac{dy}{d\tau}\right)d\tau, \end{aligned}$$

where $\tau_0$ and $\tau_1$ are the initial and final values of the new parameter, corresponding to the parametric values $t_0$ and $t_1$, respectively.

In this section, we have based the definition of area on the concept of the integral and have shown that this analytical definition has a truly geometrical character, since it yields a quantity independent of the co-ordinate system. However, it is easy to give a direct geometrical definition of the area bounded by a closed curve which does not intersect itself, as follows: The area is the upper bound of the areas of all polygons lying interior to the curve. The proof that the two definitions are equivalent is quite simple, but will not be given here.

As an example of the application of our formulae for the area, we consider the ellipse $y = \frac{b}{a}\sqrt{(a^2 - x^2)}$. In order to find its area, we take the upper and lower halves of the ellipse separately and in this way express its area by the integral

$$2\frac{b}{a}\int_{-a}^{+a}\sqrt{(a^2 - x^2)}\,dx.$$

However, if we use the parametric representation $x = a\cos t$, $y = b\sin t$, we find immediately that its area is given by

$$ab\int_0^{2\pi}\sin^2 t\,dt.$$

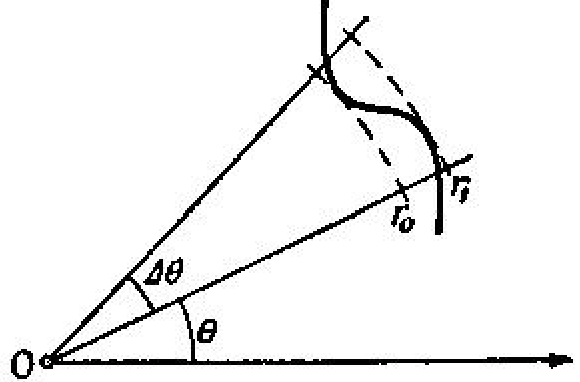


Fig. 10.—Element of area in polar co-ordinates

This can be integrated as in 4.2.3; its value is $ab\pi$.

**5.2.4 Areas in Polar Co-ordinates:** For many purposes, it is important to be able to calculate areas using polar co-ordinates. Let $r = f(\theta)$ be the equation of a curve in polar co-ordinates. Let $A(\theta)$ be the area of the region which is bounded by the $x$-axis (that is, the line $\theta = 0$), the line through the origin forming an angle $\theta$ with the $x$-axis, and the portion of the curve between these two Lines. Then

$$ab \int_0^{2\pi} \sin^2 t \, dt.$$

In fact, if we consider the radius vector corresponding to the angle $\theta$ and that corresponding to the angle $\theta+\Delta\theta$, and denote the smallest radius vector in this angular interval (Fig.10) by $r_0$ and the largest by $r_1$, the sector lying between the radius vector $\theta$ and the radius vector $\theta + \Delta\theta$ will have an area $\Delta A$ which lies between the bounds $\frac{1}{2}r_0^2\Delta\theta$ and $\frac{1}{2}r_1^2\Delta\theta$). Consequently,

$$\frac{1}{2}r_0^2 \leqq \frac{\Delta A}{\Delta\theta} \leqq \frac{1}{2}r_1^2,$$

and on passing to the limit as $\Delta\theta \to 0$, we obtain the above relation. By the fundamental theorem of the integral calculus, the area of the sector between the polar angles $\alpha$ and $\beta$ is then given by

$$\frac{1}{2}\int_\alpha^\beta r^2 \, d\theta.$$

If $\beta > \alpha$, this expression cannot be less than zero. Since we readily see that, as $\theta$ increases, the point with co-ordinates $(r,\theta)$ describes the boundary of the region in the positive sense, this is in agreement with our previous sign convention.

As an example, consider the area bounded by one loop of a lemniscate/. Its equation is $r^2 = 2\,a^2\cos 2\theta$ (cf. A2.1) and we obtain one loop by letting $\theta$ vary from $-\pi/4$ to $+\pi/4$. This gives us the expression

$$a^2 \int_{-\pi/4}^{\pi/4} \cos 2\theta \, d\theta$$

for the area. This can be integrated at once by introducing the new variable $u=2\theta$; we find the value of the integral to be $a^2$.

**5.2.5 Length of a Curve:** Another important geometrical concept - the **length of arc** - leads to integration.

To start with, we shall explain geometrically how we are led to a definition of the length of an arbitrary curve. The elementary process of measuring a length consists of comparing the length to be measured with rectilinear standards of length. The simplest method is to apply our standard length to the curve, with its ends on the curve, and count the number of times that we have to repeat the process in order to pass from the beginning to the end of the curve; we can refine the method as required by using smaller and smaller standards of length. By analogy with this elementary intuitive idea, we set up the definition of the length of a curve as follows: We assume that our curve is given by the equations $x = x(t)$, $y = y(t)$, $\alpha \leq t \leq \beta$ (This includes curves in the form $y = f(x)$, since these can be written as $y = f(t)$, $x = t$.) In the interval between $\alpha$ and $\beta$, we choose points $t_0 = \alpha$, $t_1$, $t_2$, $\cdots$ , $t_n = \beta$ in that order. We join the points on the curve, corresponding to these values of $t$, in order of the line segments, thus obtaining part of a polygon inscribed in the curve; we now measure the perimeter of this polygon. This length will depend on the way in which the points $t_\nu$ , or, as we may also say, the vertices of the polygon are chosen. We now let the number of the points $t_\nu$ increase beyond all bounds in such a way that the length of the longest subinterval in the interval $\alpha \leq t \leq \beta$ at the same time tends to 0; this causes the number of sides of our polygon to increase without limit, while the length of the longest side tends to 0. The length of the curve is then defined to be the limit of the perimeters of these inscribed polygons, provided that such a limit exists and is independent of the particular way in which the polygons are chosen. It is only when this assumption that the limit exists (the assumption of **rectifiability** is fulfilled that we can speak of the length of the curve. We shall soon see that very wide classes of curves can be proved to be rectifiable.

In order to express the length analytically by an integral, we think, in fact, of the curve as being represented in the first instance by a function $y = f(x)$ with a continuous derivative $y'$. We subdivide by the points $a = x_1$, $x_2$, $\cdots$ , $x_n = b$ the interval $a \leq x \leq b$ of the $x$-axis, above which lies our curve, into $(n - 1)$ intervals of lengths $\Delta x_1$, $\cdots$ , $\Delta x_{n-1}$. We inscribe in the curve a polygon the vertices of which lie vertically above these points. By Pythagoras' theorem, the total length of this inscribed polygon is given by (Fig. 11)

$$\sum_{\nu=1}^{n-1} \sqrt{(\Delta x_\nu^2 + \Delta y_\nu^2)} = \sum_{\nu=1}^{n-1} \sqrt{\left\{1 + \left(\frac{\Delta y_\nu}{\Delta x_\nu}\right)^2\right\}}\, \Delta x_\nu.$$

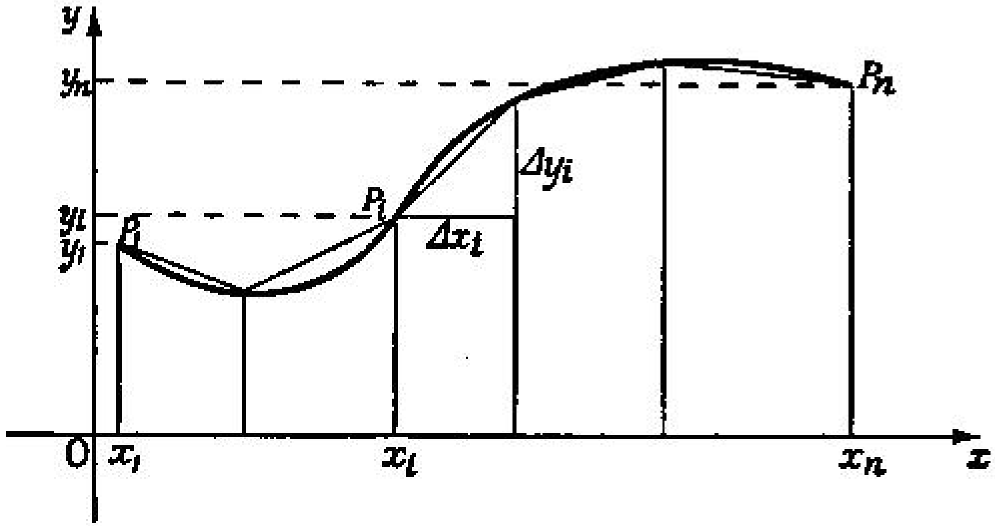



Fig. 11.—Rectification of curves

However, by the mean value theorem of the differential calculus, the difference quotient $\Delta y_\nu/\Delta x_\nu$ is equal to $f'(\xi_\nu)$, where $\xi_\nu$ is an intermediate value in the interval $\Delta x_\nu$. If we now let $n$ increase beyond all bounds and at the same time let the length of the longest sub-interval $\Delta x$ tend to zero, then, by the definition of the integral, our expression will tend to the limit

$$\int_a^b \sqrt{(1 + y'^2)}\, dx.$$

Since this passage to the limit always leads us to the same result, namely, the integral, no matter how the subdivision of the interval is made, we have established the theorem:

Every curve $y = f(x)$, for which the derivative $f'(x)$ is continuous, is a rectifiable curve and its length between $x = a$ and $x = b$ $(b \geq a)$ is given by

$$s(a, b) = \int_a^b \sqrt{(1 + y'^2)}\, dx.$$

If we denote by $s$ the length of arc measured from an arbitrary fixed point to the point with abcissa $x$, the above equation yields for the derivative of the length of arc with respect to $x$:

$$\frac{ds}{dx} = \sqrt{(1 + y'^2)}.$$

Our expression for the length of arc is still subject to the special and artificial assumption that the curve consists of one single-valued branch above the *x*-axis. Parametric representation removes this restriction. If a curve of the kind under consideration is given in parametric form by the equations $x=x(t)$, $y=y(t)$, then we obtain by introduction of the parameter $t$ into the above expression the parametric form of the length of arc

$$s(\alpha, \beta) = \int_\alpha^\beta \sqrt{(\dot{x}^2 + \dot{y}^2)}\, dt,$$

where $\alpha$ and $\beta$ are the values of $t$ which correspond to the points $x=a$ and $x=b$ of the curve, respectively.

This parametric expression for the length of a curve has a considerable advantage over the previous form in that it is not restricted to single-valued branches of the curves, represented by the equation $y=f(x)$, but instead it holds for any arbitrary arcs of curves, including closed curves, provided that the derivatives $\dot{x}$ and $\dot{y}$ are continuous along the arcs.

We recognize this most readily by going back again to the formula for the length of the inscribed polygon. We assume that $\dot{x}$ and $\dot{y}$ are continuous along the arc. As in the definition, we subdivide the interval $\alpha \le t \le \beta$ by points $t_0=\alpha$, $t_1$, $\cdots$, $t_n = \beta$, with the differences $\Delta t_\nu$ and use the corresponding points on the curve as vertices of an inscribed polygon; in the passage to the limit $n\to\infty$, we assume that the greatest difference $\Delta t_\nu$ tends to 0. If we now write the length of the polygon in the form

$$\sum_{\nu=1}^{n} \sqrt{(\Delta x_\nu^2 + \Delta y_\nu^2)} = \sum_{\nu=1}^{n} \sqrt{\left\{\left(\frac{\Delta x_\nu}{\Delta t_\nu}\right)^2 + \left(\frac{\Delta y_\nu}{\Delta t_\nu}\right)^2\right\}}\, \Delta t_\nu,$$

we see at once that this sum tends to the integral

$$\int_\alpha^\beta \sqrt{(\dot{x}^2 + \dot{y}^2)}\, dt;$$

we need only recall the generalized method of formation of an integral (A2.1). If the curve is composed of several arcs of this type, which may join one another at corners, the expression for the length of the curve is simply the sum of the corresponding integrals. Collecting the results, we have the statement:

If in the interval $\alpha \le t \le \beta$ the functions $x(t)$ and $y(t)$ are continuous as well as their derivatives $\dot{x}$ and $\dot{y}$, except perhaps for a finite number of jump discontinuities, the arc of $x = x(t)$, $y = y(t)$ has the length

$$\int_\alpha^\beta \sqrt{(\dot{x}^2 + \dot{y}^2)}\, dt,$$

where this integral, if necessary, is to be taken as an improper integral in the sense of 4.8.

By virtue of this formula, in which $\alpha$ must be less than $\beta$, there is a meaning in ascribing a negative length, given by the same formula, to an arc of a curve traversed in the direction in which the value of the parameter $t$ decreases. The sign of the length of arc therefore depends on the choice of the parameter. If we introduce a new parametric expression for the same curve which does not reverse the sense of description, that is, if we introduce a new parameter by the equation $\tau = \tau(t)$, where $d\tau/dt > 0$, we see *a priori* that our integral formula should give the same value no matter whether $t$ or $\tau$ is used as parameter, because the two integrals yield the length of the same curve and must therefore be equal. However, this may also be verified directly, because

$$\begin{aligned}\int \sqrt{(\dot{x}^2 + \dot{y}^2)}\, dt &= \int \sqrt{\left\{\left(\frac{dx}{d\tau}\right)^2 \left(\frac{d\tau}{dt}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 \left(\frac{d\tau}{dt}\right)^2\right\}}\, dt \\ &= \int \sqrt{\left\{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2\right\}}\, d\tau.\end{aligned}$$

We now present the expression for the length of arc when the curve is expressed in polar co-ordinates. In the last expression, we need only substitute for $\dot{x}$ and $\dot{y}$ their values as given by the formula ($a$) in 5.1.4 in order to obtain

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2,$$

whence

$$s(\alpha, \beta) = \int_{\alpha}^{\beta} \sqrt{(\dot{r}^2 + r^2\dot{\theta}^2)}\, dt.$$

If we now step over from the parametric expression to the equation in the form $r = f(\theta)$ by introducing as parameter $t = \theta$ itself, so that $\dot{\theta} = 1,$ we find for the length of arc

$$s(\theta_0, \theta_1) = \int_{\theta_0}^{\theta_1} \sqrt{(\dot{r}^2 + r^2)}\, d\theta.$$

A simple example of the explicit calculation of the length of an arc is given by the parabola $y = x^2/2$; we obtain immediately for its length of arc the integral

$$\int_a^b \sqrt{(1 + x^2)}\, dx,$$

which with the substitution $x = \sinh u$ becomes

$$\int_{\operatorname{ar\,sinh} a}^{\operatorname{ar\,sinh} b} \cosh^2 u\, du = \frac{1}{2}\int_{\operatorname{ar\,sinh} a}^{\operatorname{ar\,sinh} b} (1 + \cosh 2u)\, du = \frac{1}{2}(u + \sinh u \cosh u)\Big|_{\operatorname{ar\,sinh} a}^{\operatorname{ar\,sinh} b},$$

so that the length of arc of the parabola between the abscissae $x=a$ and $x=b$ is given by

$$s(a, b) = \frac{1}{2}\{\operatorname{ar\,sinh} b + b\sqrt{(1 + b^2)} - \operatorname{ar\,sinh} a - a\sqrt{(1 + a^2)}\}.$$

For the **catenary** $y = \cosh x$, we find

$$s(a, b) = \int_a^b \sqrt{(1 + \sinh^2 x)}\, dx = \int_a^b \cosh x\, dx, \quad \text{or} \quad s(a, b) = \sinh b - \sinh a.$$

Finally, note that it is convenient in many cases to introduce as parameter the length of arc reckoned from some fixed point $P_0$ on the curve, that is, to take $x = x(s)$ and $y = y(s)$. Points of the curve on opposite sides of $P_0$ will correspond to values of $s$ with opposite signs. In this case, we have

$$\dot{x}^2 + \dot{y}^2 = \left(\frac{ds}{dt}\right)^2 = 1,$$

whence by differentiation

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0;$$

these two relations are applied frequently.

**5.2.6 Curvature of a Curve:** The axes and the length of arc of a curve depend on its complete course. We now insert a discussion of a concept which has reference only to the behaviour of a curve in the neighbourhood of a point - its curvature.

If we think of a curve as being described uniformly in the positive sense in such a way that equal lengths of arc are passed over in equal periods of time, the direction of the curve will vary at a definite rate, which we take as a measure of its curvature. Hence, denoting the angle between the positive direction of the tangent (5.1.3) and the positive *x*-axis by $\alpha$ and thinking of $\alpha$ as a function of the length of arc *s*, we shall define the curvature *k* at the point corresponding to the length of arc *s* by the equation $k = d\alpha/ds$. We know that $\alpha = \text{artan } y'$, whence, by the chain rule,

$$\frac{d\alpha}{ds} = \frac{d\alpha}{dx} \div \frac{ds}{dx} = \frac{y''}{1+y'^2} \cdot \frac{1}{\sqrt{(1+y'^2)}}$$

(where the positive sign of the square root means that increasing values of *x* correspond to increasing values of *s*). Hence, the curvature is given by

$$k = \frac{y''}{(1+y'^2)^{3/2}}$$

Using the parametric formulae for *y*' and *y*'', we obtain the simple expression for the curvature of a parametrically represented curve:

$$k = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}},$$

which, of course, can also be found directly from the equation

$$\alpha = \operatorname{arc} \tan \frac{\dot{y}}{\dot{x}} = \operatorname{arc} \cot \frac{\dot{x}}{\dot{y}}.$$

In contrast to the previous expression, which depends on the equation $y = f(x)$ and consequently involves a special assumption about the position of the arc with respect to the $x$-axis, the parametric expression for the curvature holds for all arcs along which $\dot{x}$, $\dot{y}$, $\ddot{x}$, and $\ddot{y}$ are continuous functions of $t$ and $\dot{x}^2 + \dot{y}^2 \neq 0$. In particular, it holds for points where $\dot{x} = 0$, i.e., where $dy/dx$ becomes infinite.

If we introduce the length of arc $s$ as parameter and recall that

$$\dot{x}^2 + \dot{y}^2 = 1 \text{ and } \dot{x}\ddot{x} + \dot{y}\ddot{y} = 0,$$

we find

$$k = \dot{x}\ddot{y} - \dot{y}\ddot{x} = \ddot{y}\left(\dot{x} + \dot{y}\frac{\dot{y}}{\dot{x}}\right) = \frac{\ddot{y}}{\dot{x}} = -\frac{\ddot{x}}{\dot{y}}.$$

We thus obtain a particularly simple expression for the curvature.

The sign of the curvature is changed, if we reverse the sense of description of the curve, that is, if we replace the parameter $t$ or $s$ by the new parameter $\tau = -t$ or $\sigma = -s$, because then $\dot{x}$ and $\dot{y}$ change their sign, but not $\ddot{x}$, $\ddot{y}$, $\dot{x}^2$ or $\dot{y}^2$, as follows from a simple calculation:

$$\frac{d}{d\tau}x\{t(\tau)\} = \frac{dx}{dt}\,\frac{dt}{d\tau} = (\dot{x})\,(-1);$$

$$\frac{d^2}{d\tau^2}x\{t(\tau)\} = \frac{d}{d\tau}[-\dot{x}\{t(\tau)\}] = -\frac{d\dot{x}}{dt}\,\frac{dt}{d\tau} = (-\ddot{x})\,(-1).$$

(A similar calculation can be made for *y*.) In the case of the expression $k = \frac{y''}{(1+y'^2)^{3/2}}$ found first, this fact is concealed, since it is natural and customary to think of a curve as described from the left to the right hand side, in which case the square root can only be positive.

As an example, consider the curvature of a positively described circle with radius *a*. If we start from the parametric representation $x = a\cos t$, $y = a\sin t$, we obtain immediately

$$k = \frac{1}{a}.$$

Hence the curvature of a positively described circle is the reciprocal of its radius. This result assures us that our definition of curvature is really suitable, because, in the case of a circle, we naturally think of the reciprocal of the radius as a measure of its curvature.

Let us set $\rho = 1/k$. In general, the quantity $|\rho| = 1/|k|$ is called the **radius of curvature** of a curve at the point in question. For a given point on a curve, that circle which touches the curve at the point and has there the same sense of description and the same curvature as the curve and, moreover, has its centre on the positive or negative side of the normal according to whether *k* is positive or negative, is called the **circle of curvature**, corresponding to the point. Let us think of the equation of the circle (or an arc of the circle containing the point in question) as being written in the form $y=g(s)$. Then, at the point in question, we do not only have $f(x) = g(x)$ and $f'(x) = g'(x)$, as follows from the fact that the circle and curve touch, but, by virtue of the relation

$$\frac{f''(x)}{\sqrt{\{1+f'(x)^2\}^3}} = k = \frac{g''(x)}{\sqrt{\{1+g'(x)^2\}^3}},$$

we also have

$$f''(x) = g''(x).$$

The centre of the circle of curvature is called the **centre of curvature**, corresponding to the given point. Its co-ordinates are expressed parametrically by

$$\xi = x - \frac{\rho \dot{y}}{\sqrt{(\dot{x}^2 + \dot{y}^2)}}, \quad \eta = y + \frac{\rho \dot{x}}{\sqrt{(\dot{x}^2 + \dot{y}^2)}}.$$

In order to prove this, we need only employ the formulae for the direction cosines of the normal, on which the centre of curvature lies at a distance $1/|k|=|\rho|$ from the tangent. These formulae yield an expression for the centre of curvature in terms of the parameter $t$. As $t$ describes its range, the centre of curvature describes a curve, the socalled **evolute** of the given curve and since, together with $x$ and $y$, we must regard $\dot{x}$, $\dot{y}$, and $\rho$ as known functions of $t$, the formulae above yield parametric equations for the evolute.

For special examples, refer to 5.3 and to A5.1.

**5.2.7 Centre of Mass and Moment of a Curve:** We now come to some applications which take us into mechanics. We consider a system of $n$ particles in a plane. Let $m_1, m_2, \cdots, m_n$ be the masses of these particles, and $y_1, y_2, \cdots, y_n$ their respective ordinates. We then call

$$T = \sum_{\nu=1}^{n} m_\nu y_\nu = m_1 y_1 + m_2 y_2 + \ldots + m_n y_n$$

the **moment of the system of particles with respect to the *x*-axis**. The expression $\eta = T/M$, where $M$ denotes the total mass $m_1 + m_2 + \ldots + m_n$ of the system, gives us the **height of the centre of mass** of the system of particles above the $x$-axis. We define the moment about the $y$-axis and the abscissa of the centre of mass in a corresponding way.

We shall now see that this idea is readily extended to yield a definition of the moment of a curve along which a mass is distributed uniformly, and of the co-ordinates $\xi$ and $\eta$ of the centre of mass of such a curve. Merely for the sake of brevity, we assume that the density along the curve is constant, say $\mu$; any continuous distribution could equally well be discussed in the same manner.

In order to arrive at this extension, we return to the consideration of a system of a finite number of particles and then pass on to the limit. For this purpose, we assume that the length of arc $s$ is introduced as a parameter on the curve and that the curve is subdivided by ($n$—1) points into arcs of lengths $\Delta s_1, \Delta s_2, \cdots, \Delta s_n$. We represent as concentrated the mass $\mu \Delta s_i$ of each arc $\Delta s_i$ at an arbitrary point $s$ of the arc, say, with the co-ordinate $y_i$.

By definition, the moment of this system of particles with respect to the $x$-axis has the value

$$T = \mu \Sigma y_i \Delta s_i.$$

If now the largest of the quantities $\Delta s_i$ tends to 0, this sum tends to a definite limit given by

$$T = \mu \int_{s_0}^{s_1} y \, ds = \mu \int_{x_0}^{x_1} y \sqrt{(1 + y'^2)} \, dx,$$

which we shall therefore naturally accept as the definition of the moment of a curve with respect to the $x$-axis. Since the total mass of the curve equals its length, multiplied by $\mu$, is

$$\mu \int_{s_0}^{s_1} ds = \mu (s_1 - s_0),$$

we are immediately led to the expressions for the co-ordinates of the centre of mass of the curve:

$$\eta = \frac{\int_{s_0}^{s_1} y \, ds}{s_1 - s_0}, \quad \xi = \frac{\int_{s_0}^{s_1} x \, ds}{s_1 - s_0}.$$

These statements are actually definitions of the moment and centre of mass of a curve, but they are such straightforward extensions of the simpler case of a number of particles, so that we naturally expect that - as is actually the case - any statement in mechanics, which involves the centre of mass or the moment of a system of particles, will also apply to curves. In particular, the position of the centre of mass with respect to a curve does not depend on the co-ordinate system.

**5.2.8 Area and Volume of a Surface of Revolution:**. If we rotate the curve $y = f(x)$, for which $f(x) \geq 0$, about the $x$-axis, it describes a so-called surface of revolution. The area of this surface, the abscissae of which we assume to lie

between the bounds $x_0$ and $x_1 > x_0$, can be obtained by a discussion analogous to the preceding work. In fact, if we replace the curve by an inscribed polygon, we shall have instead of the curved surface a figure composed of a number of thin truncated cones. Following these intuitive suggestions, we define the area of a surface of revolution as the limit of the areas of these conical surfaces as the length of the longest side of the inscribed polygon tends to zero. We know from elementary geometry that the area of each truncated cone is equal to its slant height multiplied by the circumference of the circular section of mean radius. If we add these expressions and then carry out the passage to the limit, we obtain for the area the expression

$$A = 2\pi \int_{x_0}^{x_1} y\sqrt{(1+y'^2)}\,dx = 2\pi \int_{s_0}^{s_1} y\,ds.$$

Expressed in words, this result states that the area of a surface of revolution is equal to the length of the generating curve multiplied by the distance, traversed by the centre of mass (**Guldin's rule**).

In the same way, we find that the volume interior to a surface of revolution, which is bounded at the ends by the planes $x = x_0$ and $x = x_1 > x_0$, is given by

$$V = \pi \int_{x_0}^{x_1} y^2\,dx.$$

This formula is obtained by following the intuitive suggestion that the volume in question is the limit of the volumes of the above-mentioned figures consisting of truncated cones. The remainder of the proof is left to the reader.

**5.2.9 Moment of Inertia:** In the study of rotatory motion in mechanics, an important role is played by a certain quantity called a moment of inertia which will now be discussed briefly.

We suppose that a particle $m$ at a distance $y$ from the $x$-axis rotates uniformly about that axis with angular velocity $\omega$ (that is, it rotates in unit time through an angle $\omega$). The kinetic energy of the particle, expressed by half the product of the mass and the square of its velocity, is obviously

$$\frac{m}{2}(y\omega)^2.$$

We call the coefficient of $\omega^2/2$, that is the quantity $my^2$, the moment of inertia of the particle about the x-axis.

Similarly, if we have $n$ particles with masses $m_1, m_2, \cdots, m_n$ and ordinates $y_1, y_2, \cdots, y_n$, we call the expression

$$T = \sum_i m_i y_i^2$$

the **moment of inertia of the system of masses** about the $x$-axis. The moment of inertia is a quantity which belongs to the system of masses itself, without any reference to its state of motion. Its importance lies in the fact that, if the entire system is set in constant rotation about an axis, without change of the distances between pairs of particles, the kinetic energy is obtained by multiplying the moment of inertia about that axis by half the square of the angular velocity. Thus, the moment of inertia about an axis has the same role in rotation about an axis as has mass in rectilinear motion.

Suppose now that we have an arbitrary curve $y = f(x)$, lying between the abscissae $x_0$ and $x_1 > x_0$, along which a mass is distributed uniformly with unit density. In order to define the moment of inertia of this curve, we just proceed as we did in the 5.2.7; as before, we arrive at an expression for the moment of inertia about the $x$-axis, namely,

$$T_x = \int_{s_0}^{s_1} y^2\,ds = \int_{x_0}^{x_1} y^2\sqrt{(1 + y'^2)}\,dx.$$

We have for the moment of inertia about the $y$-axis the corresponding expression

$$T_y = \int_{s_0}^{s_1} x^2\,ds = \int_{x_0}^{x_1} x^2\sqrt{(1 + y'^2)}\,dx.$$

## 5.3. Examples

The theory of plane curves with its great variety of special forms and properties offers us a rich store of examples of these abstract concepts. But in order to avoid being lost in a mass of detail, we must limit ourselves to a few typical applications.

**5.3.1 The Common Cycloid:** We obtain at once from the equations $x = a(l - \cos t)$, $y = a(l - \sin t)$ the equations $\dot{x} = a(1 - \cos t)$, $\dot{y} = a \sin t$, whence the length of arc is

$$s = \int_0^\alpha \sqrt{(\dot{x}^2 + \dot{y}^2)}\, dt = \int_0^\alpha \sqrt{\{2a^2(1 - \cos t)\}}\, dt.$$

However, since 1 - cos $t$ = 2sin²$t$/2, the integrand is equal to 2$a$ sin $t$/2, whence for $0 \leq \alpha \leq 2\pi$

$$s = 2a \int_0^\alpha \sin\frac{t}{2}\, dt = -4a \cos\frac{t}{2}\Big|_0^\alpha = 4a\left(1 - \cos\frac{\alpha}{2}\right) = 8a \sin^2\frac{\alpha}{4}.$$

In particular, if we consider the length of arc between two successive cusps, we must set $\alpha = 2\pi$, since the interval $0 \leq t \leq 2\pi$ of values of the parameter corresponds to one revolution of the rolling circle. We thus obtain the value 8$a$, that is, the length of arc of the cycloid between successive cusps is equal to four times the diameter of the rolling circle.

Similarly, we calculate the area bounded by one arch of the cycloid and the $x$-axis:

$$\begin{aligned} I &= \int_0^{2\pi} y\dot{x}\, dt = a^2 \int_0^{2\pi} (1 - \cos t)^2\, dt \\ &= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt \\ &= a^2\left(t - 2\sin t + \frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} = 3a^2\pi. \end{aligned}$$

This area is therefore three times the area of the rolling circle.

For the radius of curvature $\rho = 1/k$, we have

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = -2a\sqrt{\{2(1 - \cos t)\}} = -4a\left|\sin\frac{t}{2}\right|;$$

at the points $t = 0$, $t = \pm 2\pi$, ···, this expression has the value zero. These are actually the cusps, where the cycloid meets the $x$-axis at right angles.

The area of the surface of revolution formed by rotation of an arch of the cycloid about the *x*-axis is given by our formula (cf. 5.2.8) as

$$A = 2\pi \int_0^{8a} y\, ds = 2\pi \int_0^{2\pi} a(1 - \cos t) \,.\, 2a \sin\frac{t}{2}\, dt$$

$$= 8a^2\pi \int_0^{2\pi} \sin^3\frac{t}{2}\, dt = 16a^2\pi \int_0^{\pi} \sin^3 u\, du$$

$$= 16a^2\pi \int_0^{\pi} (1 - \cos^2 u) \sin u\, du.$$

The last integral can be evaluated by means of the substitution cos $u$ = $v$; we find

$$A = 16a^2\pi \left(-\cos u + \frac{1}{3}\cos^3 u\right)\Big|_0^{\pi} = \frac{64a^2\pi}{3}.$$

As an exercise, the reader should find the height $\eta$ of the centre of mass of the cycloid above the *x*-axis as well as its moment of inertia

$$\eta = \frac{4}{3}a = \frac{A}{2\pi s} \quad \text{and} \quad T_x = \frac{256}{15}a^3.$$

**5.3.2 The Catenary:** The length of arc of the catenary has already been found as an example at the end of 5.2.5:

$$A = 2\pi \int_a^b \cosh^2 x\, dx = 2\pi \int_a^b \frac{1 + \cosh 2x}{2}\, dx$$

$$= \pi\left(b - a + \frac{1}{2}\sinh 2b - \frac{1}{2}\sinh 2a\right).$$

For the area of the surface of revolution obtained by rotating the catenary about the *x*-axis - the solaced **catenoid** - , we find

$$A = 2\pi \int_a^b \cosh^2 x \, dx = 2\pi \int_a^b \frac{1 + \cosh 2x}{2} \, dx$$

$$= \pi \left(b - a + \frac{1}{2}\sinh 2b - \frac{1}{2}\sinh 2a\right).$$

Moreover, this yields the height of the arc's centre of mass between *a* and *b*

$$\eta = \frac{A}{2\pi s} = \frac{b - a + \frac{1}{2}\sinh 2b - \frac{1}{2}\sinh 2a}{2(\sinh b - \sinh a)}$$

and, finally, the curvature

$$k = \frac{y''}{(1 + y'^2)^{3/2}} = \frac{\cosh x}{\cosh^3 x} = \frac{1}{\cosh^2 x}.$$

**5.3.3 The Ellipse and the Lemniscate:** The length of arc of these two curves cannot be reduced to elementary functions; they belong to the class of **elliptic integrals**, referred to in 4.7.1.

For the ellipse $y = \frac{b}{a}\sqrt{(a^2 - x^2)}$, we obtain

$$s = \frac{1}{a}\int \sqrt{\left\{\frac{a^4 - (a^2 - b^2)x^2}{a^2 - x^2}\right\}} \, dx = a \int \frac{1 - \kappa^2 \xi^2}{\sqrt{(1 - \xi^2)(1 - \kappa^2 \xi^2)}} \, d\xi,$$

where we have set $x/a = \xi$, $1 - b^2/a^2 = \kappa^2$. By the substitution $\xi = \sin \phi$, this integral becomes

$$s = \int \sqrt{\{a^2 - (a^2 - b^2)\sin^2 \varphi\}} \, d\varphi = a \int \sqrt{(1 - \kappa^2 \sin^2 \varphi)} \, d\varphi.$$

In order to obtain the semi-perimeter of the ellipse, we must let here *x* cover the interval from -*a* to +**a**, which corresponds to the interval

$$-1 \leqq \xi \leqq +1 \quad \text{or} \quad -\pi/2 \leqq \varphi \leqq +\pi/2.$$

For the lemniscate with the equation in polar co-ordinates $r^2=2a^2\cos 2t$, we obtain in a similar manner

$$s=\int\sqrt{(r^2+\dot{r}^2)}\,dt=\int\sqrt{\left(2a^2\cos 2t+2a^2\frac{\sin^2 2t}{\cos 2t}\right)}\,dt$$

$$=a\sqrt{2}\int\frac{dt}{\sqrt{(\cos 2t)}}=a\sqrt{2}\int\frac{dt}{\sqrt{(1-2\sin^2 t)}}.$$

If we introduce the independent variable $u = \tan t$ into the last integral, we have

$$\sin^2 t=\frac{u^2}{1+u^2},\quad dt=\frac{du}{1+u^2},$$

whence

$$s=a\sqrt{2}\int\frac{du}{\sqrt{(1-u^4)}}.$$

In a complete loop of the lemniscate, $u$ ranges from -1 to +1, whence the length of arc equals

$$a\sqrt{2}\int_{-1}^{+1}\frac{du}{\sqrt{(1-u^4)}},$$

a special elliptic integral which occupied an important place in the research of Gauss.

## Exercises 5.2:

1. Calculate the area bounded by the **semi-cubical parabola** $y = x^{3/2}$, the $x$-axis and the Lines « = $a$ and $x = b$.

2. Calculate the area of the region bounded by the line $y = x$ and the lower half of the loop of the **folium of Descartes**. (Use the parametric representation of 5.1.5 , Exercise 7.)

3. Calculate the area of a sector of the Archimedian spiral $r = a\theta\,(a > 0)$.

4. Caloulate the area of the cardioid (5.1.5 , Exercise 3.), using polar coordinates

5. Calculate the area of the astroid (5.1.5 , Exercise 6.).

6. Calculate the area of the pedal curve of the circle $x^2 + y^2 = 1$ with respect to a point $P$ ($x_0$,0} on the $x$-axis. Show that this area has a minimum when $P$ is at the origin.

7. Do the same as in 6. for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

8. Find the parametric representation of the cardioid when the length of arc is used as parameter.

9. Find the parametric representation of the cycloid when the length of arc is used as parameter.

10. Calculate the length of arc of the semi-cubical parabola $y = x^{3/2}$.

11. Calculate the length of the astroid.

12. Calculate the length of arc of:

($a$) The Archimmedian spiral $r = a\theta (a > 0)$.
($b$) The logarithmic spiral $r = e^{m\theta}$.
($c$) The cardioid (5.1.5, Exercise 3.).
(d) The curve $r = a(\theta^2 - 1)$.

13. Find the radius of curvature of ($a$) the parabola $y = x^2$; (6) the ellipse $x=a\cos\phi$, $y=b\sin\phi$ as a function of $x$ and of $\phi$, respectively. Find the maxima and minima of the radius of curvature and the points at which these maxima and minima occur.

14. Sketch the curve

$$x = \int_0^t \frac{\cos u}{\sqrt{u}}\,du, \quad y = \int_0^t \frac{\sin u}{\sqrt{u}}\,du$$

and determine its radius of curvature ($\rho$).

16. Show that the expression for the curvature of a curve $x = x(t)$, $y = y(t)$ is unaltered by a rotation of axes as well as by the change of parameter $t = \phi(\tau)$, where $\phi'(\tau) > 0$.

16. Let $r = f(\theta)$ be the equation of a curve in polar co-ordinates. Prove that the curvature is given by

$$k = \frac{2r'^2 - rr'' + r^2}{(r'^2 + r^2)^{3/2}}, \quad \text{where } r' = \frac{df}{d\theta}, \quad r'' = \frac{d^2f}{d\theta^2}.$$

17, Find the volume and surface of a zone of a sphere of radius $r$, i.e., of the portion of the sphere cut off by two parallel planes distant $h_2, h_1$ respectively, from the centre.

18. Find the volume and surface area of the **torus** or **anchor ring**, obtained by rotating a circle about a line which does not intersect it.

19. Find the area of the **catenoid**, the surface obtained by rotating an arc of the catenary $y = \cosh x$ about the $x$-axis.

20. Sketch the curve defined by the equation

$$x = \int_0^t \cos\left(\tfrac{1}{2}\pi t^2\right) dt, \quad y = \int_0^t \sin\left(\tfrac{1}{2}\pi t^2\right) dt.$$

What is the behaviour of the curve as $t$ ranges from $-\infty$ to $+\infty$? Calculate the curvature k as a function of the length of arc.

21. The curve for which the length of the tangent intercepted between the point of contact and the $y$-axis is always equal to 1. It is called the **tractrix**. Find its equation. Show that the radius of curvature at each point of the curve is inversely proportional to the length of the normal intercepted between the point on the curve and the $y$-axis. Calculate the length of arc of the tractrix and find its parametric equations in terms of the length of arc.

22. Let $x = x(t)$, $y = y(t)$ be a closed curve. A constant length $p$ is measured off along the normal to the curve. The extremity of this segment describes a curve, which is called a **parallel curve** to the original curve. Find the axes, the length of arc and the radius of curvature of the parallel curve.

23. Find the centre of mass of an arbitrary arc ($a$) of a circle of radius $r$, ($b$) of a catenary.

24. Calculate the moment of inertia about the $x$-axis of the boundary of the rectangle $a \le x \le b$, $\alpha \le y \le \beta$.

35. Calculate the moment of inertia of an arc of the catenary $y = \cosh x$ ($a$) about the $x$-axis, ($b$) about the $y$-axis.

26. The equation $y = f(») + \alpha$, $a \leq x \leq b$, represent a family of curves, one for each value of the parameter $\alpha$. Prove that m this family the curve with the smallest moment of inertia about the $x$-axis is that which has its centre of mass on the x-axis.

Answers and Hints

## 5.4 Some Very Simple Problems in the Mechanics of a Particle

Next to geometry, the differential and integral calculus are especially indebted to the science of mechanics for their early development. Mechanics rests upon certain basic principles which were first laid down by Newton; the statement of these principles involves the concept of the derivative and their application requires the theory of integration. Without analyzing these basic principles in detail, we shall illustrate by some simple examples how the integral and differential calculus are applied in mechanics.

**5.4.1 The Fundamental Hypotheses of Mechanics:** We shall restrict ourselves here to the consideration of a single **particle**, that is of a point at which a mass $m$ is imagined to be concentrated. Moreover, we shall assume that motion can only occur along a certain fixed curve, on which the position of the particle is specified by the length of arc $s$ measured from a fixed point on the curve; in particular, the curve may be a straight line, in which case we shall use the abscissa $x$ as the coordinate of the point instead of $s$. The motion of the point is determined by expressing the co-ordinate $s = \phi(t)$ as a function of the time. We shall mean by the **velocity of motion** the derivative $\phi'(t)$ or, as we shall also write,

$$\frac{ds}{dt} = \phi'(t) = \dot{s}.$$

We shall call the second derivative

$$\frac{d^2s}{dt^2} = \phi''(t) = \ddot{s}$$

the **acceleration**.

In mechanics, we start from the assumption that the motion of a point can be explained by means of **forces** of definite direction and magnitude. In the case of motion on a given curve, **Newton's second fundamental law** may be expressed as follows:

The mass multiplied by the acceleration is equal to the force acting on the particle in the direction of the curve; in symbols:

$$m\ddot{s} = F.$$

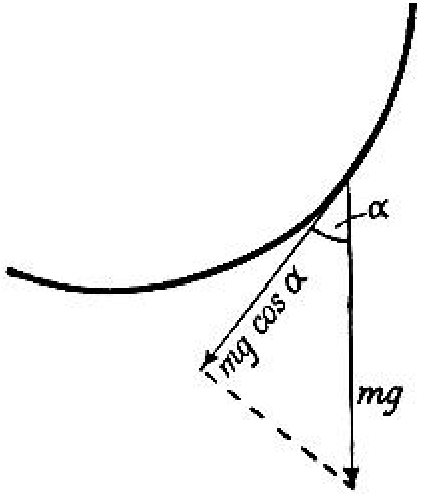


Fig. 12.—Motion on a given curve under gravity

Thus, the direction of the force is always the same as that of the acceleration; its direction is that of increasing values of *s*, if the velocity in that direction is increasing, otherwise it is opposed to the direction of increasing values of *s*.

The law of Newton is in the first instance nothing more than a definition of the concept of force. The left hand side of our equation is a quantity, which can be determined by observation of the motion, by means of which we measure the force. But this equation has a far deeper meaning. As a matter of fact, it turns out that in many cases we can determine the acting force from other physical assumptions without any consideration of the corresponding motion. The above fundamental law of Newton is then no longer a definition of the force, but instead it is a relation from which we can draw important conclusions about the motion.

The most important example of a known force is given to us by **gravity**. We know from direct measurements that the force of gravity acting on a mass $m$ is directed vertically downwards and is of magnitude $mg$, where the constant $g$, the so-called **gravitational acceleration**, is approximately equal to 981 if the time is measured in seconds and the lengths in centimetres. If a mass moves along a given curve, we learn by experiments that the force of gravity in the direction of this curve is equal to $mg\cos\alpha$, where $\alpha$ denotes the angle between the vertical and the tangent to the curve at the point under consideration (Fig. 12).

In the case of motion on our given curve, the basic problem of mechanics is: If we know the force acting on the particle (e.g., the force of gravity), we have to determine the position of the point, that is, its co-ordinate $s$ or $x$, as a function of the time.

If we restrict ourselves to the simplest case, in which this force * $mf(s)$ is known at the outset as a function of the length of arc - so that the force is independent of the time - we shall show how the course of the motion along the curve can be found from the equation

$$\ddot{s} = \frac{1}{m} F = f(s).$$

* The separation of the factor *m* in the expression for the given force is not essential, but makes the formula simpler.

We have to deal here with a differential equation, that is, an equation from which an unknown function - here $s(t)$ - is to be determined and in which the derivative of this function occurs as well as the function itself (3.7.1)

**5.4.2 Freely Falling Body. Resistance of Air:** In the case of the free fall of a particle along the vertical $x$-axis, Newton's law yields the differential equation

$$\ddot{x} = g.$$

Hence we have the equation $\dot{x}(t) = gt + v_0$, where $v_0$ is a constant of integration. Its meaning is easily found by setting $t = 0$. We then find $\dot{x}(0) = v_0$, that is, $v_0$ is the velocity of the particle at the instant from which the time is reckoned - the **initial velocity**. Another integration yields

$$x(t) = \tfrac{1}{2}gt^2 + v_0t + x_0,$$

where $x_0$ is also a constant of integration, the value of which is again found by setting $t = 0$; we thus find that $x_0$ is the **initial position**, that is, the co-ordinate of the point at the beginning of the motion.

Conversely, we can choose the initial position $x_0$ and the initial velocity $v_0$ arbitrarily and then obtain the complete representation of the motion from the equation $x = \tfrac{1}{2}gt^2 + v_0t + x_0$.

If we wish to take account of the effect of the **friction** or **air resistance** acting on the particle, we have to consider this as a force the direction of which is opposite to the direction of motion and concerning which we must make definite physical assumptions.

These assumptions must be chosen to suit the particular system under consideration; for example, the law of resistance for low speeds is not the same as that for high speeds (e.g., bullet velocities).

We shall now work out the results of different physical assumptions: (*a*) the resistance is proportional to the velocity, being given by an expression of the form $-r\dot{x}$, where $r$ is a positive constant; (*b*) the resistance is proportional to the square of the velocity, given as .$-r\dot{x}^2$. In accordance with Newton's law, we obtain for the equations of motion

$$(a)\ m\ddot{x} = mg - r\dot{x}, \quad (b)\ m\ddot{x} = mg - r\dot{x}^2.$$

If we at first consider $\dot{x} = u(t)$ as the function sought, we have $\ddot{x}(t) = \dot{u}(t)$, so that

$$(a)\ m\dot{u} = mg - ru, \quad (b)\ m\dot{u} = mg - ru^2.$$

Instead of determining by these equations *u* as a function of *t* , we determine *t* as a function of *u*, writing our differential equations in the form

$$(a)\ \frac{dt}{du} = \frac{1}{g - ru/m}, \quad (b)\ \frac{dt}{du} = \frac{1}{g - ru^2/m}.$$

With the methods, given in the Chapter IV, we can immediately carry out the integrations and obtain

$$(a)\ t(u) = -\frac{m}{r}\log\left(1 - \frac{r}{mg}u\right) + t_0,$$

$$(b)\ t(u) = -\frac{1}{2}k\log\frac{kg - u}{kg + u} + t_0,$$

where we have set $\sqrt{(m/rg)} = k$ and $t_0$ is a constant of integration. Solving these equations for *u*, we find

$$(a)\ u(t) = -\frac{mg}{r}\left(e^{-r(t-t_0)/m} - 1\right),$$

$$(b)\ u(t) = -gk\frac{e^{-2(t-t_0)/k} - 1}{e^{-2(t-t_0)/k} + 1}.$$

These equations at once reveal an important property of the motion. The velocity does not increase with time beyond all bounds, but tends to a definite limit which depends on the mass *m*. We have for

$$(a)\ \lim_{t\to\infty} u(t) = \frac{mg}{r}, \quad (b)\ \lim_{t\to\infty} u(t) = \sqrt{\frac{mg}{r}}.$$

A second integration, performed on our expressions for $u(t) = \dot{x}$ by the methods of the Chapter IV yields the results (which may be verified by differentiation)

$$(a)\ x(t) = \frac{m^2}{r^2} g e^{-r(t-t_0)/m} + \frac{mg}{r} t + c,$$

$$(b)\ x(t) = \frac{m}{r} \log \cosh \sqrt{\frac{rg}{m}} (t - t_0) + c,$$

where $c$ is a new constant of integration. The two constants of integration $t_0$ and $c$ are readily determined, if we know the initial position $x(0) = x_0$ and the initial velocity $\dot{x}(0) = u(0) = v_0$ of the falling particle.

**5.4.3 The Simplest Type of Elastic Vibration:** As a second example, we consider the motion of a particle which moves along the $x$-axis and is pulled back towards the origin by an **elastic force**. As regards this force, we assume that it is always directed towards the origin and that its magnitude is proportional to the distance from the origin. In other words, we assume the force to be equal to $-kx$, where the coefficient $k$ is a measure of the stiffness of the elastic link. Since $k$ is assumed to be positive, the force is negative when $x$ is positive and positive when $x$ is negative. Newton's law now says

$$m\ddot{x} = -kx.$$

We cannot expect that this differential equation will determine the motion completely, but it is plausible to assume that for a given instant of time, say $t = 0$, we can arbitrarily assign the initial position $x(0) = x_0$ and the initial velocity $\dot{x}(0) = v_0$; in physical language, it says that we can launch the particle from an arbitrary position with an arbitrary velocity and that thereafter the motion is determined by the differential equation. Mathematically speaking, this is expressed by the fact that the general solution of our differential equation contains two constants of integration, at first undetermined, the values of which we find by means of the initial conditions. We shall prove this fact immediately.

We can easily state directly such a solution. If we set $\omega = \sqrt{(k/m}$, we may at once verify by differentiation that our differential equation is satisfied by the above expression. This expression is readily written in the form

$$x(t) = a \sin\omega(t - \delta) = -a \sin\omega\delta \cos\omega t + a \cos\omega\delta \sin\omega t;$$

we need only write $-a \sin \omega\delta = c_1$ and $a \cos \omega\delta = c_2$, thus introducing instead of $c_1$ and $c_2$ the new constants $a$ and $\delta$. Motions of this type are said to be **sinusoidal** or **simple harmonic**. Such motions are periodic; any state, i.e. position $x(t)$ and velocity $\dot{x}(t)$, is repeated after the time $T=2\pi/\omega$, which is called the **period**, since the functions sin $\omega t$ and cos $\omega t$ have the period $T$. The number $a$ is called the **maximum displacement** or **amplitude** of the

oscillation. The number $1/T = \omega/2\pi$ is called the **frequency of the oscillation**; it measures the number of oscillations per unit time. We shall return to the theory of oscillations in Chapter XI.

**5.4.4 Motion on a Given Curve:** Finally, we shall discuss the most general form of the problem stated above, namely, the problem of the motion along a given curve under an arbitrary pre-assigned force $mf(s)$.

The objective here is the determination of the function $s(t)$ as a function of $t$ by means of the differential equation

$$\ddot{s} = f(s),$$

where $f(s)$ is a given function. This differential equation in $s$ can be solved completely by the following device.

We begin by considering any primitive function $F(s)$ of $f(s)$, so that $F'(s)=f(s)$, and multiply both sides of the equation $\ddot{s} = f(s) = F'(s)$ by $\dot{s}$. We can then write the left hand side in the form $\frac{d}{dt}\left(\frac{1}{2}\dot{s}^2\right)$, as we see at once by differentiating the expression $\dot{s}^2$; however, he right hand side $F'(s)\dot{s}$ is the derivative of $F(s)$ with respect to the time $t$, if we regard in $F(s)$ the quantity $s$ as a function of $t$. Hence, we have immediately

$$\frac{d}{dt}\left(\frac{1}{2}\dot{s}^2\right) = \frac{d}{dt}F(s),$$

or, by integration,

$$\frac{1}{2}\dot{s}^2 = F(s) + c,$$

where $s$ denotes a constant yet to be determined.

Let us write this equation in the form $\frac{ds}{dt} = \sqrt{2(F(s)+c)}$. We see that we cannot find $s$ immediately from this equation as a function of $t$ by integration. However, we arrive at a solution of the problem, if we at first content ourselves with finding the inverse function $t(s)$, that is, the time taken by the particle to reach a definite position $s$. We have for this the equation

$$\frac{dt}{ds} = \frac{1}{\sqrt{2\{F(s) + c\}}};$$

thus, the derivative of the function $t(s)$ is known and we have

$$t = \int \frac{ds}{\sqrt{2\{F(s) + c\}}} + c_1,$$

where $c_1$ is another constant of integration. As soon as we have performed the last integration, we have solved the problem, for which we have not the position $s$ as a function of $t$; we have found inversely the time $t$ as a function of the position $s$. The fact that the two constants of integration $c$ and $c_1$ are still available, allows us to fit the general solution to special initial conditions.

In the above example of elastic motion, we have to identify $x$ with $s$; we have $f(s) = -\omega^2 s$ and correspondingly, say, $F(s) = -\omega^2 s^2$. We therefore obtain

$$\frac{dt}{ds} = \frac{1}{\sqrt{(2c - \omega^2 s^2)}},$$

whence

$$t = \int \frac{ds}{\sqrt{(2c - \omega^2 s^2)}} + c_1.$$

However, this Integral is easily evaluated by introducing $\omega s/\sqrt{2c}$ as a new variable; we thus obtain

$$t = \frac{1}{\omega} \arcsin \frac{\omega s}{\sqrt{2c}} + c_1,$$

or, forming the inverse function

$$s = \frac{\sqrt{2c}}{\omega} \sin \omega (t - c_1).$$

Thus, we are led to exactly the same statement of the solution as before.

From this example, we also see what the constants of integration mean and how they are to be determined. For example, if we require that at time $t = 0$ the particle shall be at the point $s = 0$ and at that instant have the velocity $\dot{s}(0) = 1$, we obtain the two equ»tions

$$0 = \frac{\sqrt{2c}}{\omega}\sin\omega c_1, \quad 1 = \sqrt{2c}\cos\omega c_1,$$

from which we find that the constants have the values $c_1 = 0$, $c = ½$. The constants of integration $c$ and $c_1$ can be determined in exactly the same way when the initial position $s_0$ and the initial velocity $\dot{s}_0$ are prescribed arbitrarily.

**Exercises 5.3:**

1. A point $A$ moves at constant velocity 1 along a circle with radius $r$ and centre at the origin. The point $A$ is linked to a point $B$ by a line of constant length $l$ (>r); $B$ is constrained to move on the $x$-axis (like a crank, connecting-rod and piston of a steam engine). Calculate the velocity and acceleration of $B$ as functions of the time.

2. A particle starts from the origin at velocity 4 and slides under the influence of gravity down a straight wire until it reaches the vertical line $x = 2$. What must be the slope of the path in order that the point may reach the vertical line in the shortest time?

3. A particle moves along a straight line subject to a resistance producing the retardation $ku^3$, where $u$ is the velocity and $k$ a constant. Find expressions for the velocity ($u$) and the time ($i$) in terms of $s$, the distance from the initial position, and $v_0$, the initial velocity.

4. A particle of unit mass moves along the $x$-axis and is acted upon by a force $f(x) = -\sin x$.
($a$) Determine the motion of the point if it is at time $t = 0$ at the point $x = 0$ and has the velocity $v_0 = 2$. Show that as $t \to \infty$, the particle approaches a limiting position and find this position.
(b) If the conditions are the same except that $v_0$ may have any value, show that, if $v_0 > 2$, the point moves to an infinite distance as a $t \to \infty$ and, if $v_0 < 2$, the point oscillates abont the origin.

5. Choose axes with their origin at the centre of the earth with radius $R$. By Newton's law of gravitation, a particle of unit mass lying on the $y$-axis is attracted by the earth with a force $\mu M/r^2$, where $\mu$ is the **gravitational constant** and $M$ is the mass of the earth.
($a$) Calculate the motion of the particle after it has been released at the point $y_0$(> $R$), i.e., if at time $t = 0$, it is at the point $y = y_0$ and has the velocity $v_0 = 0$. ($b$) Find the velocity with which the particle in (a) strikes the earth.

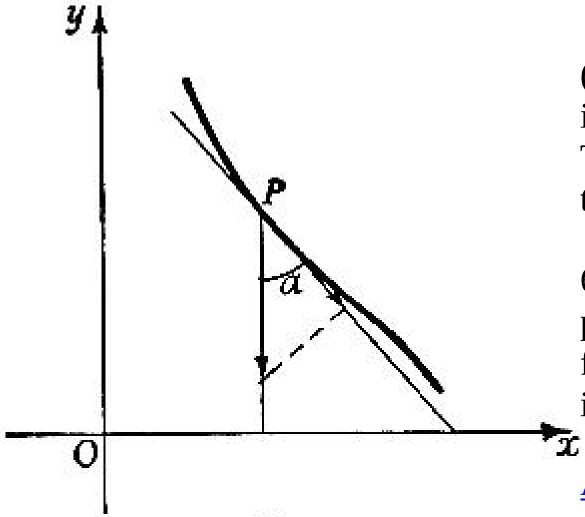



Fig. 13

(*c*) Using the result of (*b*), calculate the velocity of a particle falling to the earth from infinity.
This is the same velocity at which a projectile would have to be launched in order that it should leave the earth and never return (**escape velocity**).

6.* A particle of mass $m$ moves along the ellipse $r = k/(1 - e\cos\theta)$. The force on the particle is $cm/r^2$ and directed towards the origin. Describe the motion of the particle, find its period and show that the radius vector to the particle sweeps out equal areas in equal times.

Answers and Hints

## 5.5 Further Applications. A Particle sliding down a Curve

**5.5.1 General Remarks:** The case of a particle sliding along a frictionless curve under the influence of gravity can be treated very simply by the method just described. We shall first discuss the motion in general and then with special reference to the case of the ordinary pendulum and the cycloidal pendulum. We choose axes in such a way that the $y$-axis points vertically upwards, i.e., opposite to the direction of the force of gravity, and consider the curve as given in terms of a parameter $\theta$ by the parametric equations

$$x = \phi(\theta) = x(\theta),\ y = \psi(\theta) = y(\theta).$$

A portion of the curve, for which the motion will be studied, is shown in Fig. 13. At every point of the curve, the force of gravity acts downwards (i.e., in the direction of decreasing $y$) on the particle with magnitude $mg$. If we denote the angle between the negative $y$-axis and the tangent to the curve by $\alpha$, then, according to the hypothesis in 5.4.1, the force acting along the direction of the curve is

$$mg\cos\alpha = -mg\frac{y'}{\sqrt{(x'^2 + y'^2)}},$$

where

$$x' = \frac{d\varphi}{d\theta} = \varphi'(\theta),\quad y' = \frac{d\psi}{d\theta} = \psi'(\theta).$$

(Note that here the inverted comma ' denotes the derivative with respect to $\theta$ and not with respect to $x$.) In particular, if we introduce the length of arc $s$ as parameter in place of $\theta$, we obtain the expression - $mgdy/ds$ for the force along the curve. Hence, by Newton's law, the function $s(t)$ satisfies the differential equation

$$\ddot{s} = -g\frac{dy}{ds}.$$

The right hand side of this equation is a known function of $s$, since we know the curve and must therefore regard the quantities $x$ and $y$ to be known functions of $s$.

As in 5.4.4, we multiply both sides of this equation by $\dot{s}$. The left hand side then becomes the derivative of $\frac{1}{2}\dot{s}^2$ with respect to $t$. If we regard in the function $y(s)$ the variable $s$ to be a function of $t$, the right hand side of our equation is the derivative of $-gy$ with respect to $t$. Hence we find on integrating

$$\frac{1}{2}\dot{s}^2 = -gy + c,$$

where $c$ is a constant of integration. In order to fix the meaning of this constant, we assume that at the time $t = 0$ our particle is at the point of the curve for which the value of the parameter is $\theta_0$ and the co-ordinates are $x=\phi(\theta_0)$, $y_0=\psi(\theta_0)$, and that at this instant its velocity is zero, i.e., $\dot{s}(0) = 0$. Then, setting $t = 0$, we find immediately $-gy_0 + c = 0$, whence

$$\frac{1}{2}\dot{s}^2 = -g(y - y_0).$$

Now, instead of regarding $s$ as a function of $t$, we shall consider the inverse function $t(s)$. Then we obtain at once

$$\frac{dt}{ds} = \pm\frac{1}{\sqrt{\{2g(y_0 - y)\}}},$$

which is equivalent to

$$t = c_1 \pm \int\frac{ds}{\sqrt{\{2g(y_0 - y)\}}},$$

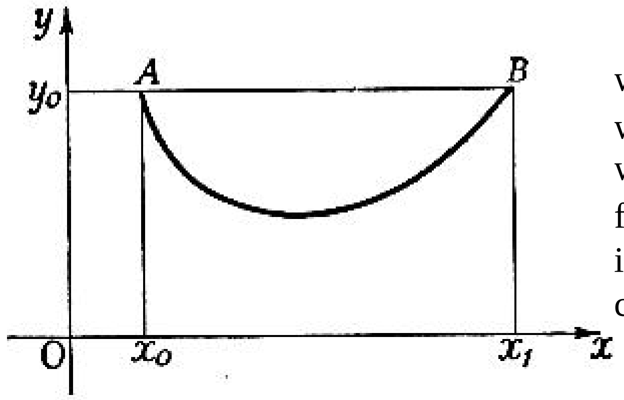



Fig. 14

where $c_1$ is a new constant of integration. As regards the sign of the square root, which is the same as the sign of $\dot{s}$, we note that, if the particle moves along an arc, which is lower than $y_0$ everywhere except at the ends, the sign cannot change. In fact, the sign of $\dot{s}$ can change only when $\dot{s} = 0$, that is, where $y - y_0 = 0$. The integrand on the right hand side is known in terms of the parameter $\theta$, since the curve is known. Introducing $\theta$ as independent variable, we obtain

$$t = c_1 \pm \int \frac{ds}{d\theta} \frac{d\theta}{\sqrt{\{2g(y_0 - y)\}}} = c_1 \pm \int \sqrt{\left(\frac{x'^2 + y'^2}{2g(y_0 - y)}\right)} d\theta,$$

where the functions $x' = \phi'(\theta)$, $y' = \psi'(\theta)$, $y = \psi(\theta)$ are known. In order to determine the constant of integration $c_1$, we note that for $t = 0$ the value of the parameter must be $\theta_0$. This yields immediately our solution in the form

$$t = \pm \int_{\theta_0}^{\theta} \sqrt{\left(\frac{x'^2 + y'^2}{2g(y_0 - y)}\right)} d\theta.$$

When integrated, the equation represents the time taken by the particle to move from the parameter value $\theta_0$ to the parameter value $\theta$. The inverse function $\theta(t)$ of this function $t(\theta)$ enables us to describe the motion completely, because we can determine at each instant $t$ the point $x = \phi\{\theta(t)\}$, $y = \psi\{\theta(t)\}$, which the particle is then passing.

**5.5.2 Discussion of the Motion:** We can deduce the general nature of the motion by simple intuitive reasoning from the equations just found, without an explicit expression for the result of the integration. We assume that our curve is of the type shown in Fig. 14, i.e., that it consists of an arc which is convex downwards; we take $s$ as increasing from the left hand side to the right hand side. If we initially release the particle at the point $A$ with the co-ordinates $x=x_0$, $y=y_0$, corresponding to $\theta = \theta_0$, the velocity increases, because the acceleration $\ddot{s}$ is positive. The particle travels from $A$ to the lowest point with increasing velocity. However, after it reaches the lowest point, the acceleration is negative, since the right hand side $-g\,dy/ds$ of the equation of motion is negative. The velocity therefore decreases. From the equation $\dot{s}^2 = -2g(y - y_0)$, we see at once that the velocity reaches the value 0 when the particle reaches the point $B$, the height of which is the same as that of the initial position $A$. Since the acceleration is still negative, the motion of the particle must be reversed at this point, so that the particle will swing back to the point $A$; this action will repeat itself indefinitely. (The reader will recall that friction has been disregarded!) In this oscillatory motion, the time which the point takes to return from $B$ to $A$ must obviously be the same as the time taken to move from $A$ to $B$. If we denote the time required for a complete journey from $A$ to $B$ and

back by $T$, the motion will obviously be periodic with the period $T$. If $\theta_0$ and $\theta_1$ are the values of the parameter corresponding to the points $A$ and $B$, respectively, the half-period is given by

$$\frac{T}{2} = \frac{1}{\sqrt{2g}}\left|\int_{\theta_0}^{\theta_1}\sqrt{\left(\frac{x'^2+y'^2}{y_0-y}\right)}\,d\theta\right|$$

$$= \frac{1}{\sqrt{2g}}\left|\int_{\theta_0}^{\theta_1}\sqrt{\left(\frac{\varphi'^2(\theta)+\psi'^2(\theta)}{\psi(\theta_0)-\psi(\theta)}\right)}\,d\theta\right|.$$

If $\theta_2$ is the value of the parameter corresponding to the lowest point of the curve, the time which the particle takes to fall from $A$ to this lowest point is

$$\frac{1}{\sqrt{2g}}\left|\int_{\theta_0}^{\theta_2}\sqrt{\left(\frac{x'^2+y'^2}{y_0-y}\right)}\,d\theta\right|.$$

**5.5.3 The Ordinary Pendulum:**. The simplest example is given by the so-called ordinary pendulum, when the curve under consideration is a circle of radius $l$:

$$x = l\sin\theta, \quad y = -l\cos\theta,$$

where the angle $\theta$ is measured in the positive sense from the position of rest. We obtain at once from the general expression above

$$T = \sqrt{\frac{2l}{g}}\int_{-\alpha}^{\alpha}\frac{d\theta}{\sqrt{(\cos\theta-\cos\alpha)}} = \sqrt{\frac{l}{g}}\int_{-\alpha}^{\alpha}\frac{d\theta}{\sqrt{\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)}},$$

where $\alpha$ $(0 < \alpha < \pi)$ denotes the amplitude of oscillation of the pendulum, i.e., the angular position from which the particle is released at time $t$=0 at velocity 0. The substitution

$$u = \frac{\sin(\theta/2)}{\sin(\alpha/2)}, \quad \frac{du}{d\theta} = \frac{\cos(\theta/2)}{2\sin(\alpha/2)}$$

yields for the period of oscillation of the pendulum

$$T = 2\sqrt{\frac{l}{g}}\int_{-1}^{1}\frac{du}{\sqrt{(1-u^2)(1-u^2\sin^2(\alpha/2))}}.$$

We have thus expressed the period of oscillation of the pendulum by an **elliptic integral**. If we assume that the amplitude of the oscillation is small, so that we may with sufficient accuracy replace the second factor under the square root by 1, we obtain

$$2\sqrt{\frac{l}{g}}\int_{-1}^{1}\frac{du}{\sqrt{(1-u^2)}}$$

as an approximation for the period of oscillation. We can evaluate this last integral by using **13.** in our table of integrals and obtain for *T* the approximate expression

$$2\pi\sqrt{\frac{l}{g}}.$$

**5.5.4 The Cycloidal Pendulum:** The fact that the period of oscillation of the ordinary pendulum is not strictly independent of the amplitude of oscillation caused Christian Huygens, in his prolonged efforts to construct accurate clocks, to look for a curve for which the period of oscillation is strictly independent of the particular position on the curve at which the oscillating particle begins its motion; the oscillations are then said to be isochronous. Huygens discovered that the cycloid is such a curve.

In order that a particle may actually be able to oscillate on a cycloid, its cusps must point in the direction opposite to that of the force of gravity, i.e., we must rotate the cycloid, considered earlier (5.1.2) about the *x*-axis (Fig. 15), whence we write the equations of the cyloid in the form

$$x = a(\theta - \sin\theta),$$
$$y = a(1 + \cos\theta),$$

which also involves a translation of the curve by a distance $2a$ in the positive *y*-direction. The time which the particle takes to travel from a point at the height

$$y_0 = a(1 + \cos\alpha) \quad (0 < \alpha < \pi)$$

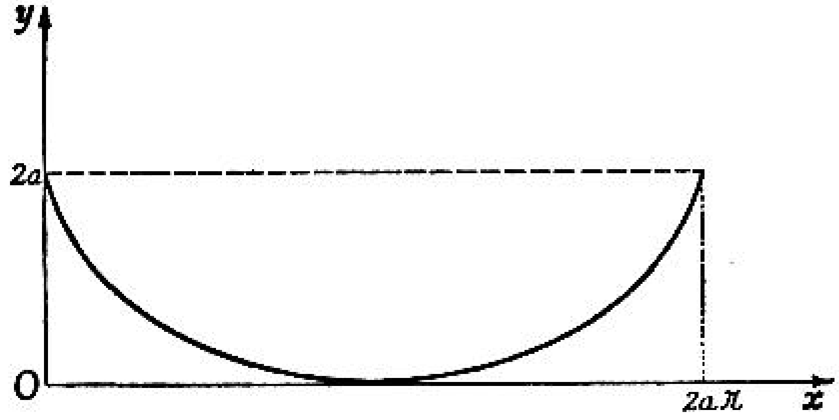



Fig. 15.—Path described by a cycloidal pendulum

down to the lowest point is, by the formula worked out in 5.5.1,

$$\frac{T}{4} = \sqrt{\frac{1}{2g}} \int_\alpha^\pi \sqrt{\left(\frac{x'^2 + y'^2}{y_0 - y}\right)}\, d\theta = \sqrt{\frac{a}{g}} \int_\alpha^\pi \sqrt{\left(\frac{1 - \cos\theta}{\cos\alpha - \cos\theta}\right)}\, d\theta.$$

We now use the equation

$$\cos\alpha - \cos\theta = 2\left(\cos^2\frac{\alpha}{2} - \cos^2\frac{\theta}{2}\right);$$

it yields

$$\frac{T}{4} = \sqrt{\frac{a}{g}} \int_\alpha^\pi \frac{\sin\dfrac{\theta}{2}}{\sqrt{\left(\cos^2\dfrac{\alpha}{2} - \cos^2\dfrac{\theta}{2}\right)}}\, d\theta.$$

We then work out the definite integral, employing the substitution

$$\cos\frac{\theta}{2} = u\cos\frac{\alpha}{2}, \quad \sin\frac{\theta}{2}\, d\theta = -2\cos\frac{\alpha}{2}\, du.$$

$$\int \frac{\sin\frac{\theta}{2}}{\sqrt{\left(\cos^2\frac{\alpha}{2} - \cos^2\frac{\theta}{2}\right)}} d\theta = -2\int \frac{du}{\sqrt{(1-u^2)}} = -2 \arc\sin u,$$

whence $$T = -8\sqrt{\frac{a}{g}} \arc\sin \frac{\cos\frac{\theta}{2}}{\cos\frac{\alpha}{2}}\Bigg|_{\alpha}^{\pi} = 4\pi\sqrt{\frac{a}{g}}.$$

The period of oscillation $T$ is therefore actually independent of the amplitude.

## 5.6 Work

**5.6.1 General Remarks:** The concept of work throws new light on the considerations of the last section and on many other questions of mechanics and physics.

Let us think again of a particle, moving on a curve under the influence of a force acting along the curve, and assume that its position is specified by the length of the arc measured from any fixed initial point. The force itself will then, as a rule, be a function of $s$. We assume that it is a continuous function $f(s)$ of the length of arc. This function will have positive values, where the direction of the force is the same as the direction of increasing values of $s$, and negative values where the direction of the force is opposite to that of increasing value of $s$.

If the magnitude of the force is constant along the path, we mean by the **work done** by the force the product of the force by the distance $(s_1 - s_0)$ traversed, where $s_1$ denotes the final point and $s_0$ the initial point of the motion. If the force is not constant, we define the work by means of a limiting process. We subdivide the interval from $s_0$ to $s_1$ into n equal or unequal sub-intervals and note that, if the sub-intervals are small, the force in each one is nearly constant; if $\sigma_\nu$ is an arbitrarily chosen point in the $\nu$-th subinterval, then throughout this subinterval the force will be approximately $f(\sigma_\nu)$. If the force throughout the $\nu$-th subinterval were exactly $f(\sigma_\nu)$, the work done by our force would be exactly

$$W = \int_{s_0}^{s_1} f(s)\,ds,$$

which we naturally call the work done by the force.

If the direction of the force and that of the motion on the curve are the same, the work done by the force is positive; we then say that the **force does work**. On the other hand, if the direction of the force and that of the motion are opposed, the work done by the force is negative; we then say that **work is done against the force**.

Note that we must here carefully distinguish the forces uder consideration. For example, in lifting a weight, the work done by the force of gravity is negative: Work is done against gravity. But from the point of view of the person doing the lifting, the work done is positive, for the person must exert a force opposed to gravity.

If we regard the co-ordinate of the position $s$ as a function of the time $t$, so that the force $f(s) = p$ is also a function of $t$, then we can plot the point in a plane with rectangular co-ordinates $s$ and $p$ with the co-ordinates $s = s(t)$, $p = p(t)$ as a function of the time. This point will describe a curve, which may be called the **work diagram of the motion**. If we are dealing with a periodic motion, as in the case of any machine, then, after a certain time $T$ (one period), the moving point $s = s(t)$, $p = p(t)$ will return to the same position, i.e., the work diagram will he a closed curve. In this case, the curve may consist simply of one and the same arc, traversed first forwards and then backwards, as happens, for instance, in elastic oscillations. But it is also possible for the curve to be a more general closed curve, enclosing an area, for example, in the case of machines in which the pressure on a piston is not the same during the forward and backward stroke. The work done in one cycle, i.e., in time $T$, will then be given simply by the negative area of the work diagram or, in other words, by the integral

$$\int_{t_0}^{t_0+T} p(t)\frac{ds}{dt}\,dt,$$

where the interval of time from $t_0$ to $t_0 + T$ represents exactly one period of the motion. If the boundary of the area is positively traversed, the work done is negative, if negatively traversed, it is positive. If the curve consists of several loops, some traversed positively and others negatively, the work done is given by the sum of the areas of the loops, each with its sign changed.

These considerations are illustrated in practice by the **indicator diagram of a steam engine**. By a suitably designed mechanical vice, a pencil is made to move over a sheet of paper; the horizontal motion of the pencil relative to the paper is proportional to the distance $s$ of the piston from its extreme position, while the vertical motion is proportional to the steam pressure, and hence proportional to the total force $p$ of the steam on the piston. The piston therefore describes the work diagram for the engine on a known scale. The area of this diagram is measured (usually by means of a planimeter) and the work done by the steam on the piston is thus found. Here we

also see that our convention for the sign of an area, as discussed in 5.2.2, is not only of theoretical interest. In fact, it does happen sometimes, when an engine is running light, that the highly expanded steam at the end of the stroke has a pressure lower than that required to expel it on the return stroke; on the diagram, this is shown by a positively traversed loop; the engine itself is drawing energy from the flywheel instead of providing energy.

5.6.2 The Mutual Attraction of Two Masses: Let a particle attract another particle according to Newton's law of attraction; as a first example, we shall consider the work done by the force of attraction as the second particle moves along the line joining the two particles. By Newton's law of gravitation, the attracting force is inversely proportional to the square of the distance. If we imagine the first particle to be at rest at the origin and the second particle at the distance $r$ from the origin, the attracting force is given by

$$f(r) = -\mu \frac{1}{r^2},$$

where $\mu$ is a positive constant. Hence the work done by this force, as the particle moves from the distance $r$ to the distance $r_1 > r$, is positive and equal to the integral

$$-\mu \int_r^{r_1} \frac{ds}{s^2} = \mu \left( \frac{1}{r_1} - \frac{1}{r} \right).$$

If the particle is moved further away from the origin from $r$ to the distance $r_1 > r$ by means of an opposing force, the work done by the force of attraction will, of course, still be given by this integral (now negative). The work done by the opposing force has the same numerical value, but the opposite sign, whence it is equal to $\mu\,(1/r - 1/r_1)$. If we think of the final position as being chosen further and further away, this approaches the limiting value $\mu/r$, which we may call the work which must be done against the force of attraction in order to move the particle from the distance $r$ to infinity. This important expression is called the **mutual potential of the two particles**. Hence, the potential here is defined as the work required to separate completely two attracting masses; for example, it is the work required to tear an electron completely away from an atom (**ionization potential**)

**5.6.3 The Stretching of a Spring:** As a second example, we consider the work done in stretching a spring. Usually, in the **theory of elasticity**, we assume (5.4.2) that the force needed to stretch the spring is proportional to $x$ - the increase in the length of the spring, i.e., $p = kx$, where $h$ is a constant. The work which must be done in order to stretch the spring from the unstressed position $x = 0$ to the final position $x = x_1$ is therefore given by the integral

$$\int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2.$$

**5.6.4 The Charging of a Condenser:**. The concept of work in other branches of physics can be treated in a similar manner. For example, we may consider the charging of a condenser. If we denote the quantity of electricity in the condenser by $Q$, its capacity by C and the **potential difference** (**voltage**) across the condenser by $V$, then we know from physics that $Q = CV$. Moreover, the work done in moving a charge $Q$ through a potential difference $V$ is equal to $QV$. Since during the charging of the condenser the difference of potential $V$ is not constant, but increases with $Q$, we perform a passage to the limit exactly analogously to that in 5.6.1 and obtain for the work done during charging a condenser

$$\int_0^{Q_1} V\,dQ = \frac{1}{C}\int_0^{Q_1} Q\,dQ = \frac{1}{2}\frac{Q_1^2}{C} = \frac{1}{2}Q_1V_1,$$

where $Q_1$ is the total quantity of electricity fed into the condenser and $V_1$ is the difference of potential across the condenser at the end of the charging process.

# Appendix to Chapter V

## A5.1 Properties of the Evolute

The parametric equations

$$\xi = x - \rho\frac{\dot{y}}{\sqrt{(\dot{x}^2+\dot{y}^2)}}, \quad \eta = y + \rho\frac{\dot{x}}{\sqrt{(\dot{x}^2+\dot{y}^2)}},$$

for the evolute of a given curve $x = x(t)$, $y = y(t)$ (5.2.6) enable us to derive some interesting geometrical relations between it and the given curve. For the sake of convenience, we use the length of arc $s$ as parameter, so that

$$\dot{x}^2+\dot{y}^2 = 1 \quad \text{and} \quad \dot{x}\ddot{x}+\dot{y}\ddot{y} = 0,$$

$$\frac{1}{\rho} = k = \frac{\ddot{y}}{\dot{x}} = -\frac{\ddot{x}}{\dot{y}},$$

or

$$\rho\ddot{y} = \dot{x} \quad \text{and} \quad \rho\ddot{x} = -\dot{y}.$$

We thus have the equations

$$\xi = x - \rho\dot{y}, \quad \eta = y + \rho\dot{x};$$

on differentiation, they yield

$$\dot{\xi} = \dot{x} - \rho\ddot{y} - \dot{\rho}\dot{y} = -\dot{\rho}\dot{y}, \quad \dot{\eta} = \dot{y} + \rho\ddot{x} + \dot{\rho}\dot{x} = \dot{\rho}\dot{x},$$

whence

$$\dot{\xi}\dot{x} + \dot{\eta}\dot{y} = 0.$$

Since the direction cosines of the normal to the curve are given by $-\dot{y}$ and $\dot{x}$, it follows that the normal to the curve is tangent the evolute at the centre of curvature, or, the tangents to the evolute are the normals of the given curve or **the evolute is the envelope of the normals** (Fig. 16).

Moreover, if we denote the length of arc of the evolute, measured from an arbitrarily fixed point, by $\sigma$, we have

$$\left(\frac{d\sigma}{ds}\right)^2 = \dot{\sigma}^2 = \dot{\xi}^2 + \dot{\eta}^2.$$

Since $\dot{x}^2 + \dot{y}^2 = 1$, we obtain from the above formulae

$$\dot{\sigma}^2 = \dot{\rho}^2,$$

so that, if we choose the direction in which $\sigma$ is measured in a suitable way, then

$$\dot{\sigma} = \dot{\rho},$$

provided that $\sigma \neq 0$, or on integration

$$\sigma_1 - \sigma_0 = \rho_1 - \rho_0.$$

Hence the length of arc of the evolute between two points is equal to the difference of the corresponding radii of curvature, provided that $\dot{\rho}$ does not vanish for the arc under consideration.

This last condition is not superfluous, because, if $\dot{\rho}$ changes sign, according to $\dot{\sigma} = \dot{\rho}$, the length of arc, on passing the corresponding point of the evolute, has a maximum or minimum, i.e., on passing this point, we do not simply continue to reckon $\sigma$ onward, but must reverse the sense in which it is measured. If we wish to avoid this, we must on passing such a point change the sign in the above formula, i.e., set $\dot{\sigma} = -\dot{\rho}$.

It may also be noted that the centres of curvature, which correspond to maxima or minima of the radius of curvature, are cusps of the evolute. (The proof will not be given here.)

The geometrical relationship, which we have just found, can be expressed in yet another way. If we imagine a flexible, inextensible thread laid along an arc of the evolute and stretched so that a part of it extends away from the curve tangentially to it, and if, in addition, the end-point *Q* of this thread lies on the original curve *C*, then, as we unwind the thread, the point *Q* will describe the curve *C*. This accounts for the name **evolute** (Latin: **evolvere**, unwind). The curve *C* is called an **involute**. On the other hand, we may start with an arbitrary

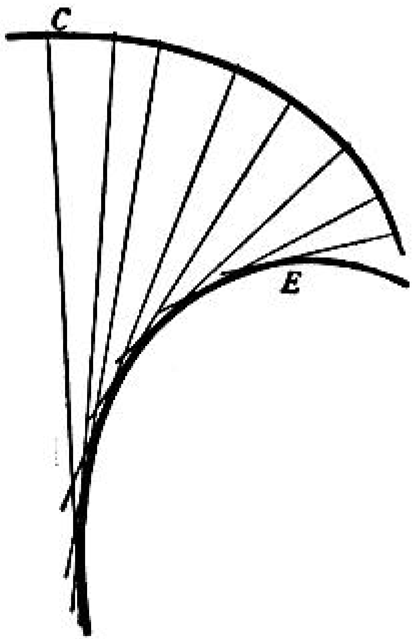



Fig. 16.—Evolute (*E*)

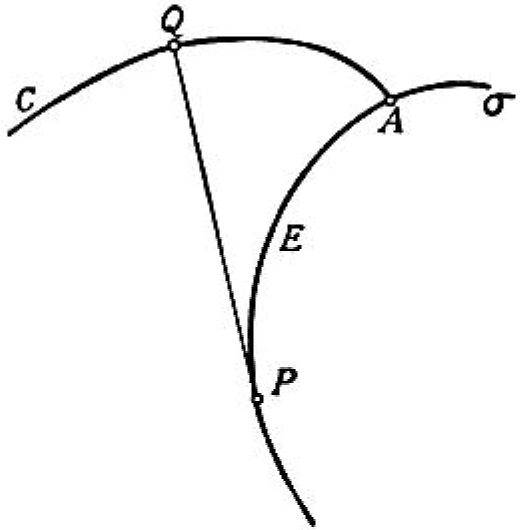



Fig. 17.—Involute (*C*)

The curve *E* and construct its involute *C* by this unwinding process. We then see that *E* conversely is the evolute of *C*.

In order to prove this, consider the curve *E*, which is now the given curve, as given in the form $\xi=\xi(\sigma)$, $\eta=\eta(\sigma)$, where the current rectangular co-ordinates are denoted by $\xi$ and $\eta$, and the parameter $\sigma$ is the length of arc. The winding is done as indicated in Fig. 17; when the thread is completely wound on to the evolute E, its end *Q* coincides with the point *A* of *E*, corresponding to the length of arc $\alpha$. If the thread is now unwound until it is tangent to the evolute at the point *P*, corresponding to the length of arc $\sigma \leq \alpha$, the length of the segment *PQ* will be $(\alpha - \sigma)$ and its direction cosines will be $\dot{\xi}$ and $\dot{\eta}$, where the dot denotes differentiation with respect to $\sigma$. Thus, for the co-ordinates *x*, *y* of the point *Q*, we obtain the expressions

$$x = \xi + (a - \sigma)\xi, \quad y = \eta + (a - \sigma)\dot{\eta},$$

which yield the equations for the involute, described by the point *Q* in terms of the parameter $\sigma$. Differentiation with respect to $\sigma$ yields

$$\dot{x} = \dot{\xi} - \dot{\xi} + (a-\sigma)\ddot{\xi} = (a-\sigma)\ddot{\xi},$$
$$\dot{y} = \dot{\eta} - \dot{\eta} + (a-\sigma)\ddot{\eta} = (a-\sigma)\ddot{\eta}.$$

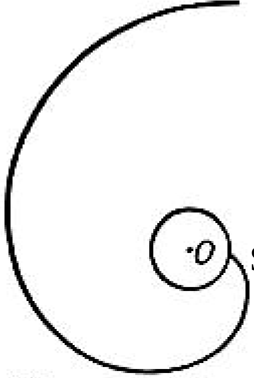


Fig. 19.—Involute of the circle

Since $\dot{\xi}\ddot{\xi} + \dot{\eta}\ddot{\eta} = 0$, we find at once that

$$\dot{\xi}\dot{x} + \dot{\eta}\dot{y} = 0,$$

which shows that the line *PQ* is normal to the involute *C*. We can therefore state that the normals to the curve *C* are tangent to the curve *E*. But this is the characteristic property of *E* - the evolute of *C*. Hence every curve is the evolute of all its involutes.

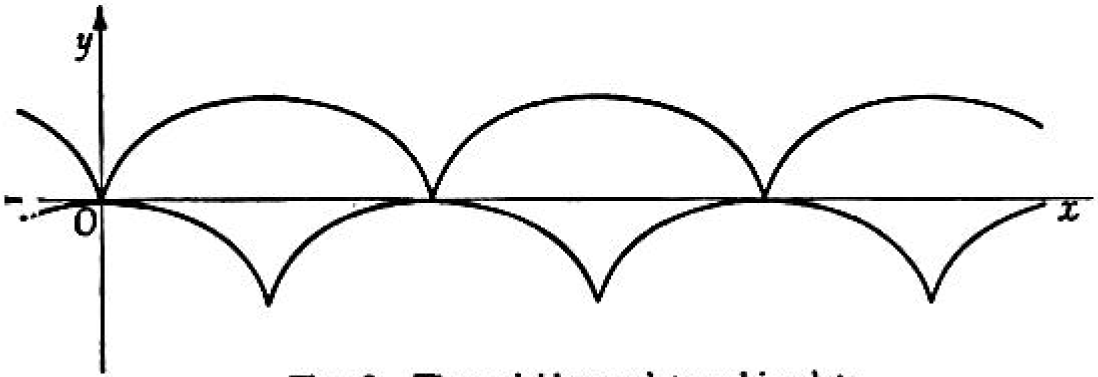


Fig. 18.—The cycloid as evolute and involute

We consider as a particular case the evolute of the **cycloid** $x=t-\sin t$, $y=t-\cos t$. By (5.2.6),

$$\xi = x - \dot{y}\,\frac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}, \quad \eta = y + \dot{x}\,\frac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y}-\dot{y}\ddot{x}};$$

hence we obtain the evolute in the form $\xi = t + \sin t$, $\eta = -t + \cos t$. If we set $t=\tau+\pi$, then $\xi - \pi = \tau - \sin\tau$, $\eta - 2 = 1 - \cos\tau$ and these equations show that the evolute is itself a cycloid which is similar to the original curve and can be obtained from it by translation, as is indicated in Fig. 18.

As a farther example we shall work out the equation for the **involute of the circle**. We begin with the circle $\xi = \cos t$, $\eta = \sin t$ and unwind the tangent, as is shown in Fig. 19. The involute of the circle is then given in the form

$$x = \cos t + t \sin t, \quad y = -\sin t + t \cos t.$$

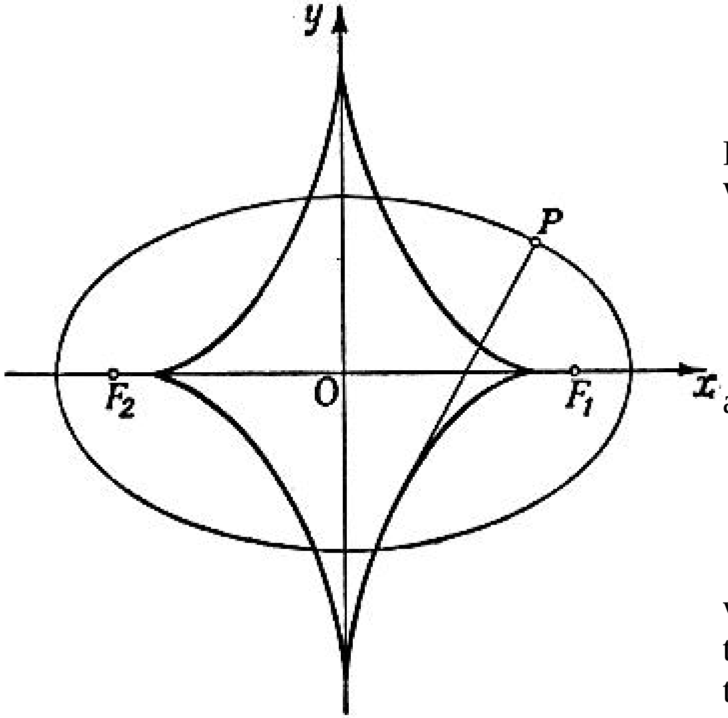



Fig. 20.—Evolute of the ellipse

Finally, we shall determine the **evolute of the ellipse** $x = a \cos t$, $y = b\sin t$. We find immediately

$$\xi = x - \dot{y}\frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \frac{a^2 - b^2}{a}\cos^3 t$$

and

$$\eta = y + \dot{x}\frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = -\frac{a^2 - b^2}{b}\sin^3 t,$$

which is a parametric representation of the evolute. If we eliminate from these equations in the usual way, we obtain the equation of the evolute in the non-parametric form

$$(a\xi)^{2/3} + (b\eta)^{2/3} = (a^2 - b^2)^{2/3}.$$

This curve is called an **astroid** (Fig.20). By means of the parametric equation, we may readily convince ourselves that the centres of curvature corresponding to the vertices of the ellipse are actually the cusps of the astroid.

**Exercises 5.4:**

1. Show that the evolute of an epicycloid (Exercise 2 above) is another epicyoloid, which can be obtained from the first by rotation and contraction.

2. Show that the evolute of a hypocycloid (Exercise 4 above) is another hypocycloid which can be obtained from the first by rotation and expansion.

**No Answers or Hints**

## A5.2 Areas bounded by closed curves

We have seen in 5.2.2 that the area bounded by a closed curve $x=x(t)$, $y=y(t)$, which nowhere intersects itself (a so-called **simple closed curve**) is given by the integral

$$-\int_{t_0}^{t_1} y(t)\dot{x}(t)\,dt,$$

where the value obtained is positive or negative according to whether the boundary is described in the positive or negative sense. We shall now extend this result to more general curves. Suppose that the curve *C*, given by the equations $x=x(t)$, $y=y(t)$ intersects itself at a finite number of points, thus dividing the plane into a finite number of sections $R_1, R_2, \cdots$ . Moreover, assume that the derivatives are continuous, except perhaps at a finite number of jump discontinuities and that $\dot{x}^2+\dot{y}^2 \neq 0$ except perhaps at a finite number of values of *t* which may correspond to corners. Finally, also assume that the curve has a finite number of **lines of support** .

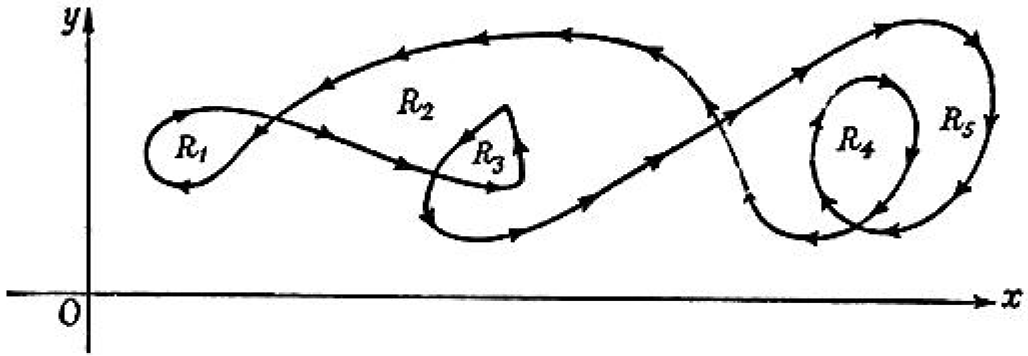



Fig. 21

We then assign to each region $R_i$ an **index** $\mu_i$, which is defined as follows: Select an arbitrary point *Q* in $R_i$, which does not lie on any line of support, and erect the line extending from *Q* upwards in the direction of the positive *y*-axis. . We count the number of times which the curve *C* crosses the half-line from right to left and subtract the number of times it crosses from left to right; the difference is the **index** $\mu_i$.

For example, the interior of the curve in Fig. 6 has the index $\mu = +1$; in Fig. 21 above, the regions $R_1, \ldots, R_5$ have the indices $\mu_1$=-l, $\mu_2$=+l, $\mu_3$=+2, $\mu_4$=-2, $\mu_{5=-1}$. These numbers $\mu_i$ actually depend on the region $R_i$ and not on the particular point *Q* chosen in $R_i$, as we readily see in the following manner. We choose any other point *Q*' in $R_i$, not on a line of support, and join *Q* to *Q*' by a broken line lying entirely in the region $R_i$. As we proceed along this broken line from *Q* to *Q*', the number of right-to-left crossings minus the number of left-to-right crossings is constant; in fact, the number of crossings between lines of support of either type is unchanged, while on crossing a line of support either the number of crossings of both types increases by one or else both the numbers decrease by one; in either case, the difference is unaltered. In the case where the line of support meets the curve at several

different points, say $A, B, \cdots, H$, we consider it as several different lines of support $FA, FB, \cdots, FH$, where $F$ is the point of the $x$-axis vertically below all the points $A, B, \cdots, H$. Our argument then applies to each of these lines. Hence the number $\mu_i$ has the same value whether we use $Q$ or $Q'$ in determining it.

In particular, if our curve does not intersect itself, the interior of the curve consists of a single region $R$ the index of which is +1 or -1 according to whether the sense in which the boundary is described is positive or negative. In order to see this, we draw any vertical line (not a line of support) intersecting the curve; on this line, we find the highest point of intersection ($P$) with the curve and choose in $R$ a point $Q$ below $P$ so near to it that no point of intersection lies between $P$ and $Q$. Then there lies above $Q$ one crossing of the curve, which, if the curve is traversed positively, must be a right-to-left crossing, so that $\mu = +1$, while otherwise $\mu_i = -1$. As we have just seen, this same value of $\mu$ holds for every other point of $R$. For such a curve, and, in fact, for all closed curves, one of the regions, the outside of the curve, extends unboundedly in all directions; we see immediately that this region has the index 0, whence we neglect it.

Our theorem about the area is now as follows: The value of the integral $-\int_{t_0}^{t_1} y\dot{x}\,dt$ is equal to the sum of the absolute areas of the regions $R_i$, each area $R_i$ being counted $\mu_i$ times, in symbols

$$-\int_{t_0}^{t_1} y\dot{x}\,dt = \Sigma\mu_i\,|\,\text{area } R_i\,|.$$

The proof is simple. We assume, as we are entitled to do, that the entire curve lies above the $x$-axis. The lines of support cut $R_i$ into a finite number of sections; let $r$ be one of them. Then, on taking the integral $-\int y\dot{x}\,dt$ for each single-valued branch of the curve, we find that the absolute area of $r$ is counted +1 times for each right-to-left branch over $r$ and -1 times for each left-to-right branch over $r$, altogether $\mu_i$ times. The same is true for every other portion of $R_i$, whence $R_i$ is counted $\mu_i$ times. Thus, the integral around the complete curve has the value $\Sigma\mu_i\,|\,\text{area } R_i\,|$, as stated. This formula agrees with what we have found for simple closed curves, as we recognize from the discussion of the values of $\mu$ for such curves.

The definition given for the index $\mu_i$ has the disadvantage of being stated in terms of a particular co-ordinate system. However, as a matter of fact, it can be shown that the value of $\mu_i$ is independent of the co-ordinate system and depends solely on the curve; however, we will not prove this here.

# Taylor's Theorem and the Approximate Expression of Functions by Polynomials

In many respects, the rational functions are the simplest functions of analysis. They are formed by a finite number of applications of the rational operations of calculation, while in the last resort the formation of every other function involves a more or less concealed passage to the limit from rational functions. The questions whether and how a given function can be expressed approximately by rational functions, in particular, by polynomials are therefore of great importance in theory as well as in practice.

## 6.1 The Logarithm and the Inverse Tangent

**6.1.1 The Logarithm:** We begin with some special cases in which the integration of the geometrical progression leads almost at once to the desired approximations. We first remind the reader of the following fact. For $q \neq 1$ and for positive integers $n$, we have

$$\frac{1}{1-q} = 1 + q + q^2 + \ldots + q^{n-1} + r_n,$$

where

$$r_n = \frac{q^n}{1-q}.$$

If $q < 1$, the remainder $r_n$ tends to 0 as $n$ increases and we then obtain (cf. 1.5.7) the **infinite geometric series**

$$1 + q + q^2 + \ldots \text{ with the sum } \frac{1}{1-q}.$$

We take as our starting-point the formula

$$\log(1+x) = \int_0^x \frac{dt}{1+t}$$

and expand the integrand in accordance with the above formula, setting $q = -t$. Then we obtain at once by integration

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \dots + (-1)^{n-1}\frac{x^n}{n} + R_n,$$

where

$$R_n = \int_0^x r_n\,dt = (-1)^n \int_0^x \frac{t^n\,dt}{1+t}.$$

Hence, for any positive integer $n$, we have expressed the function log (1 + $x$) approximately by a polynomial of the $n$-th degree, namely

$$x - \frac{x^2}{2} + \frac{x^3}{3} - + \dots + (-1)^{n-1}\frac{x^n}{n};$$

at the same time, the quantity $R_n$ - the **remainder** - specifies the size of the **error** made in the approximation.

In order to estimate the accuracy of this approximation, we need only have an estimate for the remainder $R_n$; such an estimate is given to us immediately by the integral estimates in 2.7. If we assume at first that $x \geq 0$, then the integrand is nowhere negative in the entire interval of integration and nowhere exceeds $t^n$. Consequently,

$$|R_n| \leqq \int_0^x t^n\,dt = \frac{x^{n+1}}{n+1},$$

and we therefore see that, for every value of $x$ in the interval $0 \leq x \leq 1$, this remainder can be made as small as we please by choosing $n$ large enough (cf. 1.4). If, on the other hand, the quantity $x$ is in the interval $-1 < x \leq 0$, the integrand will not change sign and its absolute value will not exceed $|t|^n/(1 + x)$, and we thus obtain the remainder estimate

$$|R_n| \leqq \frac{1}{1+x}\int_0^{|x|} t^n\,dt = \frac{|x|^{n+1}}{(1+x)(n+1)}.$$

Hence, we see that here again the remainder is arbitrarily small when we make $n$ sufficiently large. Of course, our estimate has no meaning for $x = -1$.

Summing up, we can say that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - + \ldots + (-1)^{n-1}\frac{x^n}{n} + R_n,$$

where the remainder $R_n$ tends to zero as $n$ increases, provided that $x$ lies in the interval $-1 < x \leq 0$; note that this interval is open on the left hand side and closed on the right hand side. In fact, we can find from the above inequalities an estimate for the remainder, independent of $x$, which is valid for all values of $x$ in the interval $-1 + h \leq x \leq 1$, where $h$ is a number such that $0 < h \leq 1$. In fact, then

$$|R_n| \leq \frac{1}{h}\frac{1}{n+1},$$

and this formula shows that in the entire interval the function $\log(1 + x)$ is expressed approximately by our polynomial of the $n$-th degree, the error being nowhere greater than $1/h(n + 1)$. We leave it to the reader to convince himself that for all values of $x$ for which $|x| > 1$, the remainder not only fails to approach zero, but, in fact, increases numerically beyond all bounds as $n$ increases, so that for such values of $x$ our polynomial does not yield an approximation to the logarithm.

The fact that in the above interval the remainder $R_n$ tends to zero may be expressed by saying that we have in this interval for the logarithm the **infinite series** (cf. Chapter VIII)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \ldots$$

If we insert in this series the particular value $x = 1$, we obtain the remarkable formula

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots$$

This is one of the relations the discovery of which left a deep impression on the minds of the first pioneers of the differential and integral calculus.

The above approximation for the logarithm leads us to another formula which is useful for many purposes, particularly in numerical calculations. Provided that $-1 < x < 1$, we have only to write $-x$ in place of $x$ in the above formula to obtain

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots - S_n.$$

Assuming that $n$ is even and subtracting, we find

$$\frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{n-1}}{n-1} + R_n,$$

where $R_n$ is given by

$$R_n = \frac{1}{2}(R_n + S_n) = \frac{1}{2}\int_0^x t^n\left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt = \int_0^x \frac{t^n}{1-t^2}\,dt.$$

Because

$$|R_n| \leqq \frac{|x^{n+1}|}{n+1}\,\frac{1}{1-x^2},$$

the remainder tends to zero as $n$ increases, a fact which we again express by writing the expansion as an infinite series:

$$\frac{1}{2}\log\frac{1+x}{1-x} = \operatorname{ar\,tanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots,$$

for all values of $x$ such that $|x| < 1$.

An advantage of this formula is that as $x$ traverses the interval -1 to +1, the expression $(1 + x)/(1 - x)$ ranges over all positive numbers, whence, if the value of $x$ is chosen suitably, this series enables us to calculate the value of the logarithm of any positive number with an error not exceeding the above estimate for $R$.

**6.1.2 The Inverse Tangent:** We can treat the inverse tangent in a similar way, if we begin with the formula, true for every positive integer *n*,

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - + \ldots + (-1)^{n-1} t^{2n-2} + r_n,$$

where

$$r_n = (-1)^n \frac{t^{2n}}{1+t^2}.$$

By integration, we obtain

$$\operatorname{ar\,tan} x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + R_n,$$

$$R_n = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\, dt,$$

and see at once that in the interval $-1 \leq x \leq 1$ the remainder $R_n$ tends to zero as $n$ increases, because, by the mean value theorem of the integral calculus,

$$|R_n| \leq \int_0^{|x|} t^{2n}\, dt = \frac{|x|^{2n+1}}{2n+1}.$$

Using this formula for the remainder, we can also readily show that for $|x| > 1$ the absolute value of the remainder increases beyond all bounds as $n$ increases. We have thus derived the infinite series

$$\operatorname{artan} x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \ldots,$$

valid for $|x| \leq 1$. For $x = 1$, since arc tan $1 = \pi/4$, we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - + \dots,$$

as remarkable a formula as the one found previously for log 2.

**Exercises 6.1:**

1. Prove that

$$x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \ (x > 0).$$

Hence find log 4/3 to 2 places.

2. Calculate log 6/5 to 3 places, using the series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots.$$

Prove that the result is accurate to 3 places.

3. How many terms of the series for log $(1 + x\}$ must be used in order to obtain log $(1 + x)$ to within 10 per cent, if $30 \le x \le 31$?

Answers and Hints

## 6.2 Taylor's Theorem

An approximate representation by rational functions, as in the special cases above, can also be obtained in the case of an arbitrary function $f(x)$ for which we only assume that for all values of the independent variable in an assigned closed interval the function has continuous derivatives at least up to the $(n$+l)-th order . In most of the cases, which actually occur, the existence and continuity of all the derivatives of the function are known to begin with, so that we can choose for $n$ any arbitrary integer. The approximation formula, which we shall now derive, was discovered in the early days of the differential and integral calculus by Newton's student Taylor and is known as **Taylor's theorem**.

A special case of this theorem is often referred to, without historical justification, as **Maclaurin's theorem**. We will not follow this practice.

**6.2.1 Taylor's Theorem for Polynomials:** In order to get a clear idea of the problem, we shall consider first the case where

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

is itself a polynomial of the *n*-th degree. We can then easily express the coefficients of this polynomial in terms of the derivatives of $f(x)$ at the point x=0. In fact, differentiating both sides of the equation once, twice, etc., with respect to $x$ and setting $x = 0$, we at once find that the coefficients are

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{1}{2!} f''(0), \ldots, \quad a_n = \frac{1}{n!} f^{(n)}(0).$$

Any polynomial $f(x)$ of the *n*-th degree can therefore be written in the form

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0).$$

This formula merely states that the coefficients $a_\nu$ can be expressed in terms of the derivatives at $x = 0$ and gives the expressions for them.

We can generalize this **Taylor series** for the polynomial slightly, if we replace $x$ by $\xi = x + h$ and consider the function $f(\xi) = f(x + h) = g(h)$ as a function of $h$, for moment thinking of $x$ as fixed and $h$ as the independent variable. It then follows that

$$g'(h) = f'(\xi), \quad \ldots, \quad g^{(n)}(h) = f^{(n)}(\xi),$$

and hence, if we set $h = 0$,

$$g'(0) = f'(x), \quad \ldots, \quad g^{(n)}(0) = f^{(n)}(x).$$

If we apply the previous formula to the function $f(x+h) = g(h)$, which is itself a polynomial of the $n$-th degree in $h$, we immediately obtain the Taylor series

$$f(\xi) = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x).$$

**6.2.2 Taylor's Theorem for an Arbitrary Function:** These formulae suggest that we should seek a similar formula in the case of an arbitrary function $f(x)$, which is not necessarily a polynomial; however, in this case, the formula can lead only to an approximation to the function by a polynomial.

We wish to compare the values of the function $f$ at the point $x$ and at the point $\xi=x+h$, so that $h = \xi - x$. If now $n$ is any positive integer, as a rule, the expression

$$f(x) + (\xi - x)f'(x) + \ldots + \frac{(\xi - x)^n}{n!}f^{(n)}(x)$$

will not be an exact expression for the functional value $f(\xi)$, whence we must set

$$f(\xi) = f(x) + (\xi - x)f'(x) + \frac{(\xi - x)^2}{2!}f''(x) + \ldots + \frac{(\xi - x)^n}{n!}f^{(n)}(x) + R_n,$$

where the expression $R_n$ denotes the **remainder** when $f(\xi)$ is replaced by the expression $f(x) + f'(x)(\xi - x) + \ldots$. In the first instance, this equation is nothing but a formal definition of the expression $R_n$. Its significance lies in the fact that we can easily find a neat and useful expression for $R_n$. For this purpose, we think of the quantity $\xi$ as the fixed and the quantity $x$ as the independent variable. The remainder is then the function $R_n(x)$. By the above equation, this function vanishes for $x = \xi$:

$$R_n(\xi) = 0.$$

Moreover, by differentiation, we obtain

$$R_n'(x) = -\frac{(\xi - x)^n}{n!}f^{(n+1)}(x).$$

In fact, if we differentiate this equation for the remainder with respect to $x$, we obtain 0 on the left hand side, since $f(\xi)$ does not depend on $x$ and is therefore to be regarded as a constant. On the right hand side, we differentiate each term by the rule for products and find that all the terms cancel except the last one, which is written above with a minus sign.

Now, by the fundamental theorem of the integral calculus,

$$R_n(x) = R_n(x) - R_n(\xi) = \int_\xi^x R_n'(t)\,dt = -\int_x^\xi R_n'(t)\,dt,$$

so that we obtain the formula

$$R_n(x) = \int_x^{x+h} \frac{(x+h-t)^n}{n!} f^{(n+1)}(t)\,dt.$$

If we introduce a new integration variable $\tau$ by means of the equation $\tau = t - x$, this becomes

$$R_n = \frac{1}{n!}\int_0^h (h-\tau)^n f^{(n+1)}(x+\tau)\,d\tau.$$

Collecting these results, we can make the statement:

If the function $f(x)$ has continuous derivatives up to order $(n + 1)$ in the interval under consideration, then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots + \frac{h^n}{n!}f^{(n)}(x) + R_n,$$

or (the equivalent expression for $h = \xi - x$)

$$f(\xi) = f(x) + (\xi - x)f'(x) + \frac{(\xi-x)^2}{2!}f''(x) + \dots + \frac{(\xi-x)^n}{n!}f^{(n)}(x) + R_n,$$

where the remainder $R_n$ is given by

$$R_n = \frac{1}{n!}\int_0^h (h-\tau)^n f^{(n+1)}(x+\tau)\,d\tau.$$

In particular, if we set $x = 0$ and then replace $h$ by $x$, we obtain

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + R_n$$

with the remainder

$$R_n = \frac{1}{n!}\int_0^x (x-\tau)^n f^{(n+1)}(\tau)\,d\tau.$$

These formulae are known as **Taylor's theorem**. They give expressions for the functions $f(x + h)$ and $f(x)$, respectively, in terms of polynomials of degree $n$ in $h$ and in $x$, respectively (the so-called **polynomial of approximation**), and a **remainder**. The polynomial of approximation is characterized by the fact that, when $h = 0$ (or $x = 0$, as the case may be), its value and that of its first $n$ derivatives are the same as those of the given function and its first $n$ derivatives. In contrast to the Taylor series for polynomials, which do not require a remainder, the remainder and the expression for it are here essential. The significance of the formula lies in the fact that the remainder, even though it has a more complicated form than the other terms of the formula, nevertheless provides useful means for estimating the accuracy with which the sum of the first $n + 1$ terms

$$f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0),$$

represents the function $f(x)$.

**6.2.3 Estimation of the Remainder:** Whether the first $n + 1$ terms of Taylor's series actually yield a sufficiently good approximation to the function depends naturally on whether the remainder is sufficiently small. Hence we turn now to the estimation of this remainder. Such an estimate can most easily be made by means of the mean value theorem of the integral calculus (2.7).

We employ this theorem in the form

$$\int_0^h p(\tau)\,\phi(\tau)\,d\tau = \phi(\theta h)\int_0^h p(\tau)\,d\tau,$$

where $p(\tau)$ is a continuous function which is negative nowhere in the interval of integration and $\phi(\tau)$ is merely a continuous function there, while $\theta$ is a number in the interval $0 \le \theta \le 1$. In fact, we may assume that $0 < \theta < 1$, but this is not important here. If we set in the expression for the remainder $p(\tau)=(h-\tau)$, we obtain

$$R_n = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(x+\theta h),$$

while if we set $p(\tau) = 1$, we obtain

$$R_n = \frac{h^{n+1}}{n!}(1-\theta)^n f^{(n+1)}(x+\theta h),$$

which is less important for us and is stated here only for the sake of completeness. In these formulae, $\theta$ denotes a certain number in the interval
$0 \le \theta \le 1$, the value of which, in general, cannot be specified more accurately; of course, as a rule, this value is different in the two formulae for the remainder and depends, in addition, on $n$, $x$ and $h$. The first form of the remainder was given by **Lagrange**, the second by **Cauchy,** whence they received their names.

These expressions for the remainder, as well as others, can be derived from the mean value theorem of the differential calculus and from the generalized mean value theorem (A3.3), respectively. We apply these theorems to the function $R_n = R_n(x) - R_n(\xi)$ and to the pair of functions $R_n(x)$ and $(x - \xi)^{n+1}$, respectively, where we consider $\xi$ to be fixed and employ the formula

$$R_n'(x) = -\frac{(\xi - x)^n}{n!} f^{(n+1)}(x).$$

These methods of deriving the formulae for the remainder emphasize more the fact that Taylor's theorem is a generalization of the mean value theorem; they also offer the advantage, which for many theoretical purposes is important, that we need only assume the existence and not the **continuity** of the $(n + 1)$-th derivative. However, on the other hand, we lose the advantage of having an exact expression for the remainder in the form of an integral.

Our interest will be directed chiefly towards discovering whether the remainder $R_n$ tends to zero as $n$ increases; if this is the case, the larger we choose $n$ the more accurately is $f(x + h)$ represented by the corresponding polynomial in $h$. In this case, we say that we have expanded the function in an **infinite Taylor series**

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots,$$

or, in particular, if we first set $x = 0$ and then write $x$ in place of $h$,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots.$$

We shall encounter examples in the next section. However, first of all, we wish to point out the second important point of view arising from the consideration of Taylor series. If we think in the first formula of the quantity $h$ as becoming smaller and tending to zero, then, in the terminology of 3.9.5, the various terms of the series will tend to zero at different orders of magnitude; we accordingly call the expression $f(x)$ the **term of zero order** in the Taylor series, the expression $hf'(x)$ the term of **first order**, the expression $h^2f''(x)/2!$ the term of **second order**, etc. We gather from the form of our remainder the fact:

In expanding a function as far as the term of $n$-th order, we make an error which tends to zero at order $(n+1)$ as $h \to 0$.

Many important applications depend on this fact. It shows us that the nearer the point $x + h$ lies to the point $x$, the better is the representation of the function $f(x+h)$ by the polynomial approximation and that, in a given case, the approximation can be improved in the immediate neighbourhood of the point $x$ by increasing the value of $n$.

**Examples 6.2:**

1. Let f($x$) have a continuous derivative in the interval $a \leq x \leq b$ and $f''(x) \geq 0$ for $\geq$ every value of $x$. Then, if $\xi$ is any point in the interval, the curve nowhere drops below its tangent at the point $x = \xi$, $y = f(\xi)$. (Use the Taylor expansion with three terms.)

2. Find the value of $\theta$ in Lagrange's form of the remainder $R_n$ for the expansions of $1/(1 - x)$ and $1/(1 + x)$ in powers of $x$.

## 6.3 Applications. Expansions of the Elementary Functions

We shall now use the general results of the preceding section in order to express the elementary functions approximately by polynomials and to expand them in Taylor series. However, we shall retrict ourselves to those functions for which the coefficients of the expansion in series are given by simple laws of formation. The series for certain other functions will be discussed in 8.6.

**6.3.1 The Exponential function:** The simplest example is offered by the exponential function $f(x) = e^x$, all the derivatives are which identical with $f(x)$ and therefore have the value 1 for $x = 0$. Hence, using Lagrange's form for the remainder, we at once obtain

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^{\theta x}$$

in accordance with 6.2. If we now let $n$ increase beyond all bounds, the remainder will tend to zero, no matter what fixed value of $x$ has been chosen, because $|e^{\theta x}| \le e^{|x|}$ from the start. We now choose a fixed integer $m$ large than $2|x|$, then for $n \le m$

$$\frac{|x|}{n} < \frac{1}{2}; \quad \left| \frac{x^{n+1}}{(n+1)!} \right| = \frac{|x^m|}{m!} \cdot \frac{|x|}{m+1} \cdots \frac{|x|}{n+1}$$
$$\le \frac{|x^m|}{m!} \frac{1}{2^{n+1-m}} \le \frac{|2x|^m}{m!} \frac{1}{2^n},$$

whence

$$|R_n| \le \frac{|2x|^m}{m!} e^{|x|} \frac{1}{2^n}.$$

Since the first two factors on the right hand side are independent of $x$, while the number $1/2^n$ tends to zero as $n$ increases, our statement is proved. If we think of the number $x$ as not being fixed, but free to vary in the interval $-a \le x \le a$, where $a$ is a fixed positive number, there follows from the above, if we choose $m > 2a$, that the estimate

$$|R_n| \leqq \frac{|2a|^m}{m!} e^a \frac{1}{2^n}$$

is valid provided $n \geq m$. Thus, we have specified for the remainder a bound which holds for all values of $x$ in the interval $-a \leq x \leq a$ and tends to zero as $n \to \infty$. For the function $e^x$, we can therefore write the expansion as an infinite series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!}.$$

This expansion is valid for all values of $x$. Thus, we have again proved that the number $e$ considered in 1.6.5 is the same as the base of the natural logarithm (3.6). Of course, we must employ for numerical calculations the finite form of Taylor's theorem with the remainder; for example, for $x = 1$, this yields

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \frac{e^\theta}{(n+1)!}.$$

If we wish to compute $e$ with an error of at most 1/10,000, we need only choose $n$ so large that the remainder is certainly less than 1/10,000; since this remainder is certainly less* than $3/(n + 1)!$, it is sufficient to choose n = 7, since 8!>30,000. We thus obtain the approximate value

$$e = 2.71822$$

with an error less than 0.0001. We do not take here into account the error due to neglect of the figures in the sixth decimal place.

* We have used here the fact that $e < 3$. This follows immediately from our series for $e$, because it is always true that $1/n \leq 1/2^{n-1}$, whence

$$e < 1 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots = 1 + 1/(1 - \tfrac{1}{2}) = 3.$$

**6.3.2 Sin $x$, cos $x$, sinh $x$, cosh $x$:** We find for the functions sin $x$, cos $x$, sinh $x$ and cosh $x$:

$$
\begin{array}{lcllll}
f(x) & = & \sin x & \cos x & \sinh x & \cosh x, \\
f'(x) & = & \cos x & -\sin x & \cosh x & \sinh x, \\
f''(x) & = & -\sin x & -\cos x & \sinh x & \cosh x, \\
f'''(x) & = & -\cos x & \sin x & \cosh x & \sinh x, \\
f^{iv}(x) & = & \sin x & \cos x & \sinh x & \cosh x. \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

If $f(x) = \sin x$ or $f(x) = \cos x$, the $n$-th derivative can always be represented by

$$f^{(n)}(x) = f(x + \tfrac{1}{2}n\pi).$$

Hence, the coefficients of the even powers of $x$ vanish in the polynomial approximations for $\sin x$ and sinhx, while the coefficients of the odd powers vanish in those for $\cos x$ and $\cosh x$. Thus, in the first case, the $(2n + 1)$-th and the (2n + 2)-th polynomial are identical, while, in the second case, the $2n$-th and the $(2n + 1)$-th polynomials are identical. If we employ in each case the higher order ones of these polynomials, we obtain at once Lagrange's forms of the remainders

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$+ (-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!} \cos(\theta x),$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots + (-1)^n \frac{x^{2n}}{(2n)!}$$
$$+ (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} \cos(\theta x),$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!}$$
$$+ \frac{x^{2n+3}}{(2n+3)!} \cosh(\theta x),$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!}$$
$$+ \frac{x^{2n+2}}{(2n+2)!} \cosh(\theta x),$$

where, of course, in each of the four formulae $\theta$ denotes a different number in the interval $0 \leq \theta \leq 1$, a number which, in addition, depends on $n$ and on $x$. In these formulae, we can also make the approximation as exact as we please for each value of $x$, since the remainder tends to 0 as $n$ increases. We thus obtain the four series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{x^{2\nu+1}}{(2\nu+1)!},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{x^{2\nu}}{(2\nu)!},$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{\nu=0}^{\infty} \frac{x^{2\nu+1}}{(2\nu+1)!},$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{\nu=0}^{\infty} \frac{x^{2\nu}}{(2\nu)!}.$$

The last two series can also be obtained formally from the series for $e^x$ in accordance with the definitions of the hyperbolic functions.

**6.3.3 The Binomial Series:** We may pass over the Taylor series for the functions log $(1 + x)$ and artan $x$, which we have already dealt with directly in 6.1.1. However, we must take up the generalization of the binomial theorem for arbitrary indices, which is one of the most fruitful of Newton's mathematical discoveries and which represents one of the most important cases of the expansion in Taylor series. Our objective is the expansion of the function

$$f(x) = (1 + x)^\alpha$$

in a Taylor series, where $x > -1$ and $\alpha$ is an arbitrary number, positive or negative, rational or irrational. We have chosen the function $(1 + x)^\alpha$ instead of $x^\alpha$, because it is not true at the point $x = 0$ that all the derivatives of $x^\alpha$ are continuous except in the trivial case of non-negative values of $\alpha$. We calculate first the derivatives of $f(x)$:

$$f'(x) = \alpha(1 + x)^{\alpha-1}, \quad f''(x) = \alpha(\alpha - 1)(1 + x)^{\alpha-2}, \dots,$$
$$f^{(\nu)}(x) = \alpha(\alpha - 1)\dots(\alpha - \nu + 1)(1 + x)^{\alpha-\nu}.$$

In particular, we have for $x = 0$,

$$f'(0) = \alpha, f''(0) = \alpha(\alpha - 1), \dots, f^{(\nu)}(0) = \alpha(\alpha - 1)(\alpha - \nu + 1).$$

Taylor's theorem then yields

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \dots$$
$$+ \frac{a(a-1)(a-2)\dots(a-n+1)}{n!}x^n + R_n.$$

We must still discuss the remainder. This is not very difficult, but nevertheless is not quite as simple as in the cases treated previously. Here we shall pass over the remainder estimate, since the general binomial theorem will be proved completely in a somewhat different and simpler way in 8.6.2 and A6.3. The result, which we mention here in advance, is that whenever $|x| < 1$, the remainder tends to 0, whence the expression $(1 + x)^{\alpha}$ can be expanded in the infinite **binomial series**

$$(1+x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \dots = \sum_{\nu=0}^{\infty}\binom{a}{\nu}x^{\nu},$$

where, for the sake brevity, we have introduced the **general binomial coefficients**

$$\binom{a}{\nu} = \frac{a(a-1)\dots(a-\nu+1)}{\nu!} \text{ (for } \nu > 0), \binom{a}{0} = 1.$$

**Exercises 6.3:**

1. Expand $(1+ x)^{1/2}$ into two terms plus remainder. Estimate the remainder.

2. Use the expansion of 1. (discarding the remainder) to calculate $\sqrt{2}$. What is the degree of accuracy of the approximation?

3.. What linear function approximates best $\sqrt[3]{(1+x)}$ in the neighbourhood of $x = 0$? Between what values of $x$ is the error of the approximation less than .01?

4. What quadratic function approximates best $\sqrt[3]{(1+x)}$ to ~(1 + »} in the neighbourhood of $x = 0$? What is the greatest error in the interval $-0.1 \leq x \leq 0.1$?

5. (*a*) What linear function, (*b*) what quadratic function approximates best $\sqrt[n]{(1+x)}$ in the neighbourhood of $x = 0$? What are the largest errors in $-0.1 \leq x \leq 0.1$?

6. Calculate sin 0.01 to 4 places.

7. Do the sums for (*a*) cos 0.01, (*b*) $\sqrt[3]{126}$, (*c*) $\sqrt{97}$.

8. Expand sin $(x + h)$ in a Taylor series in $h$, whence find sin 31°=sin(30°+1°) to 3 places.

Expand the functions in 9. - 18. in the neighbourhood of $x = 0$ to three terms plus remainder (writing the remainder in Lagrange's form).

9. $\sin^2 x$.

10. $\cos^2 x$.

11. $\log \cos x$.

12. $\tan x$.

13. $\log \dfrac{1}{\cos x}$.

14. $e^{-x^2}$.

15. $\dfrac{1}{\cos x}$.

16. $\cot x - \dfrac{1}{x}$.

17. $\dfrac{1}{\sin x} - \dfrac{1}{x}$.

18. $\dfrac{\log(1+x)}{1+x}$.

19. (*a*) Expand $e^{\sin x}$ to five terms plus remainder; (*b*) in the power series for $e^z$, substitute for $z$ the power series for $\sin x$, taking enough terms to secure that the coefficient of $x^4$ is correct. Compare with (*a*).

20. Find the polynomial of fourth degree which best approximates tan $x$ in the neighbourhood of * = 0. In what interval does this polynomial represent tan $x$ to within 5%?

21. Find the first 6 terms of the Taylor series for $y$ in powers of $x$ for the functions, defined by

$$(a)\ x^2 + y^2 = y,\ y(0) = 0;\ (b)\ x^2 + y^2 = y,\ y(0) = 1;$$
$$(c)\ x^3 + y^3 = y,\ y(0) = 0.$$

Answers and Hints

## 6.4 Geometrical Applications

The behaviour of a function $f(x)$ in the neighbourhood of a point $x = a$ or of a given curve in the neighbourhood of a point can be studied with increased accuracy by means of Tailor's theorem, because this theorem resolves the increment of the function on passing to a neighbouring point $x = a + h$ into a sum of quantities of the first order, second order, etc.

**6.4.1 Contact of Curves:** We shall now use this method, in order to investigate the concept of contact of two curves. If at a point, say the point $x = a$, two curves $y = f(x)$ and $y = g(x)$ do not only intersect, but also have a common tangent, we shall say that at this point the **curves touch one another** or have **contact of the first order**. The Taylor expansions of the functions $f(a + h)$ and $g(a + h)$ then have the same terms of zero and first order in $h$. If at the point $x=a$ the second derivatives of $f(x)$ and $g(x)$ are also equal to each other, we say that the curves have **contact of the second order**. In the Taylor expansions, the terms of second order are then also the same and, if we assume that both functions have at least continuous derivatives of the third order, the difference $D(x) = f(x) - g(x)$ can be expressed in the form

$$D(a+h)=f(a+h)-g(a+h)=\frac{h^3}{3!}D'''(a+\theta h)=\frac{h^3}{3!}F(h),$$

where the expression $F(h)$ tends to $f'''(a + h) - g'''(a)$ as $h$ tends to zero, whence the difference $D(a + h)$ vanishes to at least the third order in $h$.

We can proceed in this way and consider the general case, when the Taylor series for $f(x)$ and $g(x)$ are the same up to terms of the $n$-th order, i.e., when

$$f(a)=g(a), f'(a)=g'(a), f''(a)=g''(a), \ldots, f^{(n)}(a)=g^{(n)}(a).$$

We assume here that the $(n + 1)$-th derivatives are also continuous. Under these conditions, we say that at this point the curves have **contact of the *n*-th order**. The difference of the two functions will then be

$$f(a+h) - g(a+h) = \frac{h^{n+1}}{(n+1)!} F(h),$$

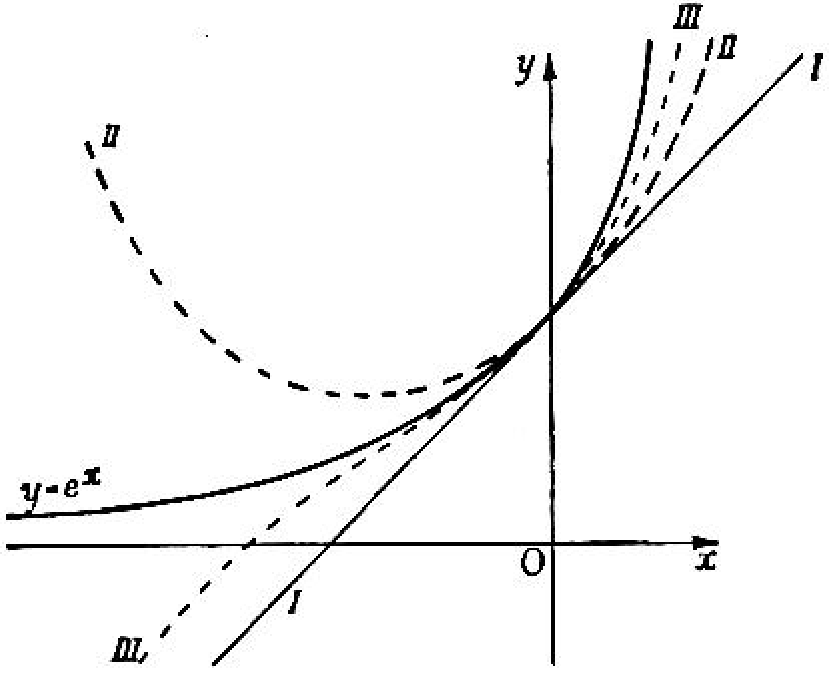



Fig. 1.—Osculating parabolas of $e^x$

where, since $0 \leq \theta \leq 1$, as $h$ tends to 0, the quantity $F(h) = D^{(n+1)}(a + \theta h)$ tends to $f^{(n+1)}(a) - g^{(n+1)}(a)$. We recognize from this formula that at the point of contact the difference $f(g) - g(x)$ vanishes at least to $(n + 1)$-th order.

The Taylor polynomials are simply defined geometrically by the fact that they are those parabolas of order $n$ which at the given point have contact of the highest possible order with the graph of the given function. Hence they are sometimes called **osculating parabolas**. For the case $y = e^x$, Fig. 1 gives the first three osculating parabolas at the point $x = 0$.

If two curves $y = f(x)$ and $y = g(x)$ have contact of order $n$, the definition does not exclude the possibility that the contact may be of a still higher order, i.e., that also the equation $f^{(n+1)}(a) = g^{(n+1)}(a)$ is true. If this is not the case, i.e., if $f^{(n+1)}(a) \neq g^{(n+1)}(a)$, we speak of a contact of exactly the $n$-th order or say that the order of the contact is exactly n.

The fact that the order of contact of two curves is a genuine geometrical relationship, which is unaffected by a change of axes, is easily verified by means of the formulae for a change of axes.

From our formula as well as from our figures, we can at once state a remarkable fact which is often overlooked by beginners. If the contact of two curves is exactly of an even order, that is, if an even number $n$ of derivatives of the two functions have the same value at the point in question, while the $(n+1)$-th derivatives differ, then, in conformity with the above formula, the difference $f(a + h) - g(a + h)$ will have different signs for small positive values and for numerically small negative values of $h$. The two curves will then cross at the point of contact. This case occurs, for example, in a contact of the second order, if the third derivatives have different values. However, if we consider in detail the case of an odd order contact, e.g., the case of an ordinary contact of the first order, the difference $f(a + h) - g(a + h)$ will have the same sign for all numerically small values of $h$, whether positive or negative; the two curves therefore will not cross in the neighbourhood of the point of contact.

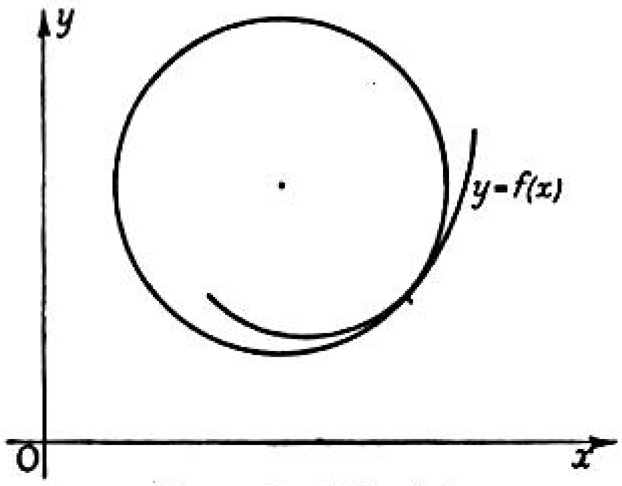


Fig. 2.—Osculating circle

The simplest example of this type is the contact of a curve with its tangent. The tangent can cross the curve only at points where the contact is at least of second order; it will actually cross the curve at points, where the order of contact is even, e.g., at an ordinary **point of inflection**, where $f''(x) = 0$, but $f'''(x) \neq 0$. At points, where the order of contact is odd, it will not cross the curve; examples are an ordinary point of the curve where the second derivative is non-zero or the curve $y = x^4$ at the origin.

**6.4.2 The Circle of Curvature as Osculating Circle:**. From this point of view, the concept of the curvature of a curve $y = f(x)$ gains a new intuitive significance. There pass through the definite point of the curve with co-ordinates $x = a$ and $y = b$ an infinite number of circles which touch the curve at the point. The centres of these circles lie on the normal to the curve and there corresponds to each point of this normal just one such tangent circle. We may expect that we can bring about by a proper choice of the centre of the circle a contact of second order between the curve and the circle.

As a matter of fact, we know from 5.2.6 that for the circle of curvature at the point $x = a$, the equation of which is, say, $y = g(x)$, we do not only have $g(a)=f(a)$ and $g'(a) = f'(a)$, but also $g''(a) = f''(a)$, whence the circle of curvature is at the same time the osculating circle at the point of the curve under consideration, i.e., it is the circle which at that point has second order contact with the curve. In the limiting case of a point of inflection or, in general, of a point at which the curvature is zero and the radius of curvature infinite, the circle of curvature degenerates into the

tangent. In ordinary cases, i.e., when the contact at the point in question does not happen to be of an order higher than the second, the circle of curvature will not merely touch the curve, but will also cross it (Fig. 2).

**6.4.3 On the Theory of Maxima and Minima:** As we have already seen in 3.5.2, a point $x = a$, at which $f'(a) = 0$, has a maximum of the function $f(x)$ if $f''(a)$ is negative, a minimum if $f''(a)$ is positive,whence these conditions are sufficient for the occurrence of a maximum or a minimum. They are by no means necessary, because there are three possibilities in the case where $f''(a)=0$; at the point in question, the function may have a maximum or a minimum or neither. Examples of the three possibilities are giver by the functions $y=-x^4$, $y=x^4$ and $y = x^3$ at the point $x = 0$. Taylor's theorem enables us immediately to make a general statement of sufficient conditions for a maximum or a minimum.We merely need expand the function $f(a + h)$ in powers of $h$; the essential point is then to discover whether the first non-vanishing term contains an even or an odd power of $h$. In the first case, we have a maximum or a minimum according to whether the coefficient of $h$ is negative or positive; in the second case, we have a horizontal **inflectional tangent** and neither a maximum nor a minimum. The reader should complete the argument by using the formula for the remainder.

However, the earlier necessary and sufficient condition (3.5.2) is more general and more convenient in applications, i.e., provided the first derivative $f'(x)$ vanishes at only a finite number of points; a necessary and sufficient condition for the occurrence of a maximum or minimu,at one of these points is that the first derivative $f'(x)$ changes sign as the point is passed.

**Exercises 6.4:**

1. What is the order of contact of the curves $y=e^x$ and $y=l+x+½\sin^2 x$ at $x=0$?

2. What is the order of contact of $y = \sin^4 x$ and $y = \tan^4 x$ at $x = 0$?

3. Determine the constants $a$, $b$, $c$, $d$ in such a way that the curses $y=e^{2x}$ and $y= a\cos x + b \sin x + c \cos 2x + d \sin 2x$ have contact of order 3 at $x = 0$.

4. What is the order of contact of the curves

$$x^3 + y^3 = xy,\ x^3 + y^3 = x$$

at their points of intersection? Plot the curves.

5. What is the order of contact of the curves

$$x^2 + y^2 = y,\ x^2 = y$$

at their points of intersection?

6. The curve $y = f(x)$ passes through the origin $O$ and touches the $x$-axis at 0. Show that the radius of curvature of the curve at $O$ is given by $\rho = \lim_{x \to 0} \frac{x^2}{2y}$.

7.* $E$ is a circle which touches a given curve at a point $P$ and passes through a neighbouring point $Q$ of the curve. Show that the limit of the circle $K$ as $Q \to P$ *is* the circle of curvature of the curve at $P$.

8.* $R$ is the point of intersection of the two normals to a given curve at the neighbouring points $P$, $Q$ of the curve. Show that, as $Q \to P$, $R$ tends to the centre of curvature of the curve for the point $P$, (The centre of curvature is the intersection of neighbouring normals.)

9.* Show that the order of contact of a curve and the osculating circle is at least three at points where the radius of curvature is a maximum or a minimum.

10. Determine the maxima and minima of the function $y = e^{-1/x^2}$ (cf. A.6).

Answers and Hints

# Appendix to Chapter VI

## A6.1 Example of a Function which cannot be expanded in a Taylor Series

The possibility of expressing a function by means of a Taylor series with a remainder of the $(n + 1)$-th order depends essentially on the differentiability of the function at the point in question. For this reason, the function log $x$ cannot be represented into a Taylor series in powers of $x$ and can the function $\sqrt[3]{x}$, the derivative of which is infinite at $x = 0$.

In fact, for a function to be capable of expansion in an infinite Taylor series **all** its derivatives must necessarily exist at the point in question; however, this condition is by no means sufficient. A function for which all the derivatives exist and are continuous throughout an interval still need not necessarily be capable of expansion in a

Taylor series, i.e., the remainder $R_n$ in Taylor's theorem may fail to tend to zero as $n$ increases, no matter how small is the interval in which we wish to expand the function.

The simplest example of this phenomenon is offered by the function $y=f(x)=e^{-1/x^2}$ for $x \neq 0$, $f(0) = 0$, which we have already encountered in A3.1.1. This function, with all its derivatives, is continuous in every interval, even at $x=0$, and we have seen that at this point all the derivatives vanish, i.e., that $f^{(n)}(0) = 0$ for every value of $n$. Hence all the coefficients of the polynomials of approximation vanish in Taylor's theorem, no matter what is the value chosen for $n$. In other words, the remainder is and remains equal to the function itself and therefore, except when $x = 0$, does not approach 0 as $n$ increases, since the function is positive for every non-zero value of $x$.

**A6.2. Proof that *e* is Irrational:** We immediately deduce from the formula $e = 2 + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!}e^\theta,$ that the number $e$ is irrational. In fact, if the contrary were true, i.e., if $e = p/q$, where $p$ and $q$ are integers, we can certainly choose $n$ larger than $q$. Then $n!e=n!p/q$ must be an integer. On the other hand,

$$n!e = 2n! + \frac{n!}{2!} + \dots + \frac{n!}{n!} + \frac{1}{n+1}e^\theta,$$

and, since $e^\theta < e < 3$, we must have $0 < e^\theta/(n + 1) < 1$, whence the integer

$$n!e = 2n! + n!/2 + \cdots 1 \text{ plus a non-vanishing proper fraction,}$$

which is impossible.

**A6.3 Proof that the Binomial Series Converges:** In 6.3.3, we have postponed the estimation of the remainder $R_n$ in the expansion of $f(x)=(1+x)^\alpha$ for $|x| < 1$. We shall now carry out this estimate. It is most convenient to separate the cases $x > 0$ and $x < 0$.

For $f^{(n+1)}(x)$, we have the expression

$$f^{(n+1)}(x) = \alpha(\alpha-1)\dots(\alpha-n)\frac{(1+x)^\alpha}{(1+x)^{n+1}}.$$

If $x > 0$, we write the remainder in Lagrange's form

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}\,\alpha(\alpha-1)\ldots(\alpha-n)\,\frac{(1+\theta x)^{\alpha}}{(1+\theta x)^{n+1}},$$

so that

$$|R_n(x)| \leqq \left|\frac{\alpha(\alpha-1)\ldots(\alpha-n)}{(n+1)!}\right| \frac{x^{n+1}(1+x)^{\alpha}}{1^{n+1}}.$$

Letting $b = [|\alpha|] + 1$, where $[|\alpha|]$ denotes the greatest integer which does not exceed $|\alpha|$, we have

$$\begin{aligned}
|R_n(x)| &\leqq 2^b\,\frac{b(b+1)\ldots(b+n)}{(n+1)!}\,x^{n+1} \\
&\leqq \frac{2^b}{(b-1)!}\,\frac{1\,.\,2\ldots(n+1)(n+2)\ldots(n+b)}{(n+1)!}\,x^{n+1} \\
&\leqq \frac{2^b}{(b-1)!}\,(n+b)^{b-1}x^{n+1},
\end{aligned}$$

and since $b$ is fixed, if $0 < x < 1$, this approaches 0 as $n$ increases.

For the case $-1 < x < 0$, we write the remainder in Cauchy's form

$$R_n(x) = \frac{x^{n+1}}{n!}(1-\theta)^n\,\alpha(\alpha-1)\ldots(\alpha-n)\,\frac{(1+\theta x)^{\alpha-1}}{(1+\theta x)^n},$$

so that

$$|R_n(x)| \leqq \frac{(1-\theta)^n}{(1-\theta|x|)^n}\,|x|^{n+1}\left|\frac{\alpha(\alpha-1)\ldots(\alpha-n)}{n!}\right|\,|(1+\theta x)^{\alpha-1}|.$$

Since $|x| < 1$, the last factor cannot exceed a constant $K$, independent of $n$. Moreover, $(1-\theta)/(1-\theta|x|) < 1$. As before, writing $b = [|\alpha|] + 1$, we have

$$|R_n(x)| \leq K\,|x|^{n+1} \frac{1}{(b-1)!}(n+2)(n+3)\ldots(n+b)$$
$$\leq \frac{K}{(b-1)!}(n+b)^{b-1}\,|x|^{n+1},$$

which approaches 0 as $n$ increases.

Thus, in either case when $|x| < 1$, the remainder tends to zero as $n$ increases, justifying the expansion in 6.3.3.

**A6.4 Zeroes and Infinities of Functions and So-called Intermediate Expressions:** The Tailor series for a function in the neighbourhood of a point $x=a$ enables us to characterize the behaviour of the function in the neighbourhood of this point as follows: We say that a function $f(x)$ at $x = a$ has an exactly $n$-tuple zero or it vanishes there exactly to the order $n$, if $f(a) = 0$, $f\,'(a) = 0$, $f\,''(a) = 0$, $\cdots$ , $f^{(n-1)}(a) = 0$, and $f^{(n)}(a) \neq 0$. We assume here that the function possesses in the neighbourhood of the point at least continuous derivatives to the $n$-th order. By our definition, we seek to indicate that the Taylor series for the function in the neighbourhood of the point can be written in the form

$$f(a+h) = \frac{h^n}{n!}F(h),$$

in which, as $h \to 0$, the factor $F(h)$ tends to a limit different from 0, namely, the value $f^{\,n}(a)$.

If a function $\phi(x)$ is defined at all points in the neighbourhood of a point $x = a$, except perhape at $x = a$ itself, and if

$$\phi(x) = \frac{f(x)}{g(x)},$$

where at the point $x = a$ the numerator does not vanish, but the denominator has a $\nu$-tuple zero, we say that the function $\phi(x)$ becomes **infinite at the $\nu$-th order** at the point $x = a$. If at the point $x = a$ the numerator also has a $\mu$-tuple zero and if $\mu > \nu$, we say that the function has a $(\mu - \nu)$-tuple zero there; while if $\mu < \nu$, we say that the function has a $(\nu - \mu)$-tuple infinity.

All these definitions are in agreement with the convention regarding functional behaviour already laid down (cf. 3.9.3). In order to make these relations precise, we expand the numerator and denominator by Taylor's theorem, using Lagrange's form of the remainder; the function then has the form

$$\phi(a+h)=\frac{f(a+h)}{g(a+h)}=\frac{\nu!}{\mu!}\frac{h^{\mu}f^{(\mu)}(a+\theta h)}{h^{\nu}g^{(\nu)}(a+\theta_1 h)},$$

in which $\theta$ and $\theta_1$ are two numbers between 0 and 1 and the factors by which $h^{\mu}/\mu!$ and $h^{\nu}/\nu!$ are multiplied do not tend to zero as $h$ does, since they approach the limits $+f^{(\mu)}(a)$ and $g^{(\nu)}(a)$, respectively, which differ from zero. If $\mu > \nu$, we have then

$$\lim_{h\to 0}\phi(a+h)=\lim_{h\to 0}\frac{\nu!}{\mu!}h^{\mu-\nu}\frac{f^{(\mu)}(a)}{g^{(\nu)}(a)}=0.$$

Accordingly, the expression $\phi(x)$ vanishes to the order $\mu - \nu$. If $\nu > \mu t$, we see that the expression $\phi(a + h)$ becomes infinite at the order $\nu - \mu$ as $h \to 0$. If $\mu = \nu$, we obtain the equation

$$\lim_{h\to 0}\phi(a+h)=\frac{f^{(\mu)}(a)}{g^{(\nu)}(a)}.$$

We can express the content of the last equation as follows: If the numerator and denominator of a function $\phi(x) = f(x)/g(x)$ both vanish at $x = a$, we can determine the limiting value as $x \to a$ by differentiating the numerator and denominator an equal number of times until at least one of the derivatives is non-zero. If this happens for the numerator and denominator simultaneously, the limit, which we are seeking, is equal to the quotient of these two derivatives. If we encounter a non-vanishing derivative in the denominator earlier than in the numerator, the fraction tends to zero. If we encounter a non-vanishing derivative in the numerator earlier than in the denominator, the absolute value of the fraction increases beyond all bounds.

We thus have a rule for evaluating the socalled **indeterminate expressions** 0/0 - a subject that is discussed at exaggerated length in many textbooks on the differential and integral calculus. In reality, the point in question is merely the very simple determination of the **limiting value** of a quotient in which the numerator and the denominator tend to zero. The name indeterminate expression, usually found in the literature, is misleading and vague.

We can arrive at our results in a somewhat different way by basing the proof on the generalized mean value theorem instead of on Taylor's theorem (A2.2).

Thi method of deriving our rule has the advantage that there is no use made of the existence of the derivative at the point $x = a$ itself; moreover, it includes the case in which $\phi(x)$ is defined for $x \geq a$ only, so that the passage to the limit $x \to a$ or $h \to 0$ is made from one side only.

According to this, if $g'(x) \neq 0$, we have

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)}=\frac{f'(a+\theta h)}{g'(a+\theta h)},$$

where $\theta$ is the same in both the numerator and denominator. Hence, in particular, when $f(a) = 0 = g(a)$,

$$\frac{f(a+h)}{g(a+h)}=\frac{f'(a+\theta h)}{g'(a+\theta h)},$$

where $\theta$ is a value in the interral $0 < \theta < 1$ and, if we set $k = \theta h$, we obtain

$$\lim_{h\to 0}\frac{f(a+h)}{g(a+h)}=\lim_{k\to 0}\frac{f'(a+k)}{g'(a+k)},$$

it being assumed that the limit on the right hand side exists. If

$$f'(a)=0=g'(a),$$

we can proceed in the same manner until we come to the first index for which it is no longer true that $f^{(\mu)}(a) = 0 = g^{(\mu)}(a)$. Then

$$\lim_{h\to 0}\frac{f(a+h)}{g(a+h)}=\lim_{l\to 0}\frac{f^{(\mu)}(a+l)}{g^{(\mu)}(a+l)},$$

where we also include the case in which both sides have the limit infinity.

As examples, consider

$$\frac{\sin x}{x},\quad \frac{1-\cos x}{x},\quad \frac{e^{2x}-1}{\log(1+x)},\quad \frac{x^2\tan x}{\sqrt{(1-x^2)}-1}$$

as $x \to 0$. We have

$$\lim_{x\to 0}\frac{\sin x}{x}=\frac{\cos 0}{1}=1;\quad \lim_{x\to 0}\frac{1-\cos x}{x}=\frac{\sin 0}{1}=0;$$

$$\lim_{x\to 0}\frac{e^{2x}-1}{\log(1+x)}=\lim_{x\to 0}\frac{2e^{2x}}{1/(1+x)}=2;$$

$$\lim_{x\to 0}\frac{x^2\tan x}{\sqrt{(1-x^2)}-1}=\lim_{x\to 0}\frac{2x\tan x+x^2/\cos^2 x}{-x/\sqrt{(1-x^2)}}$$

$$=-\lim_{x\to 0}\left(2\tan x+\frac{x}{\cos^2 x}\right)\sqrt{(1-x^2)}=0.$$

Moreover, note that other so-called indeterminate forms can alao be reduced to the case considered; for example, the limit of $\frac{1}{\sin x}-\frac{1}{x}$ as $x \to 0$, being the limit of the difference of two expressions both of which beome infinite, is an indeterminate form $\infty - \infty$. By the transformation

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x}$$

we arrive at once at an expression the limit of which as $x \to 0$ is determined by our rule

$$\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}=\lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}=0.$$

## Exercises 6.5:

Evaluate the limits 1-12:

1. $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a}$.

2. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^3}$.

3. $\lim\limits_{x \to 0} \dfrac{24 - 12x^2 + x^4 - 24 \cos x}{(\sin x)^6}$.

4. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{\sin x}$.

5. $\lim\limits_{x \to 0} \dfrac{\operatorname{arc\,sin} x}{x}$.

6. $\lim\limits_{x \to \pi/2} \dfrac{\tan 5x}{\tan x}$.

7. $\lim\limits_{x \to 1} \left( \dfrac{2}{x^2 - 1} - \dfrac{1}{x - 1} \right)$.

8. $\lim\limits_{x \to 0} \left( \dfrac{1}{\sin^2 x} - \dfrac{1}{x^2} \right)$.

9. $\lim\limits_{x \to 0} x^{\sin x}$.

10. $\lim\limits_{x \to 0} (1 + x)^{1/x}$.

11. $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{\log(1 + x)}$.

12. $\lim\limits_{x \to 0} \dfrac{x \tan x}{\sqrt{(1 - x^2)} - 1}$.

13. Prove that the function $y = (x^2)^x$~, $y(0) = 1$ is continuous at $x = 0$.

Answers and Hints

# Numerical Methods

## Preliminary Remarks

Anyone who has to use analysis as an instrument for investigating physical or technical phenomena is faced by the question whether and how the theory can be adapted to yield useful practical methods for actual numerical calculations. Yet, even from the point of view of the theoretical person, who only desires to recognize the link between natural phenomena and not to conquer them, these questions are of no trifling interest. For a systematic treatment of numerical methods, we refer the reader to special textbooks on the subject.* Here we can only discuss some particularly important points which are more or less closely linked to the preceding ideas. We wish to direct special attention to the fundamental fact that the meaning of an approximate calculation is not precise unless it is accompanied by a definite knowledge of the degree of accuracy attained.

* For example, Whittaker and Robinson, "The Calculus of Observations" (Blakie & Son, Ltd., 1929).

## 7.1 Numerical Integration

We have seen that even relatively simple functions cannot be integrated in terms of elementary functions and that it is futile to make this unattainable goal the aim of the integral calculus. On the other hand, the definite integral of a continuous function does exist and this fact raises the problem of finding methods for calculating it numerically. Here we shall discuss the simplest and most obvious of these methods with the aid of geometrical intuition and thenconsider error estimation.

Our objective is to calculate the integral

$$I = \int_a^b f(x)\,dx,$$

where $a < b$. We imagine the interval of integration to have been subdivided into $n$ equal parts of length $h = (b - a)/n$ and denote the points of subdivision by $x_1=a$, $x_2=a+h$, $\cdots$ , $x_n=b$, the values of the function at the points of subdivision by $f_0$, $f_1$, $\cdots$ , $f_n$ and, similarly, the values of the function at the midpoints of the intervals by $f_{1/2}$, $f_{3/2}$, $\cdots$ , $f_{(2n-1)/2}$. We interpret our integral as an area and cut up the region under the curve into strips of width $h$ in the usual manner. We must now obtain an approximation for each such strip of surface, that is, for the integral

$$I_\nu = \int_{x_\nu}^{x_\nu + h} f(x)\,dx.$$

**7.1.1 The Rectangle Rule:** The crudest and most obvious method of approximating the integral $I$ is directly linked to the definition of the integral; we replace the area of the strip $I_\nu$ by the rectangle of area $f_\nu h$ and obtain for the integral the approximate expression

$$I \approx h(f_0 + f_1 + \ldots + f_{n-1}).$$

From here on, the symbol ≈ means **is approximately equal to.**

$$x_\nu,\ x_{\nu+1} = x_\nu + h,\ \text{and}\ x_{\nu+2} = x_\nu + 2h$$

**7.1.2 The Trapezoid and Tangent Formulae:** We obtain a closer approximation with no greater trouble if we do not replace the area of the strip $I_\nu$ , as above, by a rectangular area, but by the trapezoid of area $(f_\nu+f_{\nu+1})h/2$ (Fig. 1.) For the whole integral, this process yields the approximate expression

$$I \approx h(f_1 + f_2 + \ldots + f_{n-1}) + \frac{h}{2}(f_0 + f_n)$$

(trapezoid formula), since, when the areas of the trapezoids are added, each value of the function except the first and the last occurs twice.

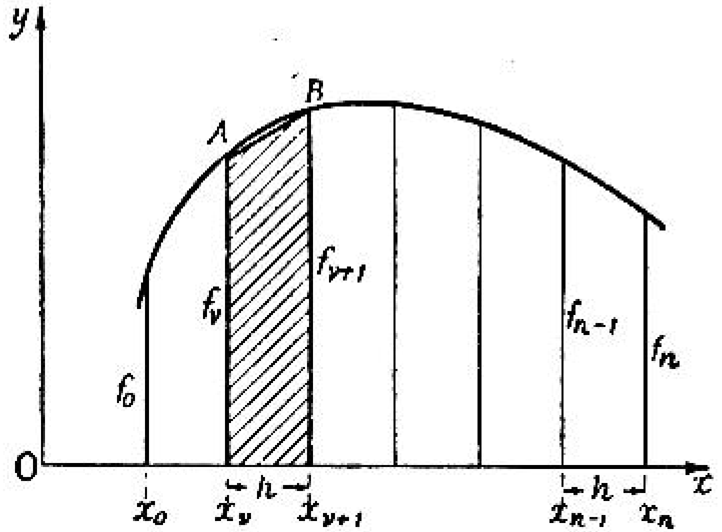


Fig. 1.—The trapezoid formula

As a rule, the approximation becomes even better if, instead of choosing the trapezoid under the chord *AB* as an approximation to the area of I, we select the trapezoid under the tangent to the curve at the point with the abscissa $x = x_\nu + h/2$. The area of this trapezoid is simply $hf_{\nu+½}$, so that the approximation for the entire integral is

$$I \approx h(f_{1/2} + f_{3/2} + \ldots + f_{(2n-1)/2}),$$

which is called the tangent formula.

**7.1.3 Simpson's Rule:** By means of Simpson's rule, we arrive with very little more trouble at a numerical result which is generally much more exact. This rule depends on estimating the area $I_\nu + I_{\nu+1}$ of the double strip between the abscissae $x = x_\nu$ and $x = x_\nu + 2 = x_{\nu+2}$ by considering the upper boundary to be no longer a straight line but a parabola - in order to be specific, that parabola which passes through the three points of the curve with the abscissae $x_\nu$, $x_{\nu+1} = x_\nu + h$, and $x_{\nu+2} = x_\nu + 2h$ (Fig. 2). The equation of this parabola is

$$y = f_\nu + (x - x_\nu)\frac{f_{\nu+1} - f_\nu}{h} + \frac{(x - x_\nu)(x - x_\nu - h)}{2}\,\frac{f_{\nu+2} - 2f_{\nu+1} + f_\nu}{h^2}.$$

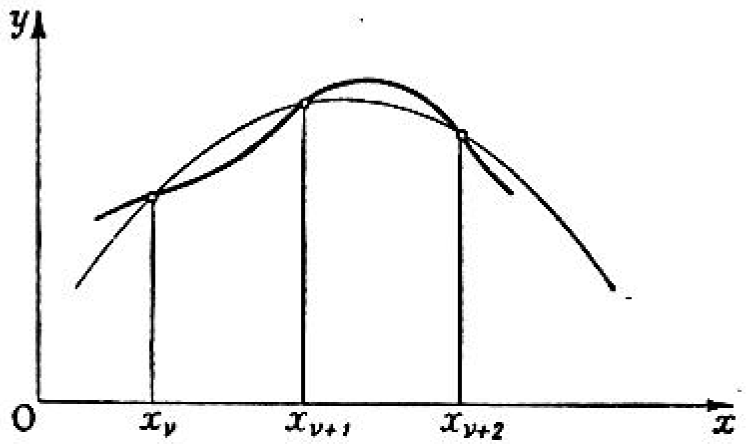



Fig. 2.—Simpson's rule

The student should verify by direct substitution that for the three values of $x$ in question this eqnation gives the proper values of $y$, i.e., $f_\nu$ , $f_{\nu+1}$ and $f_{\nu+2}$, respectively.

If we integrate this second degree polynomial between the $x_\nu$ and $x_\nu + 2h$, we obtain, after a brief calculation, for the area under the parabola

$$\int_{x_\nu}^{x_\nu+2h} y\,dx = 2hf_\nu + 2h(f_{\nu+1} - f_\nu) + \frac{1}{2}\left(\frac{8}{3}h - 2h\right)(f_{\nu+2} - 2f_{\nu+1} + f_\nu)$$
$$= \frac{h}{3}(f_\nu + 4f_{\nu+1} + f_{\nu+2}).$$

This represents the required approximation to the area of our strip $I_\nu + I_{\nu+1}$.

If we now assume that $n = 2m$, i.e., that $n$ is an even number, we obtain, by addition of the areas of such strips, Simpson's rule

$$I \approx \frac{4h}{3}(f_1 + f_3 + + \ldots f_{2m-1}) + \frac{2h}{3}(f_2 + f_4 + \ldots + f_{2m-2}) + \frac{h}{3}(f_0 + f_{2m}).$$

**7.1.4 Examples:** We now apply these methods to the calculation of

$\log_e 2 = \int_1^2 \frac{dx}{x}$. If we divide this integral from 1 to 2 into ten equal parts, $h$ will be equal 1/10 and we obtain by the trapezoidal rule

$$\begin{array}{ll}
x_1 = 1{\cdot}1 & f_1 = 0{\cdot}90909 \\
x_2 = 1{\cdot}2 & f_2 = 0{\cdot}83333 \\
x_3 = 1{\cdot}3 & f_3 = 0{\cdot}76923 \\
x_4 = 1{\cdot}4 & f_4 = 0{\cdot}71429 \\
x_5 = 1{\cdot}5 & f_5 = 0{\cdot}66667 \\
x_6 = 1{\cdot}6 & f_6 = 0{\cdot}62500 \\
x_7 = 1{\cdot}7 & f_7 = 0{\cdot}58824 \\
x_8 = 1{\cdot}8 & f_8 = 0{\cdot}55556 \\
x_9 = 1{\cdot}9 & f_9 = 0{\cdot}52632 \\
\hline
 & \text{Sum} \quad 6{\cdot}18773 \\
x_0 = 1{\cdot}0 & \tfrac{1}{2}f_0 = 0{\cdot}5 \\
x_{10} = 2{\cdot}0 & \tfrac{1}{2}f_{10} = 0{\cdot}25 \\
\hline
 & 6{\cdot}93773 \times \tfrac{1}{10} \\
\hline
 & \log_e 2 \approx 0{\cdot}69377
\end{array}$$

The value, as was to be expected, is too large, since the curve has its convex side turned towards the $x$-axis.

By the tangent rule, we find

$$
\begin{array}{ll}
x_0 + \frac{1}{2}h = 1{\cdot}05 & f_{1/2} = 0{\cdot}95238 \\
x_1 + \frac{1}{2}h = 1{\cdot}15 & f_{3/2} = 0{\cdot}86957 \\
x_2 + \frac{1}{2}h = 1{\cdot}25 & f_{5/2} = 0{\cdot}80000 \\
x_3 + \frac{1}{2}h = 1{\cdot}35 & f_{7/2} = 0{\cdot}74074 \\
x_4 + \frac{1}{2}h = 1{\cdot}45 & f_{9/2} = 0{\cdot}68966 \\
x_5 + \frac{1}{2}h = 1{\cdot}55 & f_{11/2} = 0{\cdot}64516 \\
x_6 + \frac{1}{2}h = 1{\cdot}65 & f_{13/2} = 0{\cdot}60606 \\
x_7 + \frac{1}{2}h = 1{\cdot}75 & f_{15/2} = 0{\cdot}57143 \\
x_8 + \frac{1}{2}h = 1{\cdot}85 & f_{17/2} = 0{\cdot}54054 \\
x_9 + \frac{1}{2}h = 1{\cdot}95 & f_{19/2} = 0{\cdot}51282 \\
\hline
 & 6{\cdot}92836 \times \frac{1}{10} \\
\hline
 & \log_e 2 \approx 0{\cdot}69284
\end{array}
$$

Owing to the convexity of the curve, this value is too small.

For the same set of sub-divisions, we obtain the most accurate result by means of Simpson's rule:

| | | | |
|---|---|---|---|
| $x_1 = 1.1$ | $f_1 = 0.90909$ | $x_2 = 1.2$ | $f_2 = 0.83333$ |
| $x_3 = 1.3$ | $f_3 = 0.76923$ | $x_4 = 1.4$ | $f_4 = 0.71429$ |
| $x_5 = 1.5$ | $f_5 = 0.66667$ | $x_6 = 1.6$ | $f_6 = 0.62500$ |
| $x_7 = 1.7$ | $f_7 = 0.58824$ | $x_8 = 1.8$ | $f_8 = 0.55556$ |
| $x_9 = 1.9$ | $f_9 = 0.52632$ | | |
| | Sum $3.45955 \times 4$ | | Sum $2.72818 \times 2$ |
| | $13.83820$ | | $5.45636$ |
| | | | $13.83820$ |
| | | $x_0 = 1.0$ | $f_0 = 1.0$ |
| | | $x_{10} = 2.0$ | $f_{10} = 0.5$ |
| | | | $20.79456 \times \frac{1}{30}$ |
| | | | $\log_e 2 \approx 0.69315$ |

As a matter of fact,

$$\log_e 2 = .693147\cdots.$$

**7.1.5 Estimation of the Error:**. It is easy to obtain an estimate of the error for each of our methods of integration, if the derivatives of the function $f(x)$ are known throughout the interval of the integration. We take $M_1, M_2, \cdots$ as the upper bounds of the absolute values of the first, second, $\cdots$ derivative, respectively, i.e., we assume that throughout the interval $|f^{(\nu)}(x)| < M_\nu$. Then the estimates are:

For the rectangle rule:

$$|I_\nu - hf_\nu| < \frac{1}{2} M_1 h^2 \text{ or } \left| I - h \sum_{\nu=0}^{n-1} f_\nu \right| < \frac{1}{2} M_1 n h^2 = \frac{1}{2} M_1 (b-a) h.$$

For the tangent rule:

$$|I_\nu - hf_{\nu+\frac{1}{2}}| < \frac{M_2}{24} h^3 \quad \text{or} \quad \left| I - h \sum_{\nu=0}^{n-1} f_{\nu+\frac{1}{2}} \right| < \frac{M_2}{24} (b-a) h^2.$$

For the trapezoid rule:

$$\left| I_\nu - \frac{h}{2}(f_\nu + f_{\nu+1}) \right| < \frac{M_2}{12} h^3.$$

For Simpson's rule

$$\left| I_\nu + I_{\nu+1} - \frac{h}{3}(f_\nu + 4f_{\nu+1} + f_{\nu+2}) \right| < \frac{M_4}{90} h^5$$

The last two estimates also lead to estimates for the entire integral *1*. We see that Simpson's rule has an error of much higher order in the small quantity *h* than the other rules, so that when $M_4$ is not too large, it is very advantageous for practical calculations. In order to avoid tiring the reader with the details of the proofs of these estimates, which are fundamentally quite simple, we shall restrict ourselves to the proof for the tangent formula. For this purpose, we expand the function $f(x)$ in the $(\nu + 1)$-th strip by Taylor's theorem:

$$f(x) = f_{\nu+\frac{1}{2}} + \left(x - x_\nu - \frac{h}{2}\right) f'\left(x_\nu + \frac{h}{2}\right) + \frac{1}{2}\left(x - x_\nu - \frac{h}{2}\right)^2 f''(\xi),$$

where $\xi$ is a certain intermediate value in the strip. If we integrate the right hand side over the interval $x_\nu \leq x \leq x_\nu + h$, the integral of the middle term is zero. Since, as is easily verified,

$$\frac{1}{2}\int_{x_\nu}^{x_\nu+h}\left(x - x_\nu - \frac{h}{2}\right)^2 dx = \frac{h^3}{24},$$

it follows immediately that

$$\left| \int_{x_\nu}^{x_\nu+h} f(x)\,dx - hf_{\nu+\frac{1}{2}} \right| < M_2 \frac{h^3}{24},$$

which proves the above assertion.

**Exercises 7.1:**

1. From the formula

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx,$$

compute $\pi$ with $h = 0.1$ by (*a*) the trapezoid rule, (*b*) Simpson's rule.

2. Compute $\int_0^\infty e^{-x^2}\,dx$ numerically to within 1/100 (**10.6.5**).

3. Compute $\int_0^1 \frac{1}{\sqrt{1+x^4}}$—- numerically with an error of less than 0.1.

Answers and Hints

## 7.2 Applications of the Mean Value theorem and of Taylor's Theorem

**7.2.1 The Calculus of Errors:** We now come to quite a different type of numerical calculations. These are applications of the mean value theorem or, more generally, of Taylor's theorem with remainder or, finally, of the infinite Taylor series. As an application which, although simple, is quite important in practice, we shall consider the calculus of errors. This rests on the idea - which lies at the root of the entire differential calculus - that a function $f(x)$, which is differentiable a sufficient number of times, can be represented in the neighbourhood of a point by a linear function with an error of order less than the first, by a quadratic function with an error of order less than the second, etc. Consider the linear approximation to a function $y=f(x)$. If $y+\Delta y=f(x+\Delta x)=f(x+h)$, we find by Taylor's theorem

$$\Delta y = hf'(x) + \frac{h^2}{2}f''(\xi),$$

where $\xi = x + \theta h$ $(0 < \theta < 1)$ is an intermediate value, which need not be known more precisely. If $h = \Delta x$ is small, we obtain as a practical approximation

$$\Delta y \approx hf'(x).$$

In other words, we replace the difference quotient by the derivative to which it is approximately equal and the increment of $y$ by the approximately equal linear expression in $h$.

We use this fairly obvious fact for practical purposes in the following way. Let two physical quantities $x$ and $y$ be linked by the relation $y = f(x)$. There arises then the question how an inaccuracy in the measurement of $x$ affects the determination of $y$. If we happen to use instead of the **true** value of $x$ the **inaccurate** value $x + h$, then the corresponding value of $y$ differs from the true value $y = f(x)$ by $\Delta y = f(x + h) - f(x)$, whence the error is given approximately by the above relation.

We shall understand the use of these relations better if we consider a few examples.

**Example 1. The tangent galvanometer:** Whren we determine an electric current by a tangent galvanometer, we use the formula $y = c \tan \alpha$, where $\alpha$ is the angle of deflection of the magnetic needle, $c$ is the constant of the apparatus and $y = I$ the intensity of the current. Then

$$\frac{dy}{d\alpha} = \frac{c}{\cos^2 \alpha},$$

whence $\Delta y \approx c\Delta\alpha/\alpha^2$. The percentage error in the measurement is given by

$$\frac{100\Delta y}{y} \approx \frac{100c\Delta\alpha}{c\cos^2\alpha\tan\alpha} = \frac{200}{\sin 2\alpha}\Delta\alpha.$$

We see from this that the accuracy reaches the largest possible value, i.e., that there corresponds to a given error in the measurement of the angle the least possible error in the determination of the current, when the angle $\alpha$ is equal to $\pi/4$ or 45º.

In particular, let it be possible to read the tangent galvanometer to within half a degree; then $|\Delta\alpha|$ is radians < ½×.01745··· and the percentage error will be $1.745/\sin 2\alpha$. If the galvanometer reads 30°, $\sin 2\alpha = \frac{1}{2}\sqrt{3} = \frac{1}{2} \times 1{\cdot}73205\ldots,$ and the percentage error is less than 2×1.745/1.732, i.e., about 2%.

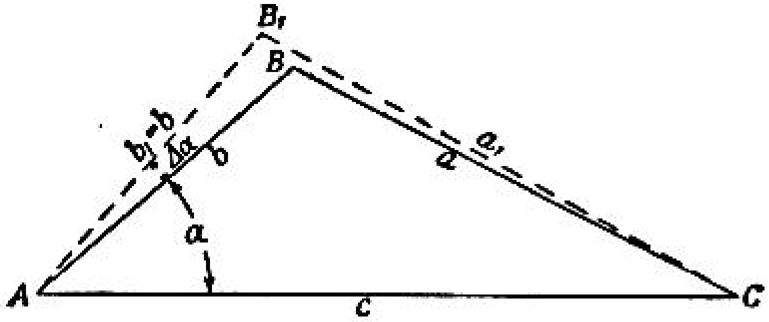



Fig. 3

**Example 2.** In a triangle *ABC* (Fig. 3), let the sides *b* and *c* have been measured accurately, while the angle $\alpha = x$ can only be measured with an error of $|\Delta x| < \delta$. Within what error range, does the value $y = a = \sqrt{(b^2 + c^2 - 2bc\cos\alpha)}$ vary?

We have

$$\Delta a \approx \frac{1}{a} bc \sin\alpha \,\Delta\alpha;$$

hence the percentage error is

$$\frac{100\,\Delta a}{a} \approx \frac{100bc}{a^2} \sin\alpha \,\Delta\alpha.$$

If we consider the special case where $b = 400$ m., $c = 500$. and $\alpha = 60°$, then, by the cosine formula, $y = a = 458.2576$ m and

$$\Delta a \approx \frac{200000}{458{\cdot}2576} \times \frac{1}{2}\sqrt{3}\,\Delta\alpha.$$

If $\Delta x$ can be measured to within ten seconds of arc, i.e., if

$$\Delta\alpha = 10'' = 4848 \times 10^{-8} \text{ radians,}$$

we find that at the worst

$$\Delta a \approx 1{\cdot}83 \text{ cm.},$$

i.e., tbe error is at most about 0.004 %.

**Example 3.** The following illustrates a type of application of the above methods by which we can often save ourselves considerable trouble in physical problems.

It is known experimentally that, if an iron rod has the length $l_0$ at temperature 0, then at temperature $t$ its length will be $l = l_0(1 + \alpha t)$, where $\alpha$ depends only on the material of the rod. Now, if a pendulum clock keeps correct time at temperature $t_1$, how many seconds will be lost per day if the temperature rises to $t_2$?

We have for the period of oscillation of a **pendulum** the formula

$$T(l) = 2\pi\sqrt{\frac{l}{g}}, \text{ whence } \frac{dT}{dl} = \frac{\pi}{\sqrt{lg}}.$$

Hence, if the change of length is $\Delta l$, the corresponding change in the period of oscillation is

$$\Delta T \approx \frac{\pi \Delta l}{\sqrt{l_1 g}},$$

where $l_1 = l_0(1 + \alpha t_1)$ and $\Delta l = \alpha\, l_0(t_2 - t_1)$. This is the time lost during each oscillation. The time lost per a second is $\Delta T/T \approx \Delta l/2l_1$, whence in one day the clock loses 43,200 $\Delta l/l_1$ seconds.

The application of our methods has saved here a number of multiplications and two extractions of square roots. Moreover, in the longer direct process, we should finally have to subtract $T(l_1)$ from the almost equal value $T(l_2)$ and a very small error in calculation would cause a relatively large percentage error in the result.

It is a point of this nature which makes the calculations of **applied optics** so extremely laborious.

In this case as well as in most cases where the function under consideration has several factors or fractional indices, we can reduce the calculation even more by taking before differentiation logarithms on both sides. In the present example, we have

$$\log T = \log 2\pi - \frac{1}{2}\log g + \frac{1}{2}\log l;$$

differentiating, we find

$$\frac{dT}{dl}\Big/ T = \frac{1}{2l}.$$

Replacement of $dT/dl$ by $\Delta T/\Delta\lambda$ yields

$$\frac{\Delta T}{T} = \frac{\Delta l}{2l},$$

in agreement with the preceding result.

**7.2.2 Evaluation of $\pi$ :** Gregory's series, also called Leibnitz' series,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \ldots,$$

which we have obtained in 6.1.2 using the series for artan, is not suitable for the calculation of $\pi$ due to the slowness of its convergence. However, we may calculate $\pi$ with comparative ease by the artifice: From the addition theorem for the tangent

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta},$$

changing to the inverse functions $\alpha$ = artan $u$, $\beta$ = artan $v$, we obtain

$$\operatorname{arc\,tan} u + \operatorname{arc\,tan} v = \operatorname{arc\,tan}\left(\frac{u+v}{1-uv}\right).$$

If we now choose $u$ and $v$ in such a manner that $(u + v)/(1 - uv) = 1$, we obtain the value $\pi/4$ on the right hand side and, if $u$ and $v$ are small numbers, we can easily compute the left hand side by means of known series.

For example, if we set $u = 1/2$, $v = 1/3$, as Euler did, we obtain

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}.$$

Moreover, if we note that

$$\left(\frac{1}{3}+\frac{1}{7}\right) \div \left(1 - \frac{1}{21}\right) = \frac{1}{2},$$

we have

$$\arctan\frac{1}{2} = \arctan\frac{1}{3} + \arctan\frac{1}{7},$$

so that

$$\frac{\pi}{4} = 2\arctan\frac{1}{3} + \arctan\frac{1}{7}.$$

Using this formula, Georg Vega (1756 - 1802) calculated the number $\pi$ to 140 places.

Moreover, using the equation

$$\left(\frac{1}{5}+\frac{1}{8}\right) \div \left(1 - \frac{1}{40}\right) = \frac{1}{3},$$

we obtain

$$\arctan\frac{1}{3} = \arctan\frac{1}{5} + \arctan\frac{1}{8}$$

or

$$\frac{\pi}{4} = 2 \operatorname{arc} \tan \frac{1}{5} + \operatorname{arc} \tan \frac{1}{7} + 2 \operatorname{arc} \tan \frac{1}{8}.$$

This expansion is extremely useful for the computation of $\pi$ by means of the series

$$\operatorname{artan} x \quad = x - \frac{x^3}{3} + \frac{x^5}{5} - + \ldots;$$

in fact, if we substitute for $x$ the values 1/5, 1/7 or 1/8, we obtain with just a few terms a high degree of accuracy, since the terms diminish rapidly. However, we can perform the calculation even more conveniently by basing it on the formula

$$\frac{\pi}{4} = \operatorname{arc} \tan \frac{120}{119} - \operatorname{arc} \tan \frac{1}{239} = 4 \operatorname{arc} \tan \frac{1}{5} - \operatorname{arc} \tan \frac{1}{239},$$

obtained by similar arguments as those above.

**7.2.3 Calculation of Logarithms:** For the computation of logarithms, we transform the series

$$\frac{1}{2} \log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots (|x| < 1),$$

where $0 < x < 1$, by the substitution

$$\frac{1+x}{1-x} = \frac{p^2}{p^2-1}, \quad x = \frac{1}{2p^2-1}$$

into the series

$$\log p = \frac{1}{2} \log (p-1) + \frac{1}{2} \log (p+1) + \frac{1}{2p^2-1} + \frac{1}{3(2p^2-1)^3} + \ldots,$$

where $2p^2 - 1 > 1$, that is, $p^2 > 1$. If $p$ is an integer and $p + 1$ can be resolved into smaller integral factors, this last series expresses the logarithm of $p$ in terms of the logarithms of smaller integers plus a series, the terms of which

diminish very rapidly and the sum of which can therefore be calculated accurately enough from only a few terms. Hence this series allows us to compute the logarithms of any prime number and hence that of any number, provided we have already calculated the value of log 2.

The accuracy of this determination of log $p$ can be estimated more easily by means of the geometric series than from the general remainder formula. In fact, we have for the remainder $R_n$ of the series, i.e., the sum of all the terms following the term $\frac{1}{n(2p^2-1)^n}$,

$$R_n < \frac{1}{(n+2)(2p^2-1)^{n+2}}\left(1+\frac{1}{(2p^2-1)^2}+\frac{1}{(2p^2-1)^4}+\dots\right)$$
$$= \frac{1}{(n+2)(2p^2-1)^n}\cdot\frac{1}{(2p^2-1)^2-1},$$

and this formula yields immediately the required error estimate.

For example, let us compute $\log_e 7$, using the first four terms of the series. We have

$$p = 7,\quad 2p^2 - 1 = 97,$$
$$\log 7 = 2\log 2 + \frac{1}{2}\log 3 + \frac{1}{97} + \frac{1}{3.97^3} + \dots,$$
$$\frac{1}{97} \approx 0{\cdot}01030928,\quad \frac{1}{3.97^3} \approx 0{\cdot}00000037,$$
$$2\log 2 \approx 1{\cdot}38629436,\quad \frac{1}{2}\log 3 \approx 0{\cdot}54930614,$$

whence

$$\log_e 7 \approx 1.945{,}910{,}15.$$

Estimation of the error yields

$$R_n < \frac{1}{5.97^5} \times \frac{1}{97^2 - 1} < \frac{1}{36 \times 10^9}.$$

However, we must note that each of the four numbers which we have added is only given to within an error of 5 $\times$ $10^{-9}$, so that the last place in the above value of $\log_e 7$ might be wrong by 2. However, as a matter of fact, the last place is also correct.

**Exercises 7.2:**

1. In order to measure the height of a hill, a tower 100 m. high on top of it is observed from the plain. The angle of elevation of the base of the tower is 42° and the tower itself subtends an angle of 6°. What are the limits of error in the determination of the height if the angle 42° is subject to an error of 1°?

2. Compute $\log_e 2$ to three decimal places by means of an expansion in series.

3. Compute $\log_e 5$ to six decimal places, using the values of $\log_e 2$ and $\log_e 3$ given in the text.

4. Compute $\pi$ to five decimal places, using any one of the formulae in 7.2.2.

Answers and Hints

## 7.3 Numerical Solution of Equations

In conclusion, we shall add some remarks about the numerical solution of the equation $f(x) = 0$, where $f(x)$ need not necessarily be a polynomial and we are, of course, only concerned with the determination of real roots. Every such numerical method is based on the scheme of starting with some known approximation $x_0$ of one of the roots and then improving this approximation, whence this first approximation for the desired root of the equation is found; the quality of the approximation is not a concern. We may perhaps take a rough guess as a first approximation or, better still, obtain it from the graph of the function $y = f(x)$, the intersection of which with the $x$-axis yields the required root (of course with an error depending on the scale and the accuracy of the drawing).

**7.3.1 Newton's Method:** The following procedure, due to Newton, is based on the fundamental principle of the differential calculus - the replacement of a curve by a straight line, the tangent - in the immediate neighbourhood of the point of contact. If we have an approximate value $x_0$ for a root of the equation $f(x) = 0$, we consider the point on the graph of the function $y = f(x)$ with the co-ordinates $x = x_0$, $y = f(x_0)$. We wish to find the intersection of the

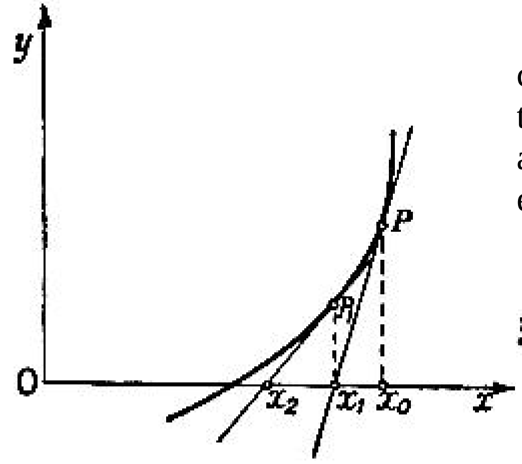



Fig. 4.—Newton's method of approximation

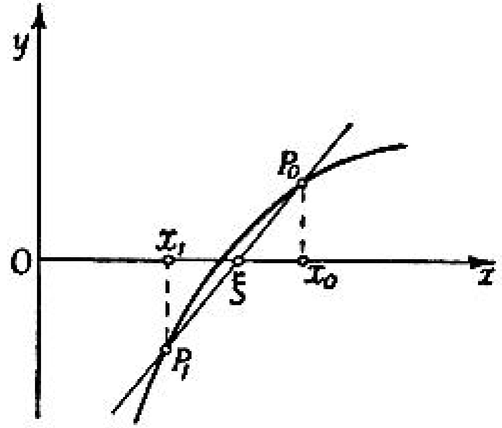



Fig. 6.—The rule of false position

curve with the $x$-axis; we find an approximation at the point $x_0$, $y_0=f(x_0)$, where the tangent intersects the $x$-axis. The abscissa $x_1$ of this intersection of the tangent with the $x$-axis then represents a new, possibly better approximation to the required root of the equation than $x_0$ .

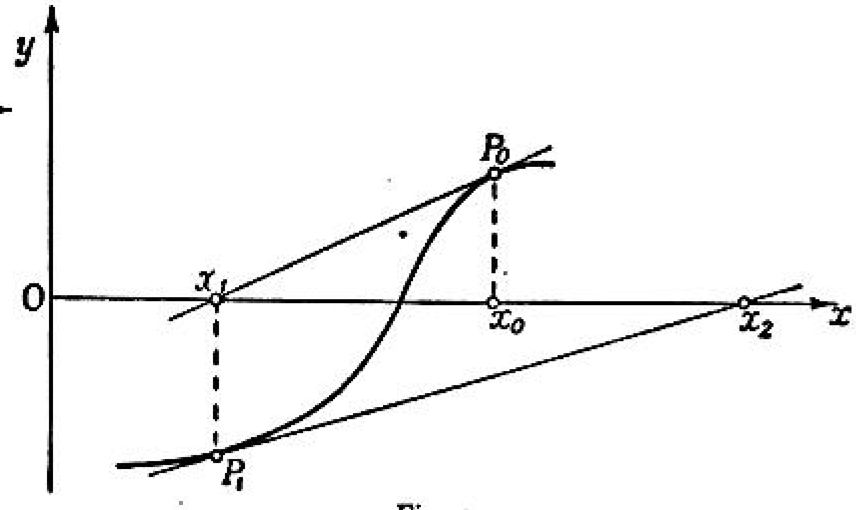



Fig. 5

By virtue of the geometrical meaning of the derivative, Fig. 4 yields at once

$$\frac{f(x_0)}{x_0 - x_1} = f'(x_0),$$

whence we obtain the formula for the calculation of the new value $x_1$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

If this procedure yields a better approximation than $x_0$, we repeat the process and find $x_2$, etc.; if the curve has the form shown in Fig. 4, these approximations will approach more and more closely to the required solution.

The usefulness of this process depends essentially on the nature of the curve $y=f(x)$. In Fig. 4, we see that the successive estimates converge with greater and greater accuracy to the required root. This is due to the fact that the curve has its convex side turned towards the $x$-axis. However, in Fig. 6, we see that if we choose the original value $x_0$ badly, our construction does not lead at all to the required root. We see from this that while using Newton's method we must examine each individual case in order to determine the degree of accuracy with which we have really solved the equation. We shall return to this topic later on.

**7.3.2 The Rule of False Position:** Newton's method, in which the tangent to the curve has a decisive role, is only the limiting case of an older method, known as the rule of false position, in which the secant takes the place of the tangent. Let us assume that we know two points $(x_0,y_0)$ and $(x_1,y_1)$ in the neighbourhood of the required intersection with the $x$-axis. If we replace the curve by the secant joining these two points, the intersection of this secant with

the $x$-axis will under certain circumstances be an improved approximation to the required root of the equation. If the abscissa of this point is denoted by $\xi$, we have (Fig. 6) the equation

$$\frac{\xi - x_0}{f(x_0)} = \frac{\xi - x_1}{f(x_1)},$$

from which we compute $\xi$:

$$\begin{aligned} \xi &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\ &= \frac{x_0 f(x_1) - x_0 f(x_0) + x_0 f(x_0) - x_1 f(x_0)}{f(x_1) - f(x_0)} \end{aligned}$$

or

$$\xi = x_0 - \frac{f(x_0)}{\{f(x_1) - f(x_0)\}/(x_1 - x_0)}.$$

This formula, which determines the next approximation $\xi$ from $x_0$ and $x_1$, is called the rule of false position. We can employ it with advantage if one value of the function is positive and the other negative, say, as in Fig. 6, where $y_0 > 0$ and $y_1 < 0$. Repetition of this process will always lead us to the required result, if we use at each step a positive and a negative value of the function, between which the required root must necessarily lie.

The above formula of Newton results from the rule of false position as a limiting case, if we let $x_1$ tend to $x_0$. In fact, the denominator of the second term on the right hand side of the statement of the rule of false position tends to $f'(x_0)$ as $x_1$ tends to $x_0$.

**7.3.3 The Method of Iteration:** Another way of approximating roots of an equation $f(x) = 0$ is the **method of iteration**. i.e., set $\phi(x) = f(x) + x$ and write it in the form $x = \phi(x)$. We then assume that $\xi$ is the true value of a solution of our equation and $x_0$ a first approximation. We obtain a second approximation $x_1$ by setting $x_1 = \phi(x_0)$, a third approximation $x_2$ by setting $x_2 = \phi(x_1)$, etc. In order to investigate the convergence of these approximations, we apply the mean value theorem; recalling that $\xi = \phi(\xi)$, we have

$$\xi - x_1 = \phi(\xi) - \phi(x_0) = (\xi - x_0)\phi'(\bar{\xi}),$$

where $\bar{\xi}$ lies between $\xi$ and $x_0$. This shows that, if for

$$|\xi - x| < |\xi - x_0|$$

the absolute value of the derivative $f'(x)$ is less than $k < 1$, then the successive approximations converge, because

$$|\xi - x_1| < k|\xi - x_0|, \quad |\xi - x_2| < k^2|\xi - x_0|, \ldots,$$
$$|\xi - x_n| < k^n|\xi - x_0|,$$

whence the errors tend to zero, The smaller is the absolute value of the derivative $\phi'(x)$ in the neighbourhood of $\xi$, the more rapid is the convergence.

If $\phi'(x) > 1$ in the neighbourhood of $\xi$, the approximations do not any longer tend to $\xi$. We can then use the inverse function or else the following device: We choose a first approximation $x_0$, calculate $A = f'(x_0)$ and write

$$\phi(x) = -\frac{1}{A}f(x) + x.$$

Then the equation $f(x)=0$ can be written in the form $x=\phi(x)$, and now $\phi'(x)=-f'(x)/A+1$, which has the value 0 at $x = x_0$, whence its absolute value will usually be less than a constant $k < 1$, provided $|\xi - x| < |\xi - x_0|$.

Returning to Newton's method, we can now investigate its suitability for application at any given point. The equation $f(x) = 0$ is equivalent to

$$x = \phi(x) = x - \frac{f(x)}{f'(x)},$$

provided that $f'(x) \neq 0$. Applying the method of iteration to this last equation, we obtain from a first approximation $x_0$ a second approximation

$$x_1 = x_0 - f(x_0)/f'(x_0);$$

in other words, the same second approximation as Newton's method gives when it is applied to the equation $f(x) = 0$. We thus see that the smaller is the value of

$$\phi'(x) = \frac{f(x)f''(x)}{(f'(x))^2},$$

the more rapidly converge the successive approximations. In words, Newton's formula converges rapidly for large values of $f'(x_0)$ and small values of $f(x_0)$ and curvature, as intuition would lead us to suspect.

We can also obtain an estimate of the accuracy of Newton's method, if we recall that, since $f(\xi) = 0$, the derivative $\phi'(\xi) = 0$. Applying Taylor's theorem, we have

$$\xi - x_1 = \phi(\xi) - \phi(x_0) = \frac{(\xi - x_0)^2}{2}\phi''(\bar{\xi}),$$

where $\bar{\xi}$ lies between $\xi$ and $x_0$. Thus, if the error of the original estimate is small, the method converges much more rapidly than the method of iteration applied directly to $f(x) = 0$.

For example, if

$$\phi''(x) = \frac{\{f'(x)\}^2 f''(x) + f'(x)f(x)f'''(x) - 2f(x)\{f''(x)\}^2}{\{f'(x)\}^3} \quad (a)$$

is everywhere less than 10, then a first approximation, which has an error by less than 0.001, will yield a second approximation with an error of less than $(0{\cdot}001)^2 \times 10 \div 2 = 0.000,005$.

## 7.3.4 Examples:

Consider as an example the equation

$$f(x) = x^3 - 2x - 5 = 0.$$

For $x_0 = 2$, we have $f(x_0) = -1$, while, for $x_1 = 2.1$, we have $f(x_1) = 0.061$. By Newton's method,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2{\cdot}1 - \frac{0{\cdot}061}{3(2{\cdot}1)^2 - 2} = 2{\cdot}1 - {\cdot}005431 = 2{\cdot}094569.$$

In order to estimate the error, we find from the expression (*a*) that the value of $\phi''(x)$ is about 1 and certainly less than 2 near $x = 2$. Moreover, the error of our first approximation is certainly less than 1/160, because the secant joining the points $x = 2$, $y = -1$ and $x = 2.1$, $y = -0.61$ cuts the $x$-axis at a distance of less than 1/160 from $x = 2.1$ and the curve, lying under the secant, cuts it even closer to 2.1. Thus, the error of our second approximation is less than

$$\frac{1}{2}\cdot\frac{2}{(160)^2} = \frac{1}{25{,}600} < {\cdot}00004.$$

Another way of estimating the error, without reference to the secant, is as follows: If we estimate the error to be less than 1/20, the error of our second approximation is less than $1/20^2 = 0.0025$. Hence the root differs from 2.1 by less than (2.1 - 2.0945) + - 0.0025 = 0.008. Hence the error was not merely less than 1/20, but less than 0.008, so that the error in $x_2$ is less than $(0.008)^2=0.000{,}064$.

If accuracy is insufficient, we can repeat the process, calculating $f(x_2)$ and $f'(x_2)$ for $x_2 = 2.094{,}569$ and obtain a third approximation $x_3$ with an error less than $1/(25{,}600)^2 < 0.000{,}000{,}002$.

As a second example, let us solve the equation

$$f(x) = x\log_{10}x - 2 = 0.$$

We have $f(3) = -0.6$ and $f(4) = +0.4$, whence we use $x_0 = 3.5$ as a first approximation. Then, using ten-figure logarithmic tables, we obtain the successive approximations

$$\begin{aligned} x_0 &= 3{\cdot}5 \\ x_1 &= 3{\cdot}598 \\ x_2 &= 3{\cdot}5972849 \\ x_3 &= 3{\cdot}5972850235. \end{aligned}$$

**Exercises 7.3:**

1. Using Newton's method, find the positive root of $x^6 + 6x - 8 = 0$ to four decimal places.

2. Find to four decimal places the root of $x = \tan x$ between $\pi$ and $2\pi$. Prove that the result is accurate to four places.

3. Using Newton's method, find the value of $x$ for which

$$\int_0^x \frac{u^2}{1+u^2}\,du = \frac{1}{2}.$$

4. Find the roots of the equation $x = 2 \sin x$ to two decimal places.

5. Determine the positive roots of the equation $x^5 - x - 0\text{-}2 = 0$ by the method of iteration.

6. Determine the least positive root of $x^4 - 3x^3 - 10x - 10 = 0$ by the method of iteration.

7. Find the roots of $x^3 - 7x^2 + 6x + 20 = 0$ to four decimal places.

Answers and Hints

# Appendix to Chapter VII

## Stirling's Formula

In very many applications, especially in statistics and in the theory of probability, we find it necessary to have a simple approximation to $n!$ as an elementary function of $n$. Such an expression is given by the following theorem which bears the name of its discoverer **James Stirling** 1692 - 1770:

As $n \to \infty$

$$\frac{n!}{\sqrt{2\pi}\, n^{n+1/2} e^{-n}} \to 1;$$

in more detail

$$\sqrt{2\pi}\, n^{n+1/2} e^{-n} < n! < \sqrt{2\pi}\, n^{n+1/2}\, e^{-n}\left(1 + \frac{1}{4n}\right).$$

In other words, the expressions $n!$ and $\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$ differ only by a small percentage for large $n$ - as we say, the two expressions are **asymptotically equal**—and at the same time the factor $1 + 1/4n$ gives us an estimate of the degree of accuracy of the approximation.

We are led to this remarkable formula, if we attempt to evaluate the area under the curve $y = \log x$. By integration(4.4.1), we find that $A_n$, the exact area under this curve between the ordinates $x = 1$ and $x = n$, is given by

$$\int_1^n \log x\, dx = x \log x - x \Big|_1^n = n \log n - n + 1.$$

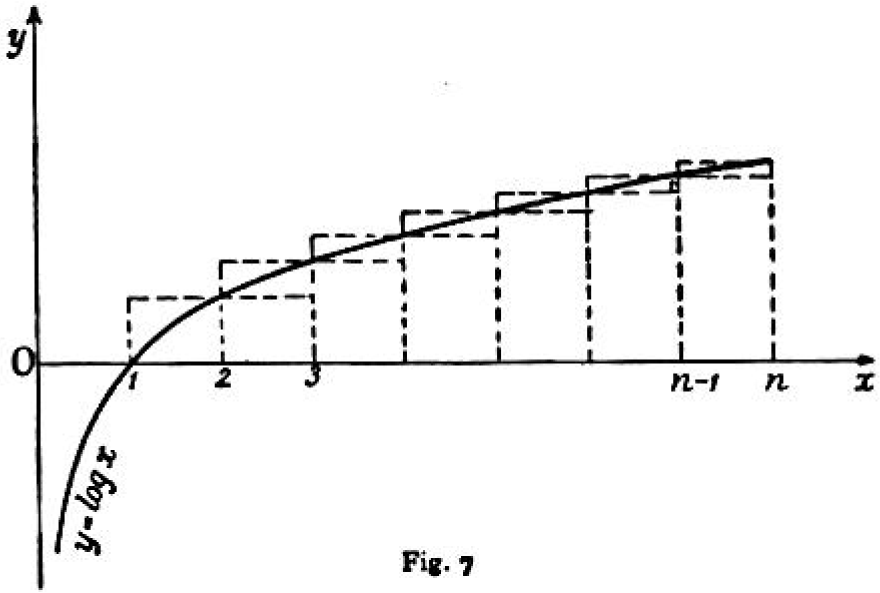


Fig. 7

However, if we estimate the area by the trapezoidal rule, erecting ordinates at $x=1$, $x=2$, $\cdots$ , $x = n$, as in Fig. 7, we obtain an approximate value $T_n$ for the area:

$$\begin{aligned} T_n &= \log 2 + \log 3 + \ldots + \log(n-1) + \frac{1}{2} \log n \\ &= \log n! - \frac{1}{2} \log n. \end{aligned}$$

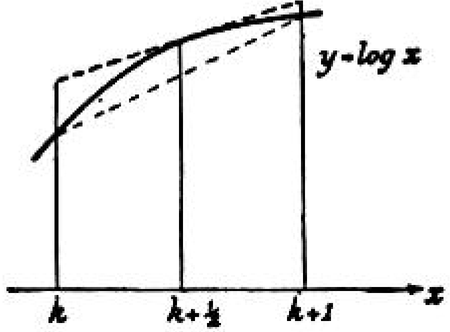


Fig. 8

If we make the reasonable assumption that $A_n$ and $T_n$ are of the same order of magnitude, we find at once that $n!$ and $n^{n+½}e^{-n}$ are of the same order of magnitude, which is essentially what Stirling's formula states.

In order to make this argument precise, we first show that $a_n = A_n - T_n$ is bounded, whence it follows immediately that $T_n = A_n\left(1 - \frac{a_n}{A_n}\right)$ is of the same order of magnitude as $A_n$.

The difference $a_{k+1} - a_k$ is the difference between the area under the curve and the area under the secant in the strip $k \le x \le k+1$. Since the curve is concave downward and lies above the secant, $a_{k+1} - a_k$ is positive, and

$$(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \dots + (a_2 - a_1) + a_1$$

is monotonic increasing. Moreover, the difference $a_{k+1} - a_k$ is clearly less than the difference between the area under the tangent at $x = k + 1/2$ and the area under the secant (Fig. 8), whence we have the inequality

$$\begin{aligned} a_{k+1} - a_k &< \log\left(k + \frac{1}{2}\right) - \frac{1}{2}\log k - \frac{1}{2}\log(k+1) \\ &= \frac{1}{2}\log\left(1 + \frac{1}{2k}\right) - \frac{1}{2}\log\left(1 + \frac{1}{2\left(k + \frac{1}{2}\right)}\right) \\ &< \frac{1}{2}\log\left(1 + \frac{1}{2k}\right) - \frac{1}{2}\log\left(1 + \frac{1}{2(k+1)}\right). \end{aligned}$$

If we add these inequalities for $k = 1, 2, \cdots, n - 1$, all the terms on the right hand side except two of them will cancel out and (since $a_1 = 0$) we have

$$a_n < \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\log\left(1 + \frac{1}{2n}\right) < \frac{1}{2}\log\frac{3}{2}.$$

Hence $a_n$ is bounded; since it is monotonic increasing, it tends to a limit $a$ as $n \to \infty$. Our inequality for $a_{k+1} - a_k$ now yields

$$a - a_n = \sum_{k=n}^{\infty} (a_{k+1} - a_k) < \frac{1}{2}\log\left(1+\frac{1}{2n}\right).$$

Since, by definition, $A_n - T_n = a_n$, we now have

$$\log n! = 1 - a_n + \left(n+\frac{1}{2}\right)\log n - n,$$

or, writing $a_n = e^{1-a}$,

$$n! = \alpha_n n^{n+\frac{1}{2}} e^{-n}.$$

The sequence $a_n$ is monotonic decreasing and tends to the limit $a = e^{1-a}$, whence

$$1 < \frac{\alpha_n}{\alpha} = e^{a-a_n} < e^{\frac{1}{2}\log(1+1/2n)} = \sqrt{\left(1+\frac{1}{2n}\right)} < 1 + \frac{1}{4n}.$$

Hence we have

$$\alpha n^{n+\frac{1}{2}} e^{-n} < n! < \alpha n^{n+\frac{1}{2}} e^{-n}\left(1+\frac{1}{4n}\right).$$

There only remains for us to find the actual value of the limit $a$. Here we employ the formula proved in 4.4.4:

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}}.$$

Replacing $n!$ by $a_n n^{n+½} e^{-n}$ and $(2n)!$ by $a_{2n} 2^{2n+½} n^{2n+½} e^{-2n}$, we immediately obtain

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{{\alpha_n}^2}{\alpha_{2n}\sqrt{2}} = \frac{\alpha^2}{\alpha\sqrt{2}},$$

whence $a = \sqrt{2\pi}$. The proof of Stirling's formula is thus complete.

In addition to its theoretical interest, Stirling's formula is a very useful tool for the numerical calculation of $n$! when $n$ is large. Instead of forming the product of a large number of integers, we have merely to calculate Stirling's expression using logarithms, which involves far fewer operations. Thus, we obtain for $n$=10 the value 3,598,696 from Stirling's expression (using seven-figure tables), while the exact value is 3628800. The percentage error is barely 5/6 %.

**Exercise:** Prove that $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$.

**No Answer and no Hint**

## Chapter VIII.

# Infinite Series and Other Limiting Processes

### Preliminary Remarks

The geometric series, Taylor series and several special examples, which we have already met, suggest that we may well study from a rather more general point of view those limiting processes under the heading of **summation of infinite series** . By its nature, any limiting value

$$S = \lim_{n\to\infty} s_n$$

can be written as an infinite series, because, if $n$ takes the values 1, 2, 3, ···, we need only set $a_n = s_n - s_{n-1}$ for $n > 1$ and $a_1 = s_1$ , in order to obtain

$$s_n = a_1 + a_2 + \ldots + a_n,$$

and the value $S$ thus appears as the limit of $s_n$ - the sum of $n$ terms as $n$ increases. We express this fact by saying that S is the **sum of the infinite series**

$$a_1 + a_2 + a_3 + \cdots.$$

Thus, an infinite series is simply a way of representing a limit, where each successive approximation is found from the preceding one by addition of one more term. In principle, the expression of a number as a decimal is merely the representation of a number $a$ in the form of an infinite series $a = a_1+a_2+a_3+ \cdots$ where, if $0 \le a \le 1$, the term $a_n$ is put equal to $a_n \times 10^{-n}$ and $a_n$ is a whole number between 0 and 9, inclusively. Since every limiting value can be written in the form of an infinite series, it may seem that a special study of series is superfluous. However, in many cases, it happens that limiting values occur naturally in the form of infinite series, which often exhibit particularly simple laws of formation. Of course, it is not true that every series has an easily recognizable law of formation. For example, the number $\pi$ can certainly be represented as a decimal, yet we know no simple law which enables us to state the value of an arbitrary digit, say its 7000-th decimal. However, if we set aside the representation of $\pi$ by a decimal and consider instead Gregory's series, we have an expression with a perfectly clear general law of formation.

Infinite series, in which approximations to the limit are found by repeated addition of new terms, are analogous to **infinite products**, in which the approximations to the limit arise from repeated multiplication by new factors. However, we shall not go deeply into the theory of infinite products; the principal subject of this chapter and of Chapter IX will be infinite series.

## 8.1 The Concepts of Convergence and Divergence

**8.1.1 The Fundamental Ideas:** We consider an infinite series the general term of which we denote * by $a_n$; the series then has the form

$$a_1 + a_2 + \ldots = \sum_{\nu=1}^{\infty} a_\nu.$$

The symbol with the summation sign on the right hand side is merely an abbreviated way of writing the expression on the left hand side.

* For formal reasons, we include the possibility that certain of the numbers $a_n$ may be zero. If **all** $a_n$ vanish onwards from a number $N$ (i.e., when $n > N$), we speak of a **terminating series**.

If, as $n$ increases, the ***n*-th partial sum**

$$s_n = a_1 + a_2 + \dots + a_n = \sum_{\nu=1}^{n} a_\nu$$

approaches a limit

$$S = \lim_{n \to \infty} s_n$$

we say that the series converges, and otherwise that it diverges. In the first case, we call $S$ the **sum of the series**. We have already met with many examples of convergent series; for instance, the geometric series $1 + q + q^2 + \dots$, which converges to the sum $1/(1-q)$ for $q < 1$, Gregory's series, the series for log 2, the series for $e$, and others converge. In the language of infinite series, Cauchy's convergence test is expressed as follows:

A necessary and sufficient condition for the convergence of a series is that the number

$$|s_m - s_n| = |a_{n+1} + a_{n+2} + \dots + a_m|$$

becomes arbitrarily small if $m$ and $n$ are chosen sufficiently large ($m > n$).

In other words: A series converges if and only if the following condition is met: Given a positive number $\varepsilon$, however small, it is possible to choose an index $N=N(\varepsilon)$, which, in general, increases beyond all bounds as $\varepsilon \to 0$, in such a way that the $|s_m - s_n|$ is less than $\varepsilon$, provided only that $m > N$ and $n > N$.

We can make the meaning of the convergence test clearer by considering the geometric series with $q = 1/2$. If we choose $\varepsilon = 1/10$, we need only take $N = 4$, because

$$\begin{aligned} |s_m - s_n| &= \frac{1}{2^n} + \dots + \frac{1}{2^{m-1}} \\ &= \frac{1}{2^{n-1}}\left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{m-n}}\right) < \frac{1}{2^{n-1}} \end{aligned}$$

and

$$\frac{1}{2^{n-1}} < \frac{1}{10}, \text{ if } n > 4.$$

If we choose $\varepsilon$ equal to 1/100, it is sufficient to take 7 as the corresponding value of $N$, as is easily verified.

Obviously, it is a **necessary condition for the convergence of a series** that

$$\lim_{n \to \infty} a_n = 0.$$

Otherwise, undoubtedly, the convergence criterion cannot be fulfilled. However, this necessary condition is by no means sufficient for convergence; on the contrary, it is easy to find infinite series the general term $a_n$ of which approaches 0 as $n$ increases, yet their sum does not exist, because the partial sum $s_n$ increases without limit as $n$ increases.

An example of this is the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots .$$

with the general term $\frac{1}{\sqrt{n}}$. We immediately see that

$$s_n > \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}.$$

The $n$-th partial sum increases beyond all bounds as $n$ increases, whence the the series diverges

The same is true for the classical example of the **harmonic series**

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots ,$$

for which

$$a_{n+1} + \dots + a_{2n} = \frac{1}{n+1} + \dots + \frac{1}{2n} > \frac{1}{2n} + \dots + \frac{1}{2n} = \frac{1}{2}.$$

Since $n$ and $m = 2n$ can be taken as large as we please, the series diverges, because Cauchy's test is not fulfilled; in fact, the $n$-th partial sum tends obviously to infinity, since all the terms are positive. On the other hand, the series, formed from the same numbers with alternating signs,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - + \dots + \frac{(-1)^{n-1}}{n} + \dots$$

converges and has the sum log 2.

It is by no means true that in every divergent series $s_n$ tends to $+\infty$ or $-\infty$. Thus, in the case of the series

$$1 - 1 + 1 - 1 + 1 + - \dots,$$

we see that the partial sum $s_n$ has alternately the values 1 and 0, and, on account of this backwards / forwards oscillation neither approaches a definite limit nor increases numerically beyond all bounds.

As the convergence and divergence of an infinite series is concerned, the following fact which, though self-evident, is very important should be noted: The convergence or divergence of a series is not changed by insertion or removal of finite numbers of terms. As far as convergence or divergence is concerned, it does not matter in the least whether we begin the series at the term $a_0$, or $a_1$, or $a_5$, or at any other arbitrarily chosen term.

**8.1.2 Absolute Convergence and Conditional Convergence:**. The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots$ diverges; but if we change the sign of every second term, the resulting series converges. On the other hand, the geometric series $1-q+q^2-q^3\cdots$ converges and has the sum $1/(1 + q)$, provided that $0 \leq q < 1$; on converting all the signs to +, we obtain the series

$$1 + q + q^2 + q^3 + \dots,$$

which also converges and has the sum $1/(1 - q)$.

There appears here a distinction which we must examine more closely. With a series all terms of which are positive, there are only two possible cases; either it converges or the partial sum increases beyond all bounds with *n*. In fact, the partial sums, being a monotonic increasing sequence, must converge, if they remain bounded. Convergence occurs, if the terms approach zero sufficiently rapidly as *n* increases; on the other hand, divergence occurs, if the terms do not approach zero at all or if they approach zero too slowly. However, in series, where some terms are positive and others negative, the changes of sign may bring about convergence, since a too great increase in the partial sums, due to the positive terms, is compensated by negative terms, as a final result of which a definite limit is approached.

In order to grasp this fact better with a series $\sum_{\nu=1}^{\infty} a_\nu$, which has positive and negative terms, we compare it with the series which has the same terms all with positive signs, that is,

$$|a_1| + |a_2| + \ldots = \sum_{\nu=1}^{\infty} |a_\nu|.$$

If this series converges, then for sufficiently large values of *n* and *m* > *n*, the expression

$$|a_{n+1}| + |a_{n+2}| + \ldots + |a_m|$$

will certainly be as small as we please; on account of the relation

$$|a_{n+1} + \ldots + a_m| \leq |a_{n+1}| + \ldots + |a_m|$$

the expression on the left hand side is also arbitrarily small, whence the original series $\sum_{\nu=1}^{\infty} a_\nu$ converges. In this case, the original series is said to be **absolutely convergent**. Its convergence is due to the numerical smallness of its terms and does not depend on the change of the signs.

On the other hand, if the series with all the terms taken positively diverges and the original series still converges, we say that the original series is **conditionally convergent**. Conditional convergence results from the terms of opposite signs compensating one another.

For conditional convergence, **Leibnitz's convergence test** is frequently of use: If the terms of a series have alternating sign and, moreover, their absolute values $|a_n|$ tend monotonically to 0 (so that $|a_{n+1}| < |a_n|$), the series $\sum_{\nu=1}^{\infty} a_\nu$ converges. (Example: Gregory's series).

In the proof, we assume that $a_1 > 0$, which does not essentially limit the generality of the argument, and write our series in the form

$$b_1 - b_2 + b_3 - + \ldots,$$

where all the terms $b_n$ are now positive, $b_n$ tends to 0 and the condition $b_{n+1} < b_n$ is satisfied. If we bracket the terms together in the two ways

$$b_1 - (b_2 - b_3) - (b_4 - b_5) - \ldots \text{ and } (b_1 - b_2) + (b_3 - b_4) + (b_5 - b_6) + \ldots$$

we see at once that the partial sums satisfy the two relations:

$$s_1 > s_3 > s_5 > \ldots > s_{2m+1} > \ldots$$
$$s_2 < s_4 < s_6 < \ldots < s_{2m} < \ldots .$$

On the other hand, $s_{2n} < s_{2n+1} < s_1$ and $s_{2n+1} > s_{2n} > s_2$, whence the odd partial sums $s_1, s_3, \cdots$ form a monotonic decreasing sequence, which in no case drops below the value $s_2$ and this sequence has a limit $L$ (c.f. A1.1.4). The even partial sums $s_2, s_4, \cdots$ likewise form a monotonic increasing sequence the terms of which in no case exceed the fixed number $s_1$, whence this sequence must have a limiting value $L'$. Since the numbers $s_{2n}$ and $s_{2n+1}$ differ only by the number $b_{2n+1}$, which approaches 0 as $n$ increases, the limiting values $L$ and $L'$ are equal to each other. In other words, the even and the odd partial sums

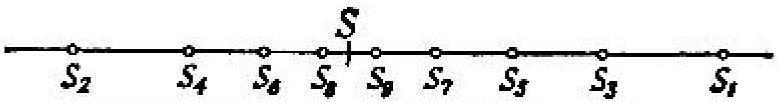



Fig. 1.—Convergence of an alternating series

approach the same limit, which we will now denote by $S$ (Fig. 1). However, this implies that our series is convergent, as has been asserted; its sum is $S$.

In conclusion, we make another general remark about the fundamental **difference** between absolute convergence and conditional convergence. We consider a convergent series $\sum_{\nu=1}^{\infty} a_\nu$ and denote the positive terms of the series by $p_1, p_2, p_3, \cdots$ and the negative terms by $-q_1, -q_2, -q_3, \cdots$. If we form the $n$-th partial sum $s_n = \sum_{\nu=1}^{n} a_\nu$ of the given series, a certain nmnber, say, $n'$ positive terms and a certain number, say $n''$ negative terms must appear, where $n'+n''=n$. Moreover, if the number of positive terms as well as that of negative terms in the series is infinite, then the two numbers $n'$ and $n''$ will increase beyond all bounds with $n$. We see immediately that the partial sum $s_n$ is simply equal to the partial sum $\sum_{\nu=1}^{n'} p_\nu$ of the positive terms of the series plus the partial sum $-\sum_{\nu=1}^{n''} q_\nu$ of the negative terms. If the given series converges absolutely, then both the series of positive terms $\sum_{\nu=1}^{\infty} p_\nu$ and the series of the absolute values of the negative terms $\sum_{\nu=1}^{\infty} q_\nu$ certainly converge. In fact, as $m$ increases, the partial sums $\sum_{\nu=1}^{m} p_\nu$ and $\sum_{\nu=1}^{m} q_\nu$ are monotonic non-decreasing with the upper bound $\sum_{\nu=1}^{\infty} |a_\nu|$.

The sum of an absolutely convergent series is then simply equal to the sum of the series consisting only of the positive terms plus the sum of the series consisting of the negative terms, or, in other words, it is equal to the difference of the two series with positive terms.

However, $\sum_{\nu=1}^{n} a_\nu = \sum_{\nu=1}^{n'} p_\nu - \sum_{\nu=1}^{n''} q_\nu$; as $n$ increases, $n'$ and $n''$ must also increase beyond all bounds, and the limit of the left hand side must therefore be equal to the difference of the two sums on the right hand side. If the series contains only a finite number of terms of one particular sign, the facts are correspondingly simplified. On the other hand, if the series does not converge absolutely, but does converge conditionally, then both the series $\sum_{\nu=1}^{\infty} p_\nu$ and $\sum_{\nu=1}^{\infty} q_\nu$ must be divergent, because, if both were convergent, the series would converge absolutely, contrary to our

hypothesis. If only one diverges, say $\sum_{\nu=1}^{\infty} p_\nu$, and the other converges, then separation into positive and negative parts, $s_n = \sum_{\nu=1}^{n'} p_\nu - \sum_{\nu=1}^{n''} q_\nu$, shows that the series could not converge; in fact, as $n$ increases, $n'$ and $\sum_{\nu=1}^{n'} p_\nu$ would increase beyond all bounds, while the term $\sum_{\nu=1}^{n''} q_\nu$ would approach a definite limit, so that the partial sum $s_n$ would increase beyond all bounds.

Hence, we see that a conditionally convergent series cannot be thought of as the difference of two convergent series, one consisting of its positive terms and the other consisting of the absolute values of its negative terms.

Closely connected with this fact, there arises another difference between absolutely and conditionally convergent series which we shall now mention briefly.

**8.1.3 Rearrangement of Terms:** It is a property of finite sums that we may change the order of the terms or, as we say, rearrange the terms at will without changing their values. There arises the question: What is the exact meaning of a change of the order of terms in an infinite series and does such a rearrangement leave the value of the sum unchanged? While in the case of finite sums there arises no difficulty, for example, in adding the terms in reverse order, in the case of infinite series such a possibility does not exist; there is no last term with which to begin. Now a change of the order of terms in an infinite series can only mean this: A series $a_1 + a_2 + a_3 + \cdots$ is rearranged into a series $b_1 + b_2 + b_3 + \cdots$, provided that every term $a_n$ of the first series occurs exactly once in the second series and conversely. For example, the amount by which $a_n$ is displaced may increase beyond all bounds as $n$ does; the only point is that it must appear somewhere in the new series. If some of the terms are moved to later positions in the series, other terms must, of course, be moved to earlier positions. For example, the series

$$1 + q + q^2 + q^4 + q^3 + q^8 + q^7 + q^6 + q^5 + q^{16} + \dots$$

is a rearrangement of the geometric series $1 + q + q^2 + \cdots$.

As regards the change of order, there is a fundamental distinction between absolutely convergent and conditionally convergent series. In absolutely convergent series, rearrangement of terms does not affect the convergence and the value of the sum of the series is not changed, exactly as in the case of finite sums.

On the other hand, in conditionally convergent series, the value of the sum of the series can be changed at will by suitable rearrangement of the series, and the series can even be made to diverge, if so desired.

The first of these facts, referring to absolutely convergent series, is easily established. Let us assume, to begin with, that our series has only positive terms and consider the $n$-th partial sum $s_n = \sum_{\nu=1}^{n} a_\nu$. All the terms of this partial sum occur in the $m$-th partial sum $t_m = \sum_{\nu=1}^{m} b_\nu$ of the rearranged series, provided only that $m$ is chosen large enough. Hence $t_m \geq s_n$. On the other hand, we can determine an index $n'$ so large that the partial sum $s_{n'} = \sum_{\nu=1}^{n'} a_\nu$ of the first series contains all the terms $b_1, b_2, \cdots, b_m$. It then follows that $t_m \leq s_{n'} \leq A$, where $A$ is the sum of the first series. Thus, for all sufficiently large values of $m$, we have $s_n \leq t_m \leq A$; since $s_n$ can be made to differ from $A$ by an arbitrarily small amount, it follows that the rearranged series also converges, in fact, to the same limit $A$ as the original series.

If the absolutely convergent series has both positive and negative terms, we may regard it as the difference of two series each of which has only positive terms. Since in the rearrangement of the original series each of these two series merely undergoes rearrangement and therefore converges to the same value as before, the same is true of the original series when rearranged. In fact, by the case just considered, the new series is absolutely convergent and is therefore the difference of the two rearranged series of positive terms.

The fact just proved may seem to the beginner to be a triviality. The fact that it really does require a proof and that in this proof absolute convergence is essential can be demonstrated by an example of the opposite behaviour of conditionally convergent series. Consider the familiar series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + - \cdots = \log 2.$$

Write below it the result of multiplication by the factor 1/2

$$\frac{1}{2} \qquad - \frac{1}{4} \qquad + \frac{1}{6} \qquad - \frac{1}{8} + - \cdots = \frac{1}{2} \log 2,$$

and add the two series, combining the terms in vertical columns. We thus obtain

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}++-\dots=\frac{3}{2}\log 2.$$

This last series can obviously be obtained by rearranging the original series and yet the value of the sum of the series has been multiplied by the factor 3/2. It is easy to imagine the effect of the discovery of this apparent paradox on the mathematicians of the Eighteenth Century, who were accustomed to operate with infinite series without regard to their convergence.

We shall give the proof of the theorem stated above concerning the change in the sum of a conditionally convergent series which arises from a change of order of its terms, although we shall have no occasion to make use of the result.

Let $p_1, p_2, \cdots$ be the positive terms and $-q_1, -q_2, \cdots$ the negative terms of the series. Since the absolute value $|a_n|$ tends to 0 as $n$ increases, the numbers $p_n$ and $q_n$ must also tend to 0 with $n$. Moreover, as we have already seen, the sum $\sum_1^\infty p_\nu$ must diverge and the same is true for $\sum_1^\infty q_\nu$.

This abbreviated notation for $\sum_1^\infty p_\nu$ and analogous expressions for other series will be used often in what follows.

We can now find easily a rearrangement of the original series which has an arbitrary number $a$ as its limit. In order to be specific, let $a$ be positive. We then add the first $n_1$ positive terms, just enough to ensure that the sum $\sum_1^{n_1} p_\nu$ is less than $a$. Since the sum $\sum_1^{n_1} p_\nu$ increases with $n_1$ beyond all bounds, it is always possible, by using enough terms, to make the partial sum larger than $a$. The sum will then differ from the exact value $a$ by at most $p_n$. We now add just enough negative terms $-\sum_1^{m_1} q$ in order to ensure that the sum $\sum_1^{n_1} p_\nu - \sum_1^{m_1} q_\nu$ is less than $a$. This is also possible, as follows from the divergence of the series from the divergence of the series $\sum_1^\infty q_\nu$. The difference between this sum and $a$ is now at most $q_{m_1}$. We now add just enough other positive terms $\sum_{n_1+1}^{n_2} p_\nu$ in order to make the partial sum

again larger than $a$, as is again possible since the series of positive terms diverges. The difference between the partial sum and $a$ is now at most $p_{n_2}$. We again add just just enough negative terms $-\sum_{m_1+1}^{m_2} q_\nu$, which start next after the last one previously used, to make the sum once more less than $a$ and continue in the same way. The values of the sums thus obtained will oscillate about the number $a$ and, when the process is carried out far enough, the oscillation will only take place between arbitrarily narrow bounds; in fact, since the terms $p_\nu$ and $q_\nu$ themselves tend to 0 when $\nu$ is sufficiently large, the length of the interval in which the oscillation occurs will also tend to 0, and the theorem is proved.

In the same manner, we can rearrange s series in such a way as to make it diverge; we merely have to choose such large numbers of positive terms as compared with the negative ones that there occurs no longer compensation.

**8.1.4 Operations with Infinite Series:** It is clear that two convergent infinite series $a_1 + a_2 + \cdots = S$ and $b_1 + b_2 + \cdots = T$ can be added term by term, i.e., that the series formed from the terms $c_n = a_n + b_n$ converges and has the sum $S + T$, because

$$\sum_{\nu=1}^{n} c_\nu = \sum_{\nu=1}^{n} a_\nu + \sum_{\nu=1}^{n} b_\nu \to S + T.$$

This theorem is really nothing more than another statement of the fact (1.6.4) that the limit of the sum of two terms is the sum of their limits.

It is also clear that, if we multiply each term of a convergent infinite series by the same factor, the series remains convergent and its sum is multiplied by the same factor.

In the cases just mentioned, it is immaterial whether the convergence is absolute or conditional. On the other hand, a further study, which is not necessary for us here, shows that, if two infinite series are multiplied by each other by the method used in multiplying finite sums, the product series will not usually converge or have the product of the two sums as its sum unless at least one of the two series is absolutely convergent.

**Exercises 8.1:** Prove that

1. $\sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots = 1.$

2. $\sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu+1)(\nu+2)} = \frac{1}{4}.$

3. $\sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{2\nu+3}{(\nu+1)(\nu+2)} = 1.$

4. For what values of $\alpha$ does the series $1 - \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} - \frac{1}{4^{\alpha}} + \ldots$ converge? 5.* Prove that, if $\sum_{\nu=1}^{\infty} a_{\nu}$ converges and $a_n = a_1 + a_2 + \cdots + a_n$, then the sequence

$$\frac{s_1 + s_2 + \ldots + s_N}{N}$$

also converges and has the limit $\sum_{\nu=1}^{\infty} a_{\nu}$.

6. Does the series $\sum_{n=1}^{\infty} \left(\frac{2n}{2n+1} - \frac{2n-1}{2n}\right)$ converge?

7. Does the series $\sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{\nu}{\nu+1}$ converge?

Answers and Hints

## 8.2 Tests for Convergence and Divergence

We have already encountered a test of a general nature for the convergence of series, which applies to series with terms of alternating signs and decreasing absolute values and which asserts that such series are at least **conditionally convergent**. In what follows, we shall only consider criteria referring to **absolute convergence**.

**8.2.1 The Comparison Test:** All such considerations of convergence depend on the comparison of the series in question with a second series; this second series is chosen in such a way that its convergence can readily be tested. The general comparison test may be stated as follows:

If all the numbers $b_1$, $b_2$ ··· are positive and the series $\sum_{\nu=1}^{\infty} b_\nu$ converges and if

$$|a_n| \leq b_n$$

for all values of $n$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If we apply Cauchy's test, the proof becomes almost trivial, because, if $m \geq n$, we have

$$|a_n + \ldots + a_m| \leq |a_n| + \ldots + |a_m| \leq b_n + \ldots + b_m.$$

Since the series $\sum_{\nu=1}^{\infty} b_\nu$ converges, the right hand side is arbitrarily small, provided that $n$ and $m$ are sufficiently large, whence for such values of $n$ and $m$ the left hand side is also arbitrarily small, so that, by Cauchy's test, the given series converges. The convergence is absolute, since our argument applies equally well to the convergence of series of absolute values $|a_n|$.

The analogous proof for the following fact will be left to the reader:

If $|a_n| \geq b_n > 0$ and the series $\sum_{\nu=1}^{\infty} b_\nu$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ is certainly not absolutely convergent.

**8.2.2 Comparison with the Geometric Series:** In applications of the test, the most frequently used comparison series is the geometric series. We obtain immdediately the theorem:

The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, if from a certain term onwards there holds a relation of the form

$$|a_n| < c\,q^n\,, \qquad \text{(I)}$$

where $c$ is a positive number independent of $n$ and $q$ is any fixed positive number less than 1.

This test is usually expressed in one of the following, weaker forms:

The series $\sum_{n=1}^{\infty} a_n$ converges absolutely, if there holds from a certain term onwards a relation of the form

$$\left|\frac{a_{n+1}}{a_n}\right| < q \,, \qquad \text{(IIa)}$$

where $q$ is again a positive number less than 1 and independent of $n$, or, if from a certain term onward there holds a relation of the form

$$\sqrt[n]{|a_n|} < q \,, \qquad \text{(IIb)}$$

where $q$ is again a positive number less than 1. In particular, the conditions of these tests are satisfied if there applies a relation of the form

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = k < 1 \quad \text{(IIIa)}$$

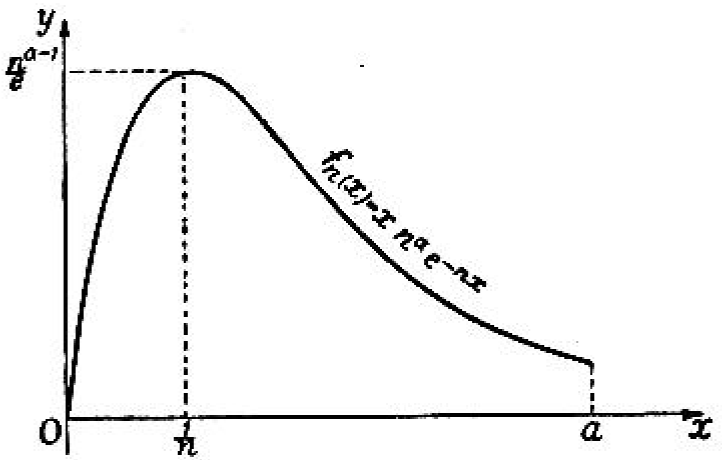



Fig. 6

These statements are easily established in the following manner:

Let the criterion II*a*, the **ratio test**, be satisfied from the suffix $n_0$ onward, i.e., when $n > n_0$. For the sake of brevity, let $a_{n_0+m+1} = b_m$ and find that

$$|b_1| < q\,|b_0|, \quad |b_2| < q\,|b_1| < q^2\,|b_0|, \quad |b_3| < q\,|b_2| < q^3\,|b_0|,$$

etc.; hence

$$|b_m| < q^m\,|b_0|,$$

which establishes our statement. For the Criterion II*b* - the **root test** -we have at once $|a_n| < q_n$, and our statement follows immediately.

Finally, in order to prove III, we consider an arbitrary number $q$ such that $k < q < 1$. Then, from a certain $n_0$ onwards, i.e.,, when $n > n_0$, it is certain that $\left|\frac{a_{n+1}}{a_n}\right| < q$ or $\sqrt[n]{|a_n|} < q$, as the case may be, since from a certain term onwards the values of $\left|\frac{a_{n+1}}{a_n}\right|$ or of $\sqrt[n]{|a_n|}$ differ from $k$ by less than $(g - k)$. The statement is then established by reference to the results already proved.

We stress the point that the four tests derived from the original criterion $|a_n|<cq^n$ are not equivalent to each other or to the original one, i.e., that they cannot be derived from one another in both directions. We shall soon see from examples that, if a series satisfies one of these conditions, it need not by any means satisfy all the other ones.

* More exactly: If III*a* is fulfilled, then II*a* ie fulfilled: if III*b*, then II*b*; if III*a*, then III*b*; if II*a*, then II*b*, and if any of the four is satisfied, then so is I. None of these statements can he reversed.

For the sake of completeness, it may be pointed out that a series certainly diverges if from a certain term onwards

$$|a_n| > c$$

for a properly chosen positive number $c$ or if from a certain term onwards

$$\sqrt[n]{|a_n|} > 1,$$

or if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = k, \quad \text{or} \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = k,$$

where $k > 1$. In fact, as we immediately recognize, in such a series the terms cannot tend to zero as $n$ increases, whence the series must diverge. (Under these conditions, the series cannot even be conditionally convergent.)

Our tests furnish sufficient conditions for the absolute convergence of a series, i.e., if they are satisfied, we can conclude that the series converges absolutely. However, they are definitely not necessary conditions, i.e., absolutely convergent series can be found which do not satisfy these conditions.

For example, the knowledge that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \quad \text{or} \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$$

does not entitle us to make any statement about the convergence of the series. Such a series may converge or diverge. For example, the series

$$\sum_{n=1}^{\infty}\frac{1}{n},$$

for which $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1,$ is divergent, as we have seen before. On the other hand, we shall soon see that the series $\sum_{n=1}^{\infty}\frac{1}{n^2},$ which satisfies the same relations, converges.

As an example of the application of our tests, we first consider the series

$$q + 2q^2 + 3q^3 + \ldots + nq^n + \ldots .$$

For this series,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = |q| \cdot \lim_{n\to\infty} \sqrt[n]{n} = |q|,$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |q| \cdot \lim_{n\to\infty} \frac{n+1}{n} = |q|.$$

It follows from the ratio test as well as from the root test even in the weaker form III that the series converges, if $|q| < 1$.

On the other hand, if we consider the series

$$1 + 2q + q^2 + 2q^3 + \ldots + q^{2n} + 2q^{2n+1} + \ldots,$$

we can no longer prove convergence by the ratio test when $\frac{1}{2} \le |q| < 1$, because then $|2q^{2n+1}/q^{2n}| = 2|q| \ge 1$. But the root test yields immediately $\lim \sqrt[n]{|a_n|} = |q|$ and shows that the series converges provided $|q| < 1$, which, of course, we could have observed directly.

### 8.2.3 Comparison with an Integral:

We now proceed to a discussion of convergence which is independent of the preceding considerations. (cf. A7.1)

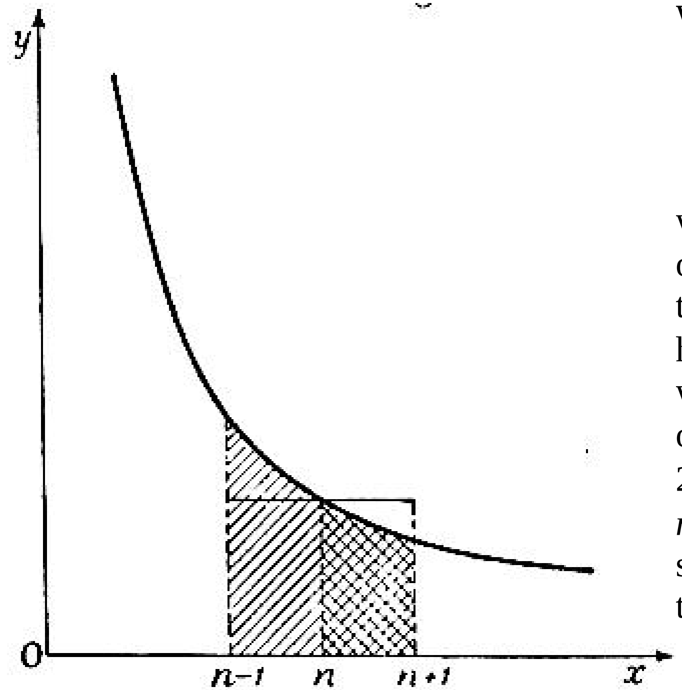


Fig. 2.—Comparison of a series with an integral

We shall carry it out for the particularly simple and important case of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots,$$

where the general term $a_n$ is $1/n^\alpha$, $\alpha > 0$. In order to investigate the properties of this series, we consider the graph of the function $y = 1/x^\alpha$ and mark off on the $x$-axis the integral values $x = 1$, $x = 2$, $\cdots$. We first construct the rectangle of height $1/n^\alpha$ over the interval $n-1 \le x \le n$ of the $x$-axis ($n > 1$) and compare it with the area of the region bounded by the same interval of the $x$-axis, the ordinates at the ends and the curve $y=1/x^\alpha$ (this region is shown shaded in Fig. 2). Secondly, we construct the rectangle of height $1/n^\alpha$ lying above the interval $n \le x \le n+1$ and similarly compare it with the area of the region lying above the same interval and below the curve (this region is cross-hatched in Fig. 2). In the first case, the area under the curve is obviously larger than the area of the

rectangle; in the second case, it is less than the area of the rectangle. In other words,

$$\int_n^{n+1} \frac{dx}{x^a} < \frac{1}{n^a} < \int_{n-1}^{n} \frac{dx}{x^a},$$

as we may also prove directly from the integral itself (cf. 2.7.2). Writing down this inequality for $n = 2$, $n = 3$, ··· , $n = m$ and summing, we obtain the estimate * for the $m$-th partial sum $s_m = \sum_{n=1}^{m} \frac{1}{n^a}$:

$$1 + \int_2^{m+1} \frac{dx}{x^a} < s_m < 1 + \int_1^{m} \frac{dx}{x^a}.$$

* From this relation follows at once that the sequence of numbers $C_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n$ is bounded below. Since we see from the inequality $\frac{1}{n+1} < \int_n^{n+1} \frac{dx}{x} = \log(n+1)$ that the sequence is monotonic decreasing, it must approach a limit

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n\right) = C.$$

The number $C$ with the value 0.577,2··· is called **Euler's constant**. In contrast to the other important special numbers of analysis, such as $\pi$ and $e$, no other expression with a simple law of formation has been found for Euler's constant.

Now, as $m$ increases, the integral $\int_1^m \frac{1}{x^a} dx$ tends to a finite limit or increases without limit according to whether $\alpha$ > l or $\alpha \le 1$. Consequently, the monotonic sequence of the numbers $s_m$ is bounded or increases beyond all bounds according to whether $\alpha$ > or $\alpha \le 1$, whence follows the theorem:

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^a} = \frac{1}{1^a} + \frac{1}{2^a} + \frac{1}{3^a} + \dots$$

converges - and, of course, absolutely - if and only if, $\alpha > 1$.

The divergence of the harmonic series, which we have proved previously in a different way, is an immediate consequence of this. In particular, we see that the series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots,$$

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

converge.

The series $\sum_{\nu=1}^{\infty}\frac{1}{\nu^{\alpha}}$, the convergence of which we have just studied, frequently serve as comparison series in investigations of convergence. For example, we see at once that for $\alpha > 1$ the series $\sum_{\nu=1}^{\infty}\frac{c_\nu}{\nu^{\alpha}}$ converges absolutely if the absolute values $|c_\nu|$ of the coefficients remain less than a fixed bound independent of $\nu$.

**Exercises 8.2:**

Find out whether the series 1. - 6. converge or do not converge:

1. $\sum_{\nu=1}^{\infty}\frac{1}{1+\nu^2}$.

2. $\sum_{\nu=1}^{\infty}\frac{\nu!}{\nu^{\nu}}$.

3. $\sum_{\nu=1}^{\infty}\frac{1}{\sqrt{\nu(\nu+1)}}$.

4.* $\sum_{\nu=2}^{\infty}\frac{1}{(\log\nu)^{\alpha}}$, $\alpha$ fixed.

5. $\sum_{\nu=2}^{\infty}\frac{1}{(\log\nu)^{\log\nu}}$.

6. $\sum_{\nu=1}^{\infty}\frac{\nu}{2^{\nu}}$.

Estimate the error after $n$ terms of the series 7. - 10.:

7. $\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{\nu^2}$.

8. $\sum_{\nu=1}^{\infty} \frac{1}{\nu!}$.

9. $\sum_{\nu=1}^{\infty} \frac{1}{\nu^\nu}$.

10. $\sum_{\nu=1}^{\infty} \frac{\nu}{2^\nu}$.

11. Prove that $\sum_{\nu=1}^{\infty} \sin^2\left[\pi\left(\nu + \frac{1}{\nu}\right)\right]$ converges.

12. Does $\sum_{\nu=-\infty}^{\infty} e^{-\nu^2}$ (that is, $1 + 2\sum_{\nu=1}^{\infty} e^{-\nu^2}$) converges.

13.* Prove that $\sum_{\nu=2}^{\infty} \frac{1}{\nu(\log \nu)^\alpha}$ converges when $\alpha > 1$ and diverges when $\alpha \leq 1$.

14.* Prove that $\sum_{\nu=3}^{\infty} \frac{1}{\nu \log \nu \, (\log \log \nu)^\alpha}$ converges when $\alpha > 1$ and diverges when $\alpha \leq 1$.

15. Prove that if $u_i \geq 0$ ($i = 1, 2, 3, \cdots$ and $\sum_{i=1}^{\infty} u_i^2$ and $\sum_{i=1}^{\infty} u_i$ converge, then $\sum_{i=1}^{\infty} u_i$ converges.

16. Show that if both $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ S converge, then $\sum_{k=1}^{\infty} a_k b_k$ also converges.

17. Prove that

$$1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \cdots + \frac{1}{3n+1} + \frac{1}{3n+2} - \frac{2}{3n+3} + \cdots = \log 3.$$

18.* Prove that, if $n$ is an arbitrary integer greater than 1,

$$\sum_{\nu=1}^{\infty} \frac{a_\nu^{\ n}}{\nu} = \log n,$$

where $a_\nu^n$ is defined as follows:

$$a_\nu^n = \begin{cases} 1 \text{ if } n \text{ is not a factor of } \nu, \\ -(n-1) \text{ if } n \text{ is a factor of } \nu. \end{cases}$$

Answers and Hints

## 8.3 Sequences and Series of Functions

**8.3.1 General Remarks:** The terms of the infinite series hitherto considered have been constants, whence these series (when convergent) always represented definite numbers. But both in theory and in applications, the series of outstanding importance are those in which the terms are functions of a variable, so that the sum of the series is also a function of the variable, as in the case of Taylor series.

We shall therefore consider a series

$$g_1(x) + g_2(x) + g_3(x) + \dots,$$

in which the functions $g_n(x)$ are defined in an interval $a \le x \le b$. We will denote the $n$-th partial sum

$$g_1(x) + g_2(x) + \dots + g_n(x),$$

by $f_n(x)$. Then the sum $f(x)$ of our series, where it exists, is simply the limit $\lim_{n\to\infty} f_n(x).$

We may therefore regard the sum of an infinite series of functions as the limit of a sequence of functions $f_1(x)$, $f_2(x)$, ··· , $f_n(x)$, ···. Conversely, for any such sequence of functions $f_1(x)$, $f_2(x)$, ···, we can form an equivalent series by setting $g_1(x) = f_1(x)$ ···and $g_n(x) = f_n(x) - f_{n-1}(x)$ for $n > 1$. Hence, when it is convenient, we can pass from the consideration of series to that of sequences and conversely.

**8.3.2 Limiting Processes with Functions and Curves:** We shall now state exactly what we mean by saying that a function $f(x)$ is the limit of a sequence $f_1(x)$, $f_2(x)$, ··· , $f_n(x)$, ··· in an interval $a \le x \le b$. The definition is: The sequence $f_1(x)$, $f_2(x)$, ··· converges in that interval to the **limit function** $f(x)$, if at each point $x$ of the interval the values $f_n(x)$ converge in the usual sense to the value $f(x)$. . In this case, we shall write $\lim_{n\to\infty} f_n(x) = f(x).$

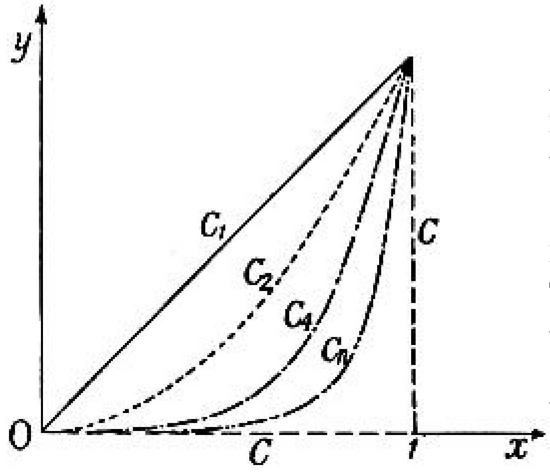


Fig. 3.—Limit curve and limit function

According to Cauchy's test , we can express the convergence of the sequence without necessarily knowing or stating the limit function $f(x)$. In fact, our sequence of functions will converge to a limit function if, and only if, at each point $x$ in our interval and for every positive number $\varepsilon$ the quantity $|f_n(x) - f_m(x)|$ is less than $\varepsilon$, provided that the numbers $n$ and $m$ are chosen large enough, i.e., larger than a certain number $N = N(\varepsilon)$. This number $N(\varepsilon)$ usually depends on $\varepsilon$ and $x$ and increases beyond all bounds as $\varepsilon$ tends to zero.

We have frequently encountered cases of limits of sequences of functions. We mention only the definition of the power $x^\alpha$ for irrational values of $\alpha$ by the equation

$$x^\alpha = \lim_{n\to\infty} x^{r_n},$$

where $r_1, r_2, \cdots, r_n, \cdots$ is a sequence of rational numbers tending to $\alpha$ or the equation

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n,$$

where the functions $f_n(x)$ on the right hand side are polynomials of degree $n$.

The graphical representation of functions by means of curves suggests that we can also speak of limits of **sequences of curves**, saying, for example, that the graphs of the above limit functions $x^\alpha$ and $e^x$ are to be regarded as the **limit curves** of the graphs of the functions $x^{r_n}$ and $\left(1 + \frac{x}{n}\right)^n$, respectively. However, there is a fine distinction between the passages to the limit with functions and with curves. Until the middle of the Nineteenth Century, this distinction was not sufficiently recognized and only by having a clear idea of it can we avoid apparent paradoxes. We shall illustrate this point by an example.

Consider the functions

$$f_n(x) = x^n \quad (n = 1, 2, \ldots)$$

in the interval $0 \le x \le 1$. All these functions are continuous and the limit function

$$\lim_{n \to \infty} f_n(x) = f(x)$$

exists. However, this limit function is not continuous. On the contrary, since for all values of *n* the value of the function $f_n(1) = 1$, the limit

$$f(1) = 1,$$

while, on the other hand, for $0 \le x < 1$, the limit $f(x) = \lim_{n \to \infty} f_n(x) = 0,$ as we have seen in 1.5.6 . The function *f*(*x*) is therefore a discontinuous function which at *x*=1 has the value 1 and for all other values of *x* in the interval has the value 0.

This discontinuity becomes clear, if we consider the graphs $C_n$ of the functions $y = f_n(x)$. These (cf. Fig. 16 in 1.5.6) are continuous curves, all of which pass through the origin and the point *x* = 1, *y* = 1 and which draw in closer and closer to the *x*-axis as *n* increases. The curves possess a **limit curve** *C* which is not at all discontinuous, but consists (Fig. 3 above) of the portion of the *x*-axis between *x*=0 and *x*=1 and the portion of the line *x*=1 between *y*=0 and *y*=1. The curves therefore converge to a continuous limit curve with a vertical portion, while the functions converge to a discontinuous limit function. We thus recognize that this discontinuity of the limit function expresses itself by the occurrence in the limit curve of a portion perpendicular to the *x*-axis. Such a portion must involve a discontinuity in the limit function and, in fact, such a portion is always present when the limit function is discontinuous. This limit curve is not the graph of the limit function nor can any curve with a vertical portion be the graph of any single-valued function *y* = *f*(*x*); in fact, corresponding to the value of *x* at which the vertical portion occurs, the curve gives an infinite number of values of *y* and the function only one. Hence the limit of the graphs of the functions $f_n(x)$ is not the same as the graph of the limit *f*(*x*) of these functions.

Naturally, corresponding statements also apply to infinite series.

## 8.4 Uniform and Non-uniform Convergence

**8.4.1 General Remarks and Examples:** The distinction between the concepts of convergence of functions and of convergence of curves introduces a phenomenon of which it is essential that the student should clearly recognize it. This is the so-called non-uniform convergence of sequences or infinite series of functions. Since it is well known that, as a rule, beginners encounter difficulties here, we should discuss the matter in some detail.

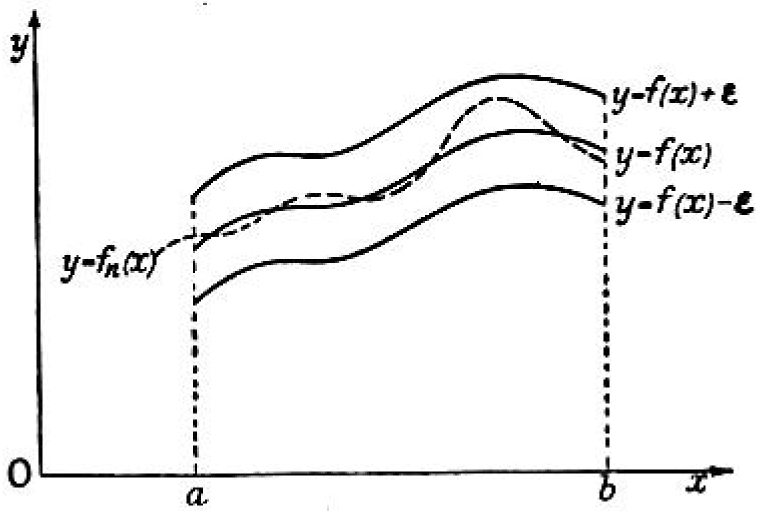


**Fig. 4.—To illustrate uniform convergence**

By definition, a function $f(x)$ is the limit of a sequence $f_1(x)$, $f_2(x)$, ··· in an interval $a \leq x \leq b$, only means that the usual limit relationship $f(x) = \lim_{n\to\infty} f(x)$ holds at each point $x$ of the interval. From a naive point of view, one might expect that it would follow automatically from this concept of convergence that if we assign an arbitrary degree of accuracy, say $\varepsilon$=1/1000 or $\varepsilon$=1/100, then from a certain index $N$ onwards all the functions $f_n(x)$ would lie between $f(x) + \varepsilon$ and $f(x) - \varepsilon$ for all values of $x$, so that their graphs $y = f_n(x)$ will lie entirely in the strip indicated in Fig. 4. Thus, for every positive $\varepsilon$, there is a corresponding number $N = N(\varepsilon)$, which, of course, will ordinarily increase beyond all bounds as $\varepsilon \to 0$, such that, for $n > N$, the difference $|f(x) — f_n(x)| < \varepsilon$, no matter where $x$ is chosen in the interval. (If this condition is met, then $|f_n(x) - f_m(x)| < 2\varepsilon$ for all values of $x$, provided that both $n$ and $m$ are greater than $N$.)

If the accuracy of the approximation can be made at least equal to a pre-assigned number $\varepsilon$ everywhere in the interval at the same time, i.e., by everywhere choosing the same number $N(\varepsilon)$ independent of $x$, we may say that the approximation is **uniform**. One is at first astonished to find that the naive assumption that convergence is necessarily uniform is altogether wrong; in other words, that convergence may very well be **non-uniform**.

**Example 1:** Non-uniform convergence occurs in the case of the sequence of functions $f_n(x) = x^n$ just considered; in the interval $0 \leq x \leq 1$, this sequence converges to the limit function $f(x) = 0$ for $0 \leq x < 1$, $f(1) = 1$. Convergence occurs at every point in the interval, i.e., if $\varepsilon$ is any positive number and if we select any definite fixed value $x = \xi$, the inequality $|\xi^n - f(\xi)| < \varepsilon$ certainly holds for sufficiently large $n$. Yet, this approximation ie not uniform, because, if we choose $\varepsilon = ½$, then, no matter how large the number $n$ is chosen, we can find a point $x = \eta$ at which $|\eta^n - f(\eta)| = \eta^n > ½$; in fact, this is true for all points $x=\eta$, where $1 > \eta > \sqrt[n]{\tfrac{1}{2}}$. Hence it is impossible to choose the number $n$ so large that the difference between $f(x)$ and $f_n(x)$ is less than ½ throughout the entire interval.

This behaviour becomes intelligible, if we refer to the graphs of these functions in Fig. 3. We see that, no matter how large a value of $n$ we choose, for values of $\xi$ only a little less than 1 the value of the function $f_n(\xi)$ will be very close to 1, whence there cannot be a good approximation to $f(\xi)$ which is 0.

A similar behaviour is exhibited by the functions

$$f_n(x) = \frac{1}{1 + x^{2n}}$$

in the neighbourhood of the points $x = 1$ and $x = -1$; this is easily shown (1.8.2.).

**Example 2:** In the two examples discussed above, the non-uniformity of the convergence is linked to the fact that the limit function is discontinuous. It is also easy to construct a sequence of continuous functions which do converge to a continuous limit function, however, not uniformly. We will restrict our attention to the interval $0 \le x \le 1$ and the definitions for $n \ge 2$:

$$f_n(x) = xn^\alpha \text{ for } 0 \leqq x \leqq \frac{1}{n},$$

$$f_n(x) = \left(\frac{2}{n} - x\right) n^\alpha \text{ for } \frac{1}{n} \leqq x \leqq \frac{2}{n},$$

$$f_n(x) = 0 \text{ for } \frac{2}{n} \leqq x \leqq 1,$$

where, to begin with, we can choose any value for $\alpha$, but must then keep this value of $\alpha$ fixed for all terms of the sequence. Graphically, our functions are represented by a roof-shaped figure made by two line segments lying over the interval $0 \le x \le 2/n$ of the $x$-axis, while from $x = 2/n$ onwards the graph is the $x$-axis itself (Fig. 5).

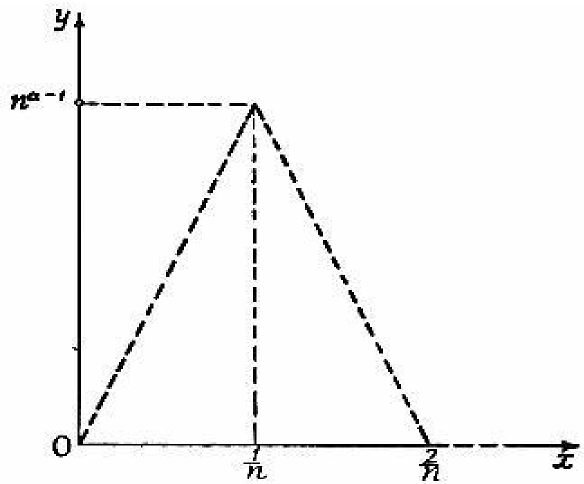


Fig. 5 To illustrate non-uniform convergence

If $\alpha < 1$, the altitude of the highest point of the graph, which has, in general, the value $n^{\alpha-1}$, will tend to 0 as $n$ increases; the curve will then tend towards the $x$-axis and the function $f_n(x)$ will converge uniformly to the limit function $f_n(x)=0$.

If $\alpha = 1$, the peak of the graph will have the height 1 for every value of $n$. If $\alpha > 1$, the height of the peak will increase beyond all bounds with $n$.

However, no matter how $\alpha$ is chosen, the sequence $f_1(x), f_2(x), \cdots$ always tends to the limit function $f(x) = 0$. In fact, if $x$ is positive, we have for all sufficiently large values of $n$ that $2/x < x$, so that $x$ is not under the roof-shaped part of the graph and $f_n(x) = 0$; for $x = 0$, the functional values $f_n(x)$

are equal to 0, so that in either case $\lim_{n\to\infty} f_n(x) = 0.$

However, the convergence is certainly non-uniform if $\alpha \geq 1$, because it is plainly impossible to choose $n$ so large that the expression $|f(x)| - f_n(x)| = f_n(x)$ is less than ½ everywhere in the interval.

**Example 3:** An exactly similar behaviour is exhibited by the sequence of functions (Fig. 6)

$$f_n(x) = xn^{\alpha}e^{-nx},$$

where, in contrast to the preceding case, each function of the sequence is represented by a single analytical expression. Here again, the equation $\lim_{n\to\infty} f_n(x) = 0$ holds for every positive value of $x$, since, as $n$ increases, the function $e^{-nx}$ tends to 0 to a higher order than any power of l/$n$ (cf. 3.9.2). We have always for $x$ = 0 that $f_n(x)$ = 0, and thus that

$$f(x) = \lim_{n\to\infty} f_n(x) = 0$$

for every value of $x$ in the interval $0 \leq x \leq a$, where $a$ is an arbitrary positive number. But here again, the convergence to the limit function is not uniform. In fact, at the point $x = 1/n$ (where $f_n(x)$ has its maximum), we have

$$f_n(x) = f_n\left(\frac{1}{n}\right) = \frac{n^{\alpha-1}}{e},$$

and we thus realize that, if $\alpha \geq 1$, the convergence is non-uniform; in fact, every curve $y = f_n(x)$, no matter how large $n$ is chosen, will contain points (namely, the point $x = 1/n$, which varies with $n$, and neighbouring points) at which

$$f_n(x) - f(x) = f_n(x) > \frac{1}{2e}.$$

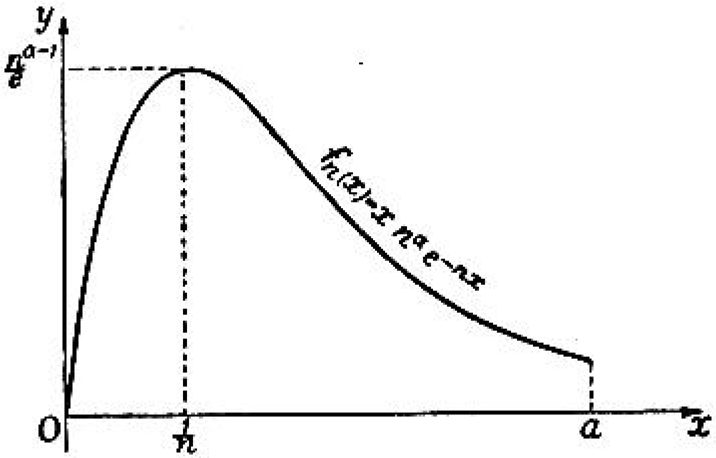



Fig. 6

**Example 4:** Naturally, the concepts of uniform and non-uniform convergence may be extended to infinite series. We say that a series

$$g_1(x) + g_2(x) + \dots$$

is uniformly convergent or not, according to the behaviour of its partial sums $f_n(x)$. A very simple example of a non-uniformly convergent series is

$$f(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3}$$

For $x = 0$, every partial sum

$$f_n(x) = x^2 + \dots + \frac{x^2}{(1+x^2)^{n-1}}$$

has the value 0, whence $f(0) = 0$. For $x \neq 0$, the series is simply a geometric series with the positive ratio $1/(1 + x^2) < 1$, whence we can sum it by the elementary rules and obtain for every $x \neq 0$ the sum

$$\frac{x^2}{1 - 1/(1+x^2)} = 1 + x^2.$$

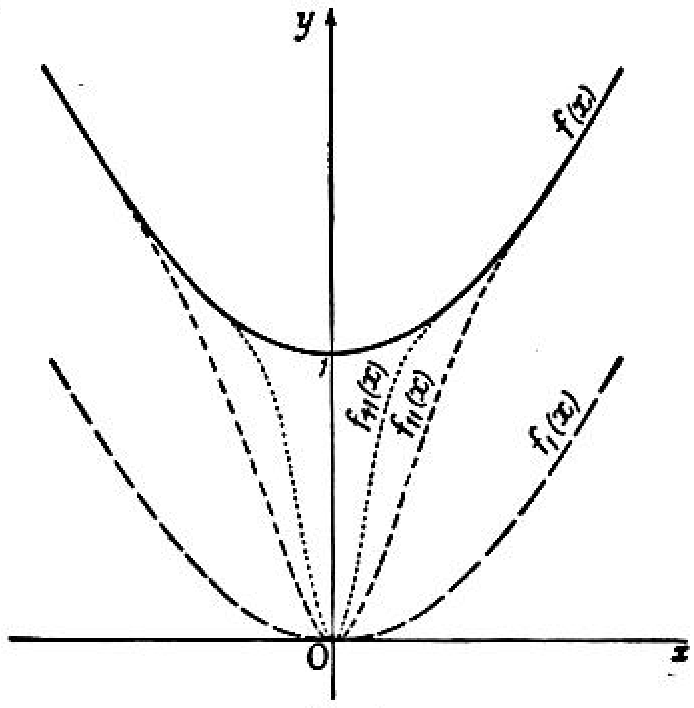



Fig. 7

Thus, the limit function $f(x)$ is given everywhere, except at $x = 0$, by $f(x)=1+x^2$, while $f(0) = 0$; it has therefore a somewhat artificial looking discontinuity at the origin.

Here again, we have in every interval containing the origin non-uniform convergence. In fact, the difference $f(x)-f_n(x) = r_n(x)$ is always 0 for $x = 0$, while, as the reader may verify, it is given for all other values of $x$ by

$$r_n(x)=1/(1+x^2)^{n-1}.$$

If we want this expression to be less than, say ½, then this can be done for each fixed value of $x$ by choosing $n$ large enough. But we can find no value of $n$ large enough to ensure that $r_n(x)$ is everywhere less than ½; in fact, if we

concentrate on any value of $x$, no matter how large, we can make $r_n(x)$ larger than ½ by taking $x$ near enough to 0, whence an uniform approximation to within ½ is impossible. This becomes clear if we consider the approximating curves (Fig. 7 ). Except near 0, as $n$ increases, these curves lie nearer and nearer to the parabola $y = 1 + x^2$; however, near $x = 0$, the curves send down a narrower and narrower extension to the origin, and as $n$ increases, this extension draws in closer and closer to a certain straight line, a portion of the $y$-axis, so that we have as a limiting curve the parabola plus a linear extension reaching vertically down to the origin.

We mention as a further example of non-uniform convergence the series $\sum_{\nu=0}^{\infty} g_\nu(x)$, where $g_\nu(x) = x^\nu - x^{\nu-1}$ for $\nu \geq 1$, $g_0(x) = 1$, defined in the interval $0 \leq x \leq 1$. The partial sums of this series are the functions $x^\nu$ which have already been considered in Example 1.

**8.4.2 A Test of Uniform Convergence:** The preceding considerations show that the uniform convergence of a sequence or series is a special property which is not possessed by all sequences and series. We shall now reformulate the concept of uniform convergence. The convergent series

$$g_1(x) + g_2(x) + \cdots$$

is said to be uniformly convergent in an interval, if the sum $f(x)$ can be approximated to within $\varepsilon$ (where $\varepsilon$ is an arbitrarily small positive number) by taking a number of terms which is sufficiently large and which is the same throughout the interval.

First of all, we assume that the series $g_1(x) + g_2(x) + \cdots$ converges at every point of a certain interval to a limit function $f(x)$; we denote by $f_n(x)$ the $n$-th partial sum of the series $f_n(x) = g_1(x) + \cdots + g_n(x)$ and by $R_n(x)$ the remainder of the series after $n$ terms:

$$R_n(x) = f(x) - f_n(x).$$

The series $g_1(x) + g_2(x) + \cdots$ is said to be uniformly convergent in the interval if there corresponds to every positive number $\varepsilon$ a number $N$, which only depends on $\varepsilon$ and not on $x$, such that for $n > N$ one has the inequality $|R(x)|=|f(x)-f_n(x)|<\varepsilon$ for all values of $x$ in the interval.

Expressed more figuratively, the partial sum $f_n(x)$ represents the sum $f(x)$ to within an error of less than $\varepsilon$ everywhere in the interval at the same time, provided only that $n$ is chosen large enough. By Cauchy's test, we

readily see that the series converges uniformly, if, and only if the difference $|f_n(x) - f_m(x)|$ can be made less than an arbitrary quantity $\varepsilon$ everywhere in the interval by choosing $n$ and $m$ larger than a number $N$ independent of $x$. In fact, firstly, if the convergence is uniform, we can make both $|f_n(x) - f(x)|$ and $|f_m(x) - f(x)|$ less than $\varepsilon/2$ by choosing $n$ and $m$ larger than a number $N$ independent of $x$, whence it follows that $|f_n(x) - f_m(x)| < \varepsilon$; and, secondly, if $|f_n(x) - f_m(x)| < \varepsilon$, for all values of $x$ whenever $n$ and $m$ are larger than $N$, then, on choosing any fixed value of $n > N$ and letting $m$ increase beyond all bounds, we have

$$|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leqq \epsilon,$$

for every value of $x$, whence the convergence is uniform.

If we wish to speak of the uniform convergence of a sequence of functions, we have only to make minute changes in the above definition; the sequence $f_1(x), f_2(x), \cdots$ converges uniformly to $f(x)$ in an interval, if the difference $|f(x)-f_n(x)|$ can be made less than $\varepsilon$ everywhere in the interval by choosing $n$ larger than a number $N$ independent of $x$. As above, a necessary and sufficient condition for the uniform convergence of the sequence is that $|f_n(x) - f_m(x)|<\varepsilon$ for all values of $x$ when both $n$ and $m$ are larger than a certain number $N$, which depends on $\varepsilon$ but not on $x$.

We shall soon see that it is just this condition of uniform convergence which makes infinite series and other limiting processes with functions into convenient and useful tools of analysis. Fortunately, in the limiting processes, usually encountered in the Calculus and its applications, non-uniform convergence is a sort of exceptional phenomenon which will scarcely trouble us in our present applications of analysis.

In most cases, the uniformity of the convergence of a series is established by means of the criterion:

If the terms of the series $\sum_{\nu=1}^{\infty} g_\nu(x)$ satisfy the condition $|g_\nu(x)| \leqq a_\nu$, where the numbers $a_\nu$ are constants which form a convergent series $\sum_{\nu=1}^{\infty} a_\nu$, then the series $\sum_{\nu=1}^{\infty} g_\nu(x)$ converges uniformly (and, we incidentally remark, absolutely).

In fact, we have then

$$\left|\sum_{\nu=n}^{m} g_\nu(x)\right| \leqq \sum_{\nu=n}^{m} |g_\nu(x)| \leqq \sum_{\nu=n}^{m} a_\nu,$$

and since, by Cauchy's test, the sum $\sum_{\nu=n}^{m} a_\nu$, can be made arbitrarily small by choosing $n$ and $m > n$ large enough, this expresses exactly the necessary and sufficient condition for uniform convergence.

A first example is offered by the geometric series $1 + x + x^2 + \cdots$, where $x$ is restricted to the interval $|x| \leq q$, a positive number less than 1. The terms of the series are then numerically less than or equal to the terms of the convergent geometric series $\Sigma q^\nu$.

A further example is the trigonometric series

$$\frac{c_1 \sin(x - \delta_1)}{1^2} + \frac{c_2 \sin(x - \delta_2)}{2^2} + \frac{c_3 \sin(x - \delta_3)}{3^2} + \cdots,$$

provided that $|c_n| < c$, a positive constant independent of $n$. Because we have then

$$g_n(x) = \frac{c_n \sin(x - \delta_n)}{n^2}, \text{ so that } |g_n(x)| < \frac{c}{n^2}.$$

Hence the uniform and absolute convergence of the trigonometric series follows from the convergence of the series $\sum_{\nu=1}^{\infty} \frac{c}{n^2}$.

**8.4.3 Continuity of the Sum of a Uniformly Convergent Series of Continuous Functions:** As we have already hinted, the significance of the uniform convergence of an infinite series lies in the fact that a uniformly convergent series in many respects behaves exactly like the sum of a finite number of functions. For example, the sum of a finite number of continuous functions is itself continuous and, correspondingly, we have the theorem:

If a series of continuous terms converges uniformly in an interval, its sum is also a continuous function.

The proof is quite simple. We subdivide the series

$$f(x) = g_1(x) + g_2(x) + \cdots$$

into the $m$-th partial sum $f_n(x)$ plus the remainder $R_n(x)$. As usual,

$$f_n(x) = g_1(x) + \ldots + g_n(x).$$

If now any positive number $\varepsilon$ is assigned, we can,by virtue of the uniform convergence, choose the number $n$ so large that the remainder is less than $\varepsilon/4$ throughout the entire interval, whence

$$|R_n(x+h) - R_n(x)| < \frac{\epsilon}{2}$$

for every pair of numbers $x$ and $x + h$ in the interval. The partial sum $f_n(x)$ consists of the sum of a finite number of continuous functions, whence it is continuous; thus, for each point $x$ in the interval, we can choose a positive $\delta$ so small that

$$|f_n(x+h) - f_n(x)| < \frac{\epsilon}{2},$$

provided $|h| < \delta$ and the points $x$ and $m + h$ lie in the interval. It then follows that

$$\begin{aligned} |f(x+h) - f(x)| &= |f_n(x+h) - f_n(x) + R_n(x+h) - R_n(x)| \\ &\leqq |f_n(x+h) - f_n(x)| + |R_n(x+h) - R_n(x)| < \epsilon, \end{aligned}$$

which expresses the continuity of our function.

The significance of this theorem becomes clear when we recall that the sums of non-uniformly convergent series of continuous functions are not necessarily continuous, as our previous examples show. We may conclude from the preceding theorem that, if the sum of a convergent series of continuous functions has a point of discontinuity, then in every neighbourhood of this point the convergence is non-uniform, whence every representation of discontinuous functions by series of continuous functions is based on the use of non-uniformly convergent limiting processes.

**8.4.4 Integration of Uniformly Convergent Series:** A sum of a finite number of continuous functions can be integrated term by term, i.e., the integral of the sum can be found by integrating each term separately and adding

the integrals. In the case of a convergent infinite series, the same procedure is possible, provided that the series converges uniformly in the interval of integration.

A series $\sum_{\nu=1}^{\infty} g_\nu(x) = f(x)$ which converges uniformly in an interval can be integrated term by term in that interval, or, more precisely, if $a$ and $x$ are two numbers in the interval of uniform convergence, the series $\sum_{\nu=1}^{\infty} \int_a^x g_\nu(t)\,dt$ converges and, in fact, converges uniformly with respect to $x$ for each fixed value of $a$, its sum being equal to $\int_a^x f(t)\,dt.$.

In order to prove this we write, as before,

$$f(x) = \sum_{\nu=1}^{\infty} g_\nu(x) = f_n(x) + R_n(x).$$

We have assumed that the separate terms of the series are continuous, whence, by 8.4.3, the sum is also continuous and therefore integrable. Now, if $\varepsilon$ is any positive number, we can find a number $N$ so large that, for every $n > N$, one has $|R_n(x)| < \varepsilon$ for every value of $x$ in the interval. By the first mean value theorem of the integral calculus, we have

$$\left| \int_a^x \left\{ f(t) - f_n(t) \right\} dt \right| \leqq \epsilon l,$$

where $l$ is the length of the interval of integration. Since the integration of the finite sum $f_n(x)$ can be performed term by term, this yields

$$\left| \int_a^x f(t)\,dt - \sum_1^n \int_a^x g_\nu(t)\,dt \right| < \epsilon l.$$

However, since $\varepsilon\, l$ can be made as small as we please, this yields

$$\sum_{\nu=1}^{\infty} \int_a^x g_\nu(t)\,dt = \lim_{n\to\infty} \sum_{\nu=1}^{n} \int_a^x g_\nu(t)\,dt = \int_a^x f(t)\,dt,$$

as was to be proved.

If we wish to conside instead of infinite series sequences of functions, our result can be expressed as follows:

If in an interval the sequence of functions $f_1(x)$, $f_2(x)$, $\cdots$ tends uniformly to the limit function $f(x)$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

for every pair of numbers $a$ and $b$ in the interval; in other words, we can then interchange the order of the operations of integration and going to the limit.

This fact is far from being a triviality. It is true that, from a naive point of view such as prevailed in the Eighteenth Century, the interchangeabi]ity of the two processes is hardly to be doubted; however, a glance at the examples in 8.4.1, shows that in the case of non-uniform convergence the above equation might not hold. We need only consider Example 2, in which the integral of the limit function is 0, while the integral of the function $f_n(x)$ over the interval $0 \le x \le 1$, i.e., the area of the triangle in Fig. 5 has the value

$$\int_0^1 f_n(x)\,dx = n^{\alpha-2},$$

and, when $\alpha \ge 2$, this does not tend to zero. We see here immediately from Fig.5 that the reason for the difference between $\int_0^1 f_n(x)\,dx = n^{\alpha-2}$, and $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$ lies in the non-uniformity of the convergence.

On the other hand, by considering values of $\alpha$ in the interval $1 \le \alpha \le 2$, we see that the equation $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$ can be valid, although the convergence is non-uniform. As a further example, we can integrate term by term the series $\sum_0^\infty g_n(x)$, where $g_n(x) = x^n - x^{n-1}$ for $n \ge 1$ and $g_0(x) = 1$, between 0 and 1, even

though it does not converge uniformly. Thus, while uniformity of convergence is a sufficient condition for term-by-term integrability, it is by no means a necessary condition. Neglect of this aspect may easily lead to a misunderstanding.

**8.4.5 Differentiation of Infinite Series:** The behaviour of uniformly converging series or sequences with respect to differentiation is quite different from that with respect to integration. For example, the sequence of functions $f_n(x) = \frac{\sin n^2 x}{n}$ certainly converges uniformly to the limit function $f(x) = 0$, but the derivative $f_n'(x) = n \cos n^2 x$ does not converge everywhere to the derivative of the limit function $f'(0) = 0$, as we see by considering $x = 0$. In spite of the uniformity of the convergence, we cannot therefore change the order of the process of differentiation and pass on to the limit.

Naturally, corresponding statements hold for infinite series. For example, the series

$$\sin x + \frac{\sin 2^4 x}{2^2} + \frac{\sin 3^4 x}{3^2} + \dots$$

is absolutely and uniformly convergent, because its terms are numerically not greater than the terms of the convergent series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots .$$

However, if we differentiate the series term by term, we obtain the series]

$$\cos x + 2^2 \cos 2^4 x + 3^2 \cos 3^4 x + \dots ,$$

which obviously does not converge everywhere, since, for example, it diverges at $x = 0$.

The only useful criterion which assures us in special cases that term-by-term differentiation is permissible is given by the theorem:

If, on differentiating a convergent infinite series $\sum_{\nu=0}^{\infty} G_\nu(x) = F(x)$ term by term, we obtain a uniformly convergent series of continuous terms $\sum_{\nu=0}^{\infty} g_\nu(x) = f(x)$, then the sum of this last series is equal to the derivative of the sum of the first series.

Hence, this theorem expressly requires that, after differentiating the series term by term, we must still investigate whether the result of the differentiation is a uniformly convergent series or not.

The proof of the theorem is almost trivial, because, by the theorem in 8.4.4, we can integrate term by term the series obtained by differentiation. Recalling that $g_\nu(t) = G_\nu'(t)$, we obtain

$$\int_a^x f(t)\,dt = \int_a^x \left(\sum_{\nu=0}^{\infty} g_\nu(t)\right) dt = \sum_{\nu=0}^{\infty} \int_a^x g_\nu(t)\,dt = \sum_{\nu=0}^{\infty} (G_\nu(x) - G_\nu(a))$$
$$= F(x) - F(a).$$

Since this is true for every value of $x$ in the interval of uniform convergence, it follows that

$$f(x) = F'(x),$$

as was to be proved.

**Exercises 8.3:**

1. Show by comparison with a series of constant terms that the following series converge uniformly in the intervals stated:

(a) $x - x^2 + x^3 - x^4 + \dots \quad (-\frac{1}{2} \leq x \leq \frac{1}{2})$.

(b) $\frac{1}{2}\sqrt{1-x^2} + \frac{1}{4}\sqrt{1-x^4} + \frac{1}{8}\sqrt{1-x^8} + \dots + \frac{1}{2^n}\sqrt{1-x^{2n}} + \dots$
$(-1 \leq x \leq 1)$.

(c) $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \dots + \frac{\sin nx}{n^2} + \dots$ .

(d) $e^x + e^{2x} + \dots + e^{nx} + \dots (-2 \leq x \leq -1)$.

2. Prove that $\lim f_n(x) = 0$, where $f_n(x) = \frac{nx}{1+n^2x^2}$, $-1 \leq x \leq 1$. Prove that the convergence is non-uniform.

3.* (*a*) Find $\lim_{n\to\infty} f_n(x)$, where $f_n(x) = \frac{n^2x^2}{1+n^2x^2}$, $-1 \leq x \leq 1$. Prove that the convergence la non-uniform. Prove that nevertheless

$$\lim_{n\to\infty}\int_{-1}^{1} f_n(x)\,dx = \int_{-1}^{1}\lim_{n\to\infty} f_n(x)\,dx.$$

(*b*) Discuss the convergence, uniform convergence and term-by-term integrability of the sequence given by $f_n(x) = \frac{n^a x^2}{1+n^a x^2}$ .

4.* Sketch the curves $y = f_n(x) = \frac{x^{2n}}{1+x^{2n}}$, $-2 \leq x \leq 2$, for $n = 1, 3, 10$. Find $\lim_{n\to\infty} f_n(x)$. Prove that the convergence is non-uniform.

5. Show that $\sum_{\nu=-\infty}^{\infty} e^{-(x-\nu)^2}$ converges uniformly in any fixed interval $a \leq x \leq b$.

6. Show that in the interval $0 \leq x \leq \pi$ the following sequences converge, but not uniformly:

(*a*) $\sqrt[n]{\sin x}$.
(*b*) $(\sin x)^n$.
(*c*) $\sqrt[n]{x \sin x}$.
(*d*) $[f(x)]^n$, where $f(x) = \dfrac{\sin x}{x}$, $f(0) = 1$.
(*e*) $\sqrt[n]{f(x)}$, where $f(x) = \dfrac{\sin x}{x}$, $f(0) = 1$.

7. The sequence $f_n(x)$, $n = 1, 2, \ldots$ is defined in the interval $0 \le x \le 1$ by the equations

$$f_0(x) \equiv 1, \quad f_n(x) = \sqrt{x f_{n-1}(x)}.$$

(*a*) Prove that the sequence converges in the interval $0 \le x \le 1$ to a continuous limit. (b)* Prove that the convergence is uniform.

8.* Let $f_0(x)$ be continuous in the interval $0 \le x \le a$. The sequence of functions $f_n(x)$ is defined by

$$f_n(x) = \int_0^x f_{n-1}(t)\,dt, \quad n = 1, \ 2, \ \ldots .$$

9. Sketch the curves $x^{2n} + y^{2n} = 1$ for $n = 1, 2, 4$, To what limit do these curves tend as $n \to \infty$ ?

10.* Let $f_n(x)$, $n = 1, 2, \cdots$ be a sequence of functions with continuous derivatives in the interval $a \le x \le b$. Prove that, if $f_n(x)$ converges at each point of the interval and the inequality $|f_n'(x)| < M$ (where $M$ is a constant) is satisfied for all values of $n$ and $x$, then the convergence is uniform.

Answers and Hints

## 8.5 Power Series

Among infinite series, power series occupy the main position. We mean by a power series a series of the type

$$P(x) = c_0 + c_1 x + c_2 x^2 + \ldots = \sum_{\nu=0}^{\infty} c_\nu x^\nu$$

(a power series in ***x***) or, more generally,

$$P(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots = \sum_{\nu=0}^{\infty} c_\nu (x - x_0)^\nu$$

(a power series in $(x - x_0)$), where $x_0$ is a fixed number. If we introduce into the last series $\xi = x - x_0$ as a new variable, it becomes a power series $\sum_{\nu=0}^{\infty} c_\nu \xi^\nu$ in the new variable $\xi$, whence, without any loss of generality, we can confine our attention to power series of the more special form $\sum_{\nu=0}^{\infty} c_\nu x^\nu$.

In 6.2, we have considered the approximate representation of functions by polynomials and were thus led to the expansion of functions in Taylor series, which, in fact, are power series. We shall now study power series in somewhat greater detail and obtain expansions of the most important functions in series in simpler and more convenient ways than before.

**8.5.1 Convergence Properties of Power Series:** There are power series which converge for no value of $x$, except, of course, for $x = 0$, e.g., the series

$$x + 2^2x^2 + 3^3x^3 + \ldots + n^nx^n + \ldots .$$

In fact, if $x \neq 0$, we can find an integer $N$ such that $|x| > 1/N$. Then the absolute values of all the terms $n^nx^n$, for which $n > N$, will be larger than 1 and, as $n$ increases, $nn^nx^n$ will increase beyond all bounds, so that the series fails to converge.

On the other hand, there are series which converge for every value of $x$; for example, the power series for the exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,$$

the convergence of which for every value of $x$ follows at once from the ratio test in 8.2.2. The $(n + 1)$-th term, divided by the n-th term, yields $x/n$ and, whatever number $x$ is chosen, this ratio tends to zero as $n$ increases. The behaviour of power series with regard to convergence is expressed in the **fundamental theorem**:

If a power series in $x$ converges for a value $x = \xi$, it converges absolutely for every value $x$ such that $|x| < |\xi|$ and the convergence is uniform in every interval $|x| \leq \eta$, where $\eta$ is any positive number less than $|\xi|$. Hence $\eta$ may lie as close to $|\xi|$ as we please.

The proof is simple. If the series $\sum_{\nu=0}^{\infty} c_\nu \xi^\nu$ converges, its terms tend to 0 as $n$ increases. From this follows the weaker statement that all the terms lie below a bound $M$ independent of $\nu$, i.e., $|c_\nu \xi^\nu| < M$. If now $q$ is any number such that $0<q<1$ and we restrict $x$ to the interval $|x| \leq q|\xi|$, then $|c_\nu x^\nu| \leq |c_\nu \xi^\nu| q^\nu < Mq^\nu$. Hence, in this interval, the terms of the series $\sum_{\nu=0}^{\infty} c_\nu \xi^\nu$ are smaller in absolute value than the terms of the convergent geometric series $\Sigma Mq^\nu$, whence follows, by the theorem in 8.4.2, the absolute and uniform convergence of the series in the interval $-q|\xi| \leq x \leq q|\xi|$.

If a power series does not converge everywhere, i.e., if there is a value $x = \xi$ for which it diverges, it must diverge for every value of $x$ such that $|x| > |\xi|$. In fact, if it were convergent for such a value of $x$, by the theorem above, it would have to converge for any numerically smaller value of $\xi$.

Hence, we recognize that a power series, which converges for at least one value of $x$ other than 0 and which diverges for at least one value of $x$, has an **interval of convergence**, i.e., there exists a definite positive number $\rho$ such that the series diverges for $|x| > \rho$ and converges for $|x| < \rho$. No general statement can be made for $|x| = \rho$. The limiting cases, in which the series converges only for $x=0$ and that which converges everywhere, are expressed symbolically by writing $\rho = 0$ and $\rho = \infty$, respectively.

It is possible to find this interval of convergence directly from the coefficients $c_\nu$ of the series. If the limit $\lim_{n\to\infty} \sqrt[n]{|c_n|}$ exists, then

$$\rho = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|c_n|}}.$$

In general, $\rho$ is given by the formula

$$\rho = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|c_n|}}$$

where $\overline{\lim}$ is the symbol for the upper limit, as it has been defined in A1.5.

For example, $\rho = 1$ for the geometric series $1 + x + x^2 + \cdots$; at the end-points of the interval of convergence, the series diverges. Similarly, for the series for the inverse tangent

$$\operatorname{arc}\tan x = x - x^3/3 + x^5/5 - + \dots\,.$$

we have $\rho = 1$ and at both of the interval's end-points $x = \pm 1$ the series converges, as we recognize at once from Leibnitz's test.

We deduce from the uniform convergence the important fact that within its interval of convergence (if such an interval exists) a power series represents a continuous function.

**8.5.2 Integration and Differentiation of Power Series:** On account of the uniformity of convergence, it is always permissible to integrate a power series

$$f(x) = \sum_{\nu=0}^{\infty} c_\nu x^\nu$$

term by term over any closed interval lying entirely within the interval of convergence. We thus obtain the function

$$F(x) = c + \sum_{\nu=0}^{\infty} \frac{c_\nu}{\nu+1} x^{\nu+1},$$

for which

$$F'(x) = f(x).$$

Moreover, since $|c_\nu/(\nu+1)| \le |c_\nu|$ for all values of $\nu$, the series obtained by integration converges more rapidly than the original series.

We can also differentiate a power series term by term within its interval of convergence, thus obtaining the equation

$$f'(x) = \sum_{\nu=1}^{\infty} \nu c_\nu x^{\nu-1}.$$

In order to prove this statement, we need only show that the series on the right hand side converges uniformly, if $x$ is restricted to an interval lying entirely within the interval of convergence. Now, let $\xi$ be a number, lying as close to $\rho$ as we please, for which $\sum_{\nu=1}^{\infty} c_\nu \xi^\nu$ converges; then, as we have seen before, all the numbers $|c_\nu \xi_\nu|$ lie below a bound $M$ independently of $\nu$, so that $|c_\nu \xi^{\nu-1}| < \frac{M}{|\xi|} = N.$ Now let $q$ be any number such that $0 < q < 1$; if we restrict $x$ to the interval $|x| \leq q|\xi|$, the terms of the series under discussion are not larger than those of the series $\sum_{\nu=1}^{\infty} |\nu c_\nu q^{\nu-1} \xi^{\nu-1}|$, and therefore less than those of the series $\sum_{\nu=1} N\nu q^{\nu-1}.$ However, in this last series, the ratio of the $(n + 1)$-th term to the $n$-th term is $\{(n+1)/n\}q$, which tends to $q$ as $n$ increases. Since $0<q<1$, it follows by Criterion III*a* that this series converges. Hence the series, obtained by differentiation, converges uniformly and by the theorem at the end of last section represents the derivative $f'_n(x)$ of the function $f(x)$, which proves our statement.

If we apply this result again to the power series

$$f'(x) = \sum_{\nu=1}^{\infty} \nu c_\nu x^{\nu-1},$$

we find on differentiating term by term that

$$f''(x) = \sum_{\nu=2}^{\infty} \nu(\nu-1) c_\nu x^{\nu-2}$$

and, by continuing the process, we arrive at the theorem:

Every function represented by a power series can be differentiated as often as we please within the interval of convergence and the differentiation can be performed term by term.

As an explicit expression for the $k$-th derivative, we obtain

$$f^{(k)}(x) = \sum_{\nu=k}^{\infty} \nu(\nu-1)\ldots(\nu-k+1)c_\nu x^{\nu-k},$$

or in a slightly different form

$$\frac{f^{(k)}(x)}{k!} = \sum_{\nu=k}^{\infty} \binom{\nu}{k} c_\nu x^{\nu-k} = \sum_{\nu=0}^{\infty} \binom{k+\nu}{k} c_{k+\nu} x^\nu.$$

These two formulae are frequently useful.

**8.5.3 Operations with Power Series:** The preceding theorems on the behaviour of power series are our justification for operating in the same way with power series as with polynomials. Obviously, two power series can be added or subtracted by adding or subtracting the corresponding coefficients (8.1.4). It is also clear that a power series, like any other convergent series, can be multiplied by a constant factor by multiplying each term by that factor. On the other hand, the multiplication and division of two power series requires a somewhat more detailed study, for which we refer the reader to the Appendix. We merely mention here without a proof that two power series

$$f(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu \text{ and } g(x) = \sum_{\nu=0}^{\infty} b_\nu x^\nu$$

can be multiplied by each other like polynomials. In order to be specific, we have the theorem:

Throughout the common part of the interval of convergence of two power series, their product is given by the convergent power series $\sum_{\nu=0}^{\infty} c_\nu x^\nu$ with the coefficients

$$\begin{aligned}
c_0 &= a_0 b_0,\\
c_1 &= a_0 b_1 + a_1 b_0,\\
c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0,\\
&\cdots\cdots\cdots\cdots\\
c_n &= a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0,\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

The proof is given in A8.1.

**8.5.4 Theorem of Uniqueness of Power Series:** In the theory of power series, the following fact is important:

If both power series $\sum_{\nu=0}^{\infty} a_\nu x^\nu$ and $\sum_{\nu=0}^{\infty} b_\nu x^\nu$ converge in an interval which contains the point $x = 0$ inside and if in that interval the two series represent the same function $f(x)$, then they are identical, that is, the equation $a_n = b_n$ is true for every value of $n$.

In other words:

If at all, a function $f(x)$ can be represented by a power series in $x$ in only one way,

Briefly speaking, the representation of a function by a power series is unique.

In order to prove this, we need only note that the difference of the two power series, i.e., the power series $\phi(x) = \sum_{\nu=0}^{\infty} c_\nu x^\nu$ with the coefficients $c_\nu = a_\nu - b_\nu$ represents the function

$$\phi(x) = f(x) - f(x) = 0$$

in the interval, i.e., this last power series converges to the limit 0 everywhere in the interval. In particular, for $x = 0$, the sum of the series must be 0, i.e., $c_0 = 0$, so that $a_0 = b_0$. We now differentiate the series in the interior of the interval and obtain $\phi'(x) = \sum_{\nu=1}^{\infty} \nu c_\nu x^{\nu-1}$. However, $\phi'(x)$ is also 0 throughout the interval, whence, in particular, we have for $x = 0$: $c_1 = 0$ or $a_1 = b_1$. Continuing this process of differentiation and then setting $x = 0$, we find successively that all the coefficients $c_\nu$ are equal to zero, which proves the theorem.

Moreover, we see that we can draw from the above discussion the conclusion:

If we take the $\nu$-th derivative of a series $f(x) = \sum a_\nu x^\nu$ and then set $x = 0$, we at once obtain

$$a_\nu = \frac{1}{\nu!} f^{(\nu)}(0),$$

i.e., Every power series which converges for points other than $x = 0$ is the Taylor series of the function which it represents.

The uniqueness of the expansion is here expressed by the fact that the coefficients are uniquely determined by the function itself.

## 8.6 Expansion of Given Functions in Power Series. Method of Undetermined Coefficients. Examples

Within its interval of convergence, every power series represents a continuous function with continuous derivatives of all orders. We shall now discuss the converse problem of the expansion of a given function in a power series. In theory, we can always do this by means of Taylor's theorem, however, in practice, we often encounter difficulties with the actual calculation of the $n$-th derivative and with the estimation of the remainder. But we can often reach our goal more simply by means of the device:

We first write down the statement $f(x) = \sum_{\nu=0}^{\infty} c_\nu x^\nu,$ where, to begin with, the coefficients $c_\nu$ are unknown. Then we determine by some known property of the function $f(x)$ the coefficients and prove subsequently the convergence of the series. The series represents a function and there remains only to prove that this function is identical to $f(x)$. Due to the uniqueness of the expansion in power series, we know that no other series than the one just found can be the required expansion.

We shall now consider some examples of the use of this method. In fact, we have already obtained the series for artan $x$ and log $(1+x)$ by a method, which forms part of the range of ideas of the present chapter. We simply integrated term by term the series for the derivatives of these functions, which we knew to be geometric series.

**8.6.1 The Exponential Function:** Our problem is to find a function for which $f'(x) = f(x)$ and $f(0) = 1$. If we write down the series with undetermined coefficients

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots ,$$

and differentiate it, we obtain

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \dots .$$

Since, by assumption, these two power series must be identical, we have the equation

$$nc_n = c_{n-1},$$

which is true for all values of $n \geq 1$. If we observe that, due to the relation $f(0)=1$, the coefficient $c_0$ must have the value 1, we can compute all the coefficients successively and obtain the power series

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

As is easily shown by the ratio test, this series converges for all values of $x$ and therefore represents a function for which the relations $f'(x) = f(x)$, $f(0) = 1$ are actually fulfilled. (We have avoided here intentionally making any use of what we have previously learned about the expansion of the exponential function!)

Now, the function $e^x$ certainly possesses these properties; we readily deduce that the function $f(x)$ is identical to $e^x$. In fact, if we form the quotient $\phi(x)=f(x)/e^x$ and differentiate it, we find

$$\varphi'(x) = \frac{e^x f'(x) - e^x f(x)}{e^{2x}} = 0.$$

Hence, the function $\phi(x)$ is a constant and since it has the value 1 for $x = 0$, it must be identically equal to 1, thus proving that our power series and the exponential function are identical. (cf. also the analogous discussion in 3.7).

**8.6.2 The Binomial Series:** We can now return to the binomial series (6.3.3), this time using the **method of undetermined coefficients**. We wish to expand the function $f(x) = (1+x)^\alpha$ in a power series, whence we write

$$f(x) = (1 + x)^\alpha = c_0 + c_1 x + c_2 x^2 + \dots ,$$

where the coefficients $c_\nu$ are undetermined. We now note that obviously our function satisfies the relation

$$(1 + x)f'(x) = \alpha f(x) = \sum_{\nu=0}^{\infty} \alpha c_\nu x^\nu.$$

On the other hand, if we differentiate the series for $f(x)$ term by term and multiply by $(1+x)$, we obtain

$$(1+x)f'(x) = c_1 + (2c_2 + c_1)x + (3c_3 + 2c_2)x^2 + \dots;$$

since these two power series for $f'(x)$ must be identical, we have

$$\alpha c_0 = c_1, \quad \alpha c_1 = 2c_2 + c_1, \quad \alpha c_2 = 3c_3 + 2c_2, \quad \dots .$$

It is now certain that $c_0 = 1$, since our series must have the value 1 for $x = 0$; we thus find in succession the expressions

$$c_1 = \alpha, \quad c_2 = \frac{(\alpha - 1)\alpha}{2}, \quad c_3 = \frac{(\alpha-2)(\alpha-1)\alpha}{3 \,.\, 2}, \quad \dots,$$

for the coefficients, and, in general, as is readily established,

$$c_\nu = \frac{(\alpha - \nu + 1)(\alpha - \nu + 2)\dots(\alpha - 1)\alpha}{\nu(\nu - 1)\dots 2 \,.\, 1} = \binom{\alpha}{\nu}.$$

Substituting these values for the coefficients, we have the series $\sum_{\nu=0}^{\infty} \binom{\alpha}{\nu} x^\nu$;

we must yet investigate the convergence of this series and show that it actually represents $(1 + x)^n$.

By the ratio test, we find that, when $\alpha$ is not a positive integer, the series converges if $|x| < 1$ and diverges if $|x| > 1$; in fact, then the ratio of the $(n + 1)$-th term to the $n$-th term is $\{(\alpha - n + 1)/n\}x$ and the absolute value of this expression tends to $|x|$ as $n$ increases beyond all bounds.

We state here without proof the exact conditions under which this series converges. If the index $\alpha$ is an integer $\geq 0$, the series terminates and is therefore valid for all values of $x$ (as it becomes the ordinary binomial theorem). For all other values of $\alpha$, the series is absolutely convergent for $|x|<1$ and diverges for $|x|>1$. For $x = +1$, the series converges absolutely if $\alpha > 0$, conditionally if $-1 < \alpha < 0$, and it diverges if $\alpha \leq -1$. Finally, at $x = -1$, the series is absolutely convergent if $\alpha > 0$, divergent if $\alpha < 0$.

Hence, if $|x| < 1$, our series represents a function $f(x)$ which satisfies the condition $(1 + x)f'(x) = \alpha f(x)$, as follows from the method of forming the coefficients. Moreover, $f(0) = 1$. But these two conditions ensure that the function $f(x)$ is identical to $(1 + x)^{\alpha}$. In fact, setting

$$\varphi(x) = f(x)/(1 + x)^{\alpha},$$

we find that

$$\varphi'(x) = \frac{(1 + x)^{\alpha} f'(x) - \alpha(1 + x)^{\alpha-1} f(x)}{(1 + x)^{2\alpha}} = 0;$$

hence $\phi(x)$ is a constant and is always equal to 1, since $\phi(0) = 1$. We have thus proved that for $|x| < 1$

$$(1 + x)^{\alpha} = \sum_{\nu=0}^{\infty} \binom{\alpha}{\nu} x^{\nu},$$

which is the binomial series.

We present here the following special cases of the binomial series:

The **geometric series**:

$$\frac{1}{1 + x} = (1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - + \ldots = \sum_{\nu=0}^{\infty} (-1)^{\nu} x^{\nu};$$

the series

$$\frac{1}{(1 + x)^2} = (1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + - \ldots = \sum_{\nu=0}^{\infty} (-1)^{\nu} (\nu + 1) x^{\nu},$$

which may otherwise be obtained from the geometric series by differentiation and the series

$$\sqrt{(1+x)} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{2.4}x^2 + \frac{1.3}{2.4.6}x^3 - \frac{1.3.5}{2.4.6.8}x^4 + - \ldots,$$

$$\frac{1}{\sqrt{(1+x)}} = (1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1.3}{2.4}x^2 - \frac{1.3.5}{2.4.6}x^3 + \frac{1.3.5.7}{2.4.6.8}x^4 - + \ldots,$$

the first two or three terms of which are useful expressions.

**8.6.3 The Series for arsin *x*:** This series can be established readily by expansion of the expression $1/\sqrt{(1-t^2)}$ as a binomial series

$$(1-t^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}t^2 + \frac{1.3}{2.4}t^4 + \ldots.$$

This series converges if $|t| \leq 1$, whence it converges uniformly for $|t| \leq q < 1$. On integrating term by term between 0 and $x$, we obtain

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots;$$

by the ratio test, we find that this series converges if $|x| < 1$ and diverges if $|x| > 1$.

The derivation of this series by Taylor's theorem would be decidedly less convenient owing to the difficulty of estimating the remainder.

**8.6.4 The Series for arsinh $x = \log\{x + (1+x^2)^{1/2}\}$:** We obtain this expansion by a similar method. By the binomial theorem, we write down the series for the derivative of arsinh $x$:

$$\frac{1}{\sqrt{(1+x^2)}} = 1 - \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 - \frac{1.3.5}{2.4.6}x^6 + - \ldots,$$

and then integrate term by term to obtain the expansion

$$\operatorname{ar\,sinh} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\frac{x^5}{5} - + \dots .$$

with the interval of convergence $-1 \leq x \leq 1$.

**8.6.5 Example of Multiplication of Series:** The expansion of the function

$$\frac{\log(1+x)}{1+x}$$

is a simple example of the application of the rule for the multiplication of power series. We have only to multiply the logarithmic series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \dots$$

by the geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - + \dots ;$$

the reader should verify that this yields the remarkable expansion

$$\frac{\log(1+x)}{1+x} = x - \left(1 + \frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)x^4 + - \dots$$

for $|x| < 1$.

**8.6.6 Example of Term-by-Term Integration (Elliptic Integral):**. In previous applications, we have encountered the elliptic integral

$$K = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{(1 - k^2 \sin^2 \varphi)}} \quad (k^2 < 1)$$

(the period of oscillation of a pendulum . In order to evaluate the integral, we can first expand the integrand by the binomial theorem:

$$\frac{1}{\sqrt{(1 - k^2 \sin^2 \varphi)}} = 1 + \frac{1}{2} k^2 \sin^2 \varphi + \frac{1 . 3}{2 . 4} k^4 \sin^4 \varphi + \frac{1 . 3 . 5}{2 . 4 . 6} k^6 \sin^6 \varphi + \dots .$$

Since $k^2 \sin^2 \varphi$ is never larger than $k^2$, this series converges uniformly for all values of $\varphi$ and we may integrate term by term:

$$K = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{(1 - k^2 \sin^2 \varphi)}} = \int_0^{\pi/2} d\varphi + \frac{1}{2} k^2 \int_0^{\pi/2} \sin^2 \varphi \, d\varphi + \frac{1 . 3}{2 . 4} k^4 \int_0^{\pi/2} \sin^4 \varphi \, d\varphi + \dots .$$

The integrals here have already been calculated in 4.4.4. Substitution of their values yields

$$K = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{(1 - k^2 \sin^2 \varphi)}} = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 . 3}{2 . 4}\right)^2 k^4 + \left(\frac{1 . 3 . 5}{2 . 4 . 6}\right)^2 k^6 + \dots \right\}.$$

More examples on the theory of series are given in A8.1.

**Exercises 8.4:** Determine the intervals of convergence of the series $\sum_{n=1}^{\infty} a_n x^n$, where $a_n$ is given by 1. - 20.:

1. $\frac{1}{n}$.

2. $n$.

3. $\frac{1}{\sqrt{n}}$.

4. $\sqrt{n}$.

5. $\frac{1}{n^2}$.

6. $\frac{n}{n!}$.

7. $\frac{1}{a+n}$.

8. $\frac{1}{an+b}$.

9. $\frac{1}{\log(n+1)}$.

10. $\frac{1}{\log\log 10n}$.

11. $\frac{1}{\sqrt[n]{n}}$.

12. $a^n$.

13. $a^{\sqrt{n}}$.

14. $a^{\log n}$.

15. $(\sqrt[n]{n}-1)^n$.

16. $\frac{(n!)^2}{(2n)!}$.

17. $\frac{n+\sqrt{n}}{n^2-n}$.

18. $\frac{1}{1+a^n}$.

19. $\frac{1}{\sqrt{n}}+\frac{(-1)^n}{n}$.

20. $\frac{1}{n^{1+1/n}}$.

Expand in power series the functions 21. - 26.:

21. $a^x$.

22. $\frac{x+\log(1-x)}{x^2}$.

23. $\sin^2 x$.

24. $\cos^2 x$.

25. $\sin^6 x$.

26. $\operatorname{arc}\sin x^3$.

27. Using the binomial series, compute $\sqrt{2}$ to four decimals.

28. Obtain approximations by series for the following integrals by expanding them in power series and their integration:

(a) $\int_0^1 \frac{\sin x}{x}\,dx$.

(b) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^4)}}$.

(c) $\int_0^1 \frac{\log(1+x)}{x}\,dx$.

(d) $\int_5^{10} \frac{dx}{\sqrt{(1+x^4)}}$.

29. Obtain by multiplication of power series the expansion of the following expressions up to the terms in $x^4$:

(a) $e^x \sin x$.

(b) $[\log(1+x)]^2$.

(c) $\dfrac{\text{arc}\sin x}{\sqrt{1-x}}$.

(d) $\sin^3 x$.

30.* Prove by multiplication of power series that

(a) $e^x e^y = e^{x+y}$. (b) $\sin 2x = 2 \sin x \cos x$.

31. If the interval of convergence of the power series $\Sigma a_n x^n$ is $|x| < \rho$ and that of $\Sigma b_n x^n$ is $|x| < \rho'$, where $\rho < \rho'$, what is the interval of convergence of $\Sigma(a_n + b_n)x^n$?

33. By the method of undetermined coefficients, find the function $f(x)$ which satisfies the conditions:

(a) $f(0) = 3$; (b) $f'(x) = f(x) + x$.

Answers and Hints

## 8.7 Power Series with Complex Terms

**8.7.1 Introduction of Complex Terms into Power Series:** The similarity between certain power series representing functions which are apparently unrelated led Euler to set up a purely formal link between them, found by giving complex values, in particular, pure imaginary values to the variable $x$. We shall first do this formally, unhindered by questions of rigour and investigate the results of the process.

The first striking relation of this sort is obtained, if we replace the quantity $x$ in the series for $e^x$ by a pure imaginary $i\phi$, where $\phi$ is a real number. If we recall the fundamental equation for the imaginary unit $i$, that is, $i^2 = -1$, from which it follows that $i^3 = -i$, $i^4 = 1$, $i^5 = i$, ····, then, on separating the real and imaginary terms of the series, we obtain

$$e^{i\phi} = \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + - \dots\right) + i\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + - \dots\right),$$

or, in another form,

$$e^{i\phi} = \cos\phi + i\sin\phi.$$

This is the well-known and important **Euler formula**. which is as yet purely formal. It is consistent with De Moivre's theorem which is expressed by the equation

$$(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi) = \cos(\phi + \psi) + i\sin(\phi + \psi).$$

By virtue of Euler's formula, this equation merely states that the relation

$$e^x \, . \, e^y = e^{x+y}$$

continues to hold for pure imaginary values *x* = *iϕ*, *y* = *iψ*.

If we replace the variable *x* in the power series for cos *x* by the pure imaginary *ix*, we obtain at once the series for cosh *x*; this relation can be expressed by the equation

$$\cosh x = \cos ix.$$

In the same way, we obtain

$$\sinh x = \frac{1}{i}\sin ix.$$

Since Euler's formula also yields

$$e^{-i\phi} = \cos\phi - i\sin\phi,$$

we arrive at the exponential expressions for the trigonometric functions

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

These are exactly analogous to the exponential expressions for the hyperbolic functions and are, in fact, transformed into them by the relations

$$\cosh x = \cos ix, \; \sinh x = \frac{1}{i} \sin ix.$$

Naturally, corresponding formal relations can also be obtained for the functions tan *x*, tanh *x*, cot *x* and coth *x*, which are interrelated by the equations

$$\tanh x = \frac{1}{i} \tan ix, \;\; \coth x = i \cot ix.$$

Finally, similar relations can also be found for the inverse trigonometric and hyperbolic functions. For example, we immediately find from

$$y = \tan x = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})} = \frac{e^{2ix} - 1}{i(e^{2ix} + 1)}$$

that

$$e^{2ix} = \frac{1 + iy}{1 - iy}.$$

It we take the logarithms of both sides of this equation and then write *x* instead of *y* and artan *x* instead of *x*, we obtain the equation

$$\operatorname{artan} x = \frac{1}{2i} \log \frac{1 + ix}{1 - ix},$$

which expresses a remarkable connection between the inverse tangent and the logarithm. If we replace in the known power series for ½log(1 + *x*)/(1 - *x*) (6.1.1) *x* by *ix*, we actually obtain the power series for artan *x*:

$$\operatorname{artan} x = \frac{1}{i}\left(ix + \frac{(ix)^3}{3} + \frac{(ix)^5}{5} + \dots\right) = x - \frac{x^3}{3} + \frac{x^5}{5} - + \dots .$$

The above relations have as yet a purely formal character and naturally demand a more exact statement as to the meaning which they are intended to convey. In the next sub-section, we shall indicate how this can be given with the help of function theory.

However, we shall later on only require Euler's formula $e^{i\phi} = \cos\phi + i\sin\phi$. This being so, we can avoid here a thorough analysis. We need only regard the symbol e $^{i\phi}$ as a formal abbreviation for $\cos\phi + i\sin\phi$, in which case De Moivre's formula $e^{i\phi}\cdot e^{i\psi} = e^{i(\phi+\psi)}$ appears merely as a consequence of the elementary addition theorem of trigonometry. From this formal point of view, in order to make the relation $e^{x} . e^{y} = e^{x+y}$ e*. e* remain valid for any complex arguments, we set up the further definition

$$e^{x} = e^{\xi}(\cos\eta + i\sin\eta),$$

where $x = \xi + i\eta$ ($\xi, \eta$ real).

**8.7.2 A Glance at the General Theory of functions of a Complex Variable:** Although the purely formal point of view, indicated above, is by itself free from objection, it is still desirable to recognize in the above formula something more than a mere formal connection. In order to pursue this aim, we are led to the **general theory of functions**, as (for the sake of brevity) we call the general theory of the so-called **analytic functions of a complex variable**. In this, we may use as our starting point a general discussion of the theory of power series with complex variables and complex coefficients. The construction of such a theory of power series presents no difficulties once we have defined the concept of limit in the domain of complex numbers; in fact, it follows almost exactly the theory of real power series . But as we shall not make any use of these matters in what follows, we shall content ourselves here by stating certain facts and omit the proofs. It is found that the following generalization of the theorem of 8.5.1 holds for complex power series:

If a power series converges for any complex value $x = \xi$ whatever, then it converges absolutely for every value $x$ for which $|x| < |\xi|$; if it diverges for a value $x = \xi$, then it diverges for every value $x$ for which $|x| > |\xi|$. A power series, which does not converge everywhere, but does converge for some other point in addition to $x = 0$, has a **circle of convergence**, that is, there exists a number $\rho > 0$ such that the series converges absolutely for $|x| < \rho$ and diverges for $|x| > \rho$.

Having established the concept of functions of a complex variable, represented by power series, and having developed the rules for operating with such functions, we can think of the functions $e^{x}$, sin $x$, cos $x$, artan $x$, etc., of

the complex variable $x$ as simply defined by the power series which represent them for real values of $x$. Then, all the above formal relationships reduce to trivialities.

We shall merely indicate by two examples, how this introduction of complex variables helps us to understand the elementary functions. The geometric series for $1/(1 + x^2)$ ceases to converge when $x$ leaves the interval $-1 \leq x \leq 1$ and so does the series for artan $x$, although there are no peculiarities in the behaviour of these functions at the ends of the interval of convergence; in fact, they and all their derivatives are continuous for all real values of $x$. On the other hand, we can readily understand that the series for $1/(1 - x^2)$ and $\log(1 - x)$ cease to converge as $x$ passes through the value 1, since they become infinite there. But this divergence of the series for the inverse tangent and the series

$$\sum_{\nu=0}^{\infty} (-1)^{\nu} x^{2\nu} \quad \text{for} \quad |x| > 1$$

becomes clear immediately, if we also consider complex values of $x$. In fact, we find that when $x = i$, the sum functions become infinite and so cannot be represented by a convergent series. Hence, by our theorem about the circle of convergence, the series must diverge for all values of $x$ such that $|x| > |i|$; in particular, for real values of $x$, the series diverge outside the interval $-1 \leq x \leq 1$.

Another example is given by the function $f(x) = e^{-1/x^2}$ for $x \neq 0$ (cf.A3.1.1, A6.1), which, in spite of its apparently regular behaviour, cannot be expanded in a Taylor series. As a matter of fact, this function ceases to be continuous, if we take pure imaginary values of $x = i\xi$ into account. The function then takes the form $e^{1/\xi^2}$ and increases beyond all bounds as $\xi \to 0$. It is therefore clear that no power series in $x$ can represent this function for all complex values of $x$ in a neighbourhood of the origin, no matter how small a neighbourhood we choose.

These remarks on the theory of functions and power series of a complex variable must be sufficient for us here.

# Appendix to Chapter VIII

## A8.1 Multiplication and Division of Series

**A8.1.1 Multiplication of Absolutely Convergent Series:** Let

$$A=\sum_{\nu=0}^{\infty} a_\nu, \quad B=\sum_{\nu=0}^{\infty} b_\nu$$

be two absolutely convergent series. Together with these series, consider the corresponding series of absolute values

$$\bar{A}=\sum_{\nu=0}^{\infty} |a_\nu| \quad \text{and} \quad \bar{B}=\sum_{\nu=0}^{\infty} |b_\nu|.$$

Moreover, let

$$A_n=\sum_{\nu=0}^{n} a_\nu, \quad B_n=\sum_{\nu=0}^{n} b_\nu, \quad \bar{A}_n=\sum_{\nu=0}^{n} |a_\nu|, \quad \bar{B}_n=\sum_{\nu=0}^{n} |b_\nu|$$

$$c_n=a_0b_n+a_1b_{n-1}+\ldots+a_nb_0.$$

We now assert that the series $\sum_{\nu=0}^{\infty} c_n$ is absolutely convergent and that its sum is equal to $A \cdot B$.

In order to prove this, we write down the series

$$\begin{aligned} &a_0b_0+a_1b_0+a_1b_1+a_0b_1+a_2b_0+a_2b_1\\ &+a_2b_2+a_1b_2+a_0b_2+\ldots+a_nb_0+a_nb_1\\ &+\ldots+a_nb_n+\ldots+a_1b_n+a_0b_n+\ldots, \end{aligned}$$

the $n^2$-th partial sum of which is $A_nB_n$, and we assert that it converges absolutely. In fact, the partial sums of the corresponding series with absolute values increase monotonically; the $n^2$-th partial sum is equal to $\bar{A}_n\bar{B}_n$, which is less than $\bar{A}\bar{B}$ (and which tends to $\bar{A}\bar{B}$). The series with absolute values therefore converges and the series written down above converges absolutely. The sum of the series is obviously $AB$, since its $n^2$-th partial sum is $A_nB_n$, which tends to $AB$ as $n \to \infty$. We now interchange the order of the terms, which is permissible for absolutely convergent series, and bracket successive terms together. In a convergent series, we may bracket successive terms in as many places as we desire without disturbing the convergence or altering the sum of the series, because, if we bracket, say,

the terms $(a_{n+1} + a_{n+2} + \ldots + a_m)$, then, if we form the partial sum, we shall omit those partial sums which originally fell between $s_n$ and $s_m$, which does not affect the convergence or change the value of the limit. Also, if the series was absolutely convergent before the brackets were inserted, it remains absolutely convergent. Since the series

$$\sum_{\nu=0}^{\infty} c_\nu = (a_0 b_0) + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \ldots$$

is formed in this way from the series written down above, the required proof is complete.

**A8.1.2 Multiplication and Division of Power Series:** The principal use of our theorem is found in the theory of power series. The following assertion is an immediate consequence of it:

The product of the two power series

$$\sum_{\nu=0}^{\infty} a_\nu x^\nu \quad \text{and} \quad \sum_{\nu=0}^{\infty} b_\nu x^\nu$$

is represented in the interval of convergence common to the two power series by a third power series $\sum_{\nu=0}^{\infty} c_\nu x^\nu$, the coefficients of which are given by

$$c_\nu = a_0 b_\nu + a_1 b_{\nu-1} + \ldots + a_\nu b_0.$$

As regards the division of power series, we can likewise represent the quotient of the two power series above by a power series $\sum_{\nu=0}^{\infty} q_\nu x^\nu$, provided $b_0$, the constant term in the denominator, does not vanish. (In the latter case, such a representation is, in general, not possible; indeed, it could not converge at $x$=0 on account of the vanishing of the denominator, while, on the other hand, every power series must converge at $x$ = 0.) The coefficients of the power series

$$\sum_{\nu=0}^{\infty} q_\nu x^\nu$$

can be calculated by recalling that

$$\sum_{\nu=0}^{\infty} q_\nu x^\nu \cdot \sum_{\nu=0}^{\infty} b_\nu x^\nu = \sum_{\nu=0}^{\infty} a_\nu x^\nu,$$

so that the following equations must be true:

$$\begin{aligned} a_0 &= q_0 b_0, \\ a_1 &= q_0 b_1 + q_1 b_0, \\ a_2 &= q_0 b_2 + q_1 b_1 + q_2 b_0, \\ &\cdots\cdots\cdots\cdots \\ a_\nu &= q_0 b_\nu + q_1 b_{\nu-1} + \ldots + q_\nu b_0. \end{aligned}$$

We find readily from the first of these equations $q_0$, from the second one $q_1$, from the third one (by using the values $q_0$ and $q_1$) the value of $q_2$, etc. In order to strictly justify the expression of the quotient of two power series by the third power series, we must still investigate the convergence of the formally calculated power series $\sum_{\nu=0}^{\infty} q_\nu x^\nu,$ the general investigation of which we will omit. We shall be content with the statement that the series for the quotient does actually converge, provided *x* remains within a sufficiently small interval, in which the denominator does not vanish and both the numerator and the denominator are convergent series.

## A8.2 Infinite Series and Improper integrals

The infinite series and the concepts developed in connection with them have simple applications and analogies in the theory of improper integrals. We shall confine ourselves here to the case of a convergent integral with an infinite interval of integration, say an integral of the form $\int_0^\infty f(x)\,dx.$ If we subdivide the interval of integration by a sequence of numbers $x_0 = 0, x_1, \cdots$, which tends monotonically to $\infty$, we can write the improper integral in the form

$$\int_0^\infty f(x)\,dx = a_1 + a_2 + \ldots,$$

where each term of our infinite series is an integral:

$$a_1 = \int_0^{x_1} f(x)\,dx, \quad a_2 = \int_{x_1}^{x_2} f(x)\,dx, \ldots,$$

etc. This is true no matter how we choose the points $x_\nu$, whence we can reduce the idea of a convergent improper integral to that of an infinite series in many ways.

It is especially convenient to choose the points $x_\nu$ so that the integrand does not change sign within any individual sub-interval. The series $\sum_{\nu=1}^{\infty} |a_\nu|$ will then correspond to the integral of the absolute value of our function

$$\int_0^\infty |f(x)|\,dx.$$

We are thus led naturally to the concept: An improper integral $\int_0^\infty f(x)\,dx$ is said to be absolutely convergent, if there exists the integral $\int_0^\infty |f(x)|\,dx.$ Otherwise, if our integral exists at all, we shall say that it is **conditionally convergent**.

Some of the integrals considered earlier (4.8.4) such as

$$\int_0^\infty \frac{1}{1+x^2}\,dx, \quad \int_0^\infty e^{-x^2}\,dx, \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt,$$

are absolutely convergent. On the other hand, the integral

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{A\to\infty} \int_0^A \frac{\sin x}{x}\,dx,$$

studied in 4.8.5, is a simple example of a conditionally convergent integral. In order to give a proof of the convergence of this integral, which is independent of the former proof, we subdivide the interval from 0 to $A$ at the points $x_\nu = \nu\pi$ ($\nu = 0, 1, 2, \cdots, \mu_A$), where $\mu_A$ is the largest possible integer for which $\mu_A \pi \le A$. Hence we divide the integral into terms of the form

$$a_\nu = \int_{(\nu-1)\pi}^{\nu\pi} \frac{\sin x}{x}\,dx \quad (\nu = 1, 2, \ldots),$$

and a remainder $R_A$ of the form

$$\int_{\mu_A\pi}^{A} \frac{\sin x}{x}\,dx \quad (0 \leqq A - \mu_A\pi < \pi).$$

It is clear that the quantities $a_\nu$ have alternating signs, since $\sin x$ is alternately positive and negative in consecutive intervals. Moreover, $|a_{\nu+1}| < |a_\nu|$; indeed, on applying the transformation $x = \xi - \pi$, we have

$$|a_\nu| = \int_{(\nu-1)\pi}^{\nu\pi} \frac{|\sin x|}{x}\,dx = \int_{\nu\pi}^{(\nu+1)\pi} \frac{|\sin(\xi - \pi)|}{\xi - \pi}\,d\xi = \int_{\nu\pi}^{(\nu+1)\pi} \frac{|\sin \xi|}{\xi - \pi}\,d\xi,$$

$$> \int_{\nu\pi}^{(\nu+1)\pi} \frac{|\sin \xi|}{\xi}\,d\xi = |a_{\nu+1}|.$$

Hence we see, by Leibnitz's test , that $\Sigma\alpha_\nu$ converges. Moreover, the remainder $R_A$ has the absolute value

$$|R_A| = \left| \int_{\mu_A\pi}^{A} \frac{\sin x}{x}\,dx \right| \leqq \int_{\mu_A\pi}^{(\mu_A+1)\pi} \frac{|\sin x|}{x}\,dx$$

$$\leqq \frac{1}{\mu_A\pi} \int_{\mu_A\pi}^{(\mu_A+1)\pi} |\sin x|\,dx = \frac{2}{\mu_A\pi},$$

and this tends to 0 as $A$ increases. Thus, if we let A tend to $\infty$ in the equation

$$\int_0^A \frac{\sin x}{x}\,dx = a_1 + a_2 + a_3 + \ldots + a_{\mu_A} + R_A,$$

the right hand side tends to $\Sigma\alpha_\nu$ as a limit and our integral is convergent. However, the convergence is not absolute, because

$$|a_\nu| > \int_{(\nu-1)\pi}^{\nu\pi} \frac{|\sin x|}{\nu\pi}\,dx = \frac{2}{\nu\pi}, \text{ so that } \Sigma|a_\nu| \text{ diverges.}$$

## A8.3 Infinite Products

In the introduction to this chapter (8.1), we drew attention to the fact that infinite series are only one way, although a particularly important way of representing numbers or functions by infinite processes. As an example of another such process, we shall introduce statements on infinite products without proofs.

We have dealt with Wallis' product

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$$

in which the number $\pi/2$ is expressed as an infinite product. We mean by the value of the infinite product

$$\prod_{\nu=1}^{\infty} a_\nu = a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdots$$

the limit of the sequences of partial products

$$a_1, \quad a_1 \cdot a_2, \quad a_1 \cdot a_2 \cdot a_3, \quad a_1 \cdot a_2 \cdot a_3 \cdot a_4, \quad \ldots,$$

provided it exists.

Naturally, the factors $a_1$, $a_2$. $a_3$, $\cdots$ may also be functions of a variable $x$. An especially interesting example is the infinite product for the function sin $x$

$$\sin \pi x = \pi x \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\cdots,$$

which we shall obtain in 9.4.8.

The infinite product for the **zeta function** has a very important role in the **theory of numbers**. In order to retain the notation, used in number theory, we denote here the independent variable by $s$ and define the zeta function for $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We know from 8.2.3 that the series on the right-hand side converges if $s > 1$. If $p$ is any number larger than 1, we obtain the equation

$$\frac{1}{1 - \frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots$$

by expanding the geometric series. If we imagine this series written down for all the **prime numbers** $p_1, p_2, p_3, \cdots$ in increasing order of magnitude and all the equations thus formed multiplied together, we obtain on the left-hand side a product of the form

$$\frac{1}{1 - p_1^{-s}} \cdot \frac{1}{1 - p_2^{-s}} \cdots.$$

If, without stopping to justify the process in any way, we multiply together the series on the right-hand sides of our equations and, in addition, recall that by an elementary theorem each integer $n > 1$ can be expressed in one and only one way as a product of powers of different prime numbers, we find that the product on the right hand side is again the function $\zeta(s)$, whence we obtain the remarkable **product form**

$$\zeta(s) = \frac{1}{1 - p_1^{-s}} \cdot \frac{1}{1 - p_2^{-s}} \cdot \frac{1}{1 - p_3^{-s}} \cdots.$$

This product form, the derivation of which we have only sketched here briefly, is actually an expression of the zeta function as an infinite product, since the number of prime numbers is infinite.

In the general theory of infinite products, we usually exclude the case where the product $a_1 \cdot a_2 \cdot a_3 \cdots$ has the limit zero, whence it is especially important that none of the factors $a_n$ should vanish. In order that the product may

converge, the factors $a_n$ must accordingly tend to 1 as $n$ increases. Since we can, if necessary, omit a finite number of factors (this has no bearing on the question of convergence), we may assume that $a_n > 0$. This case is the topic of the theorem:

A necessary and sufficient condition for the convergence of the product $\prod_{\nu=1}^{\infty} a_\nu$, where $a_\nu > 0$ is that the series $\sum_{\nu=1}^{\infty} \log a_\nu$ should converge.

Indeed, it is clear that the partial sums $\sum_{\nu=1}^{n} \log a_\nu = \log (a_1 a_2 \ldots a_n)$ of this series will tend to a definite limit, if and only if the partial products $a_1 a_2 a_3 \cdots a_n$ possess a positive limit.

In studying convergence, we usually apply the following criterion (a sufficient condition), where we put $a_\nu = 1 + a_\nu$ :

The product

$$\prod_{\nu=1}^{\infty} (1 + \alpha_\nu)$$

converges, if the series

$$\sum_{\nu=1}^{\infty} |\alpha_\nu|$$

converges and no factor $(1 + a_\nu)$ is zero.

In the proof, we may assume, if necessary after omission of a finite number of factors, that each $|a_\nu| < 1/2$. We have then $1 - |a_\nu| > 1/2$. By the mean value theorem,

$$\log(1 + h) = \log(1 + h) - \log 1 = h \frac{1}{1 + \theta h}$$

for $0 < \theta < 1$, whence

$$|\log(1+a_\nu)| = \left|\frac{a_\nu}{1+\theta a_\nu}\right| \leq \frac{|a_\nu|}{1-|a_\nu|} \leq 2|a_\nu|,$$

and so the convergence of the series $\sum_{\nu=1}^{\infty} \log(1+a_\nu)$ follows from the convergence of $\sum_{\nu=1}^{\infty} |a_\nu|$.

It follows from our criterion that the infinite product, given above for $\sin \pi x$, converges for all values of $x$ except for $x = 0, \pm 1, \pm 2 \cdots$, where factors of the product are zero. Moreover, for $p \geq 2$ and $s > 1$, we readily find that

$$\frac{1}{1-p^{-s}} = 1 + \frac{1}{p^s - 1}, \quad 0 < \frac{1}{p^s - 1} < \frac{2}{p^s}.$$

Now, if we let $p$ assume all prime values, the series $\Sigma \frac{1}{p^s}$ must converge, since the terms form only a part of the convergent series $\sum_{\nu=1}^{\infty} \frac{1}{\nu^s}$. The convergence of the product $\Pi \frac{1}{1-p^{-s}}$ for $s > 1$ has thus been proved.

## A8.4 Series involving Bernoulli Numbers

So far, we have given no expansions in power series for certain elementary functions, e.g., tan $x$. The reason is that the numerical coefficients which occur are not of any very simple form. We can express these coefficients and those in the series for a number of other functions in terms of the solaced Bernoulli numbers. These numbers are certain rational numbers with a not very simple law of formation which occur in many parts of analysis. We arrive at them most simply by expanding the function

$$\frac{x}{e^x - 1} = \frac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots}$$

in a power series of the form

$$\frac{x}{e^x - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} x^\nu.$$

If we write this equation in the form

$$x = (e^x - 1) \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} x^\nu$$

and substitute on the right hand side the power series for $e^x$ - 1, we obtain, as in A8.1.2, a recurrence relation which allows the computation of all the numbers $B_\nu$. These numbers are called Bernoulli numbers.

In some publications, a slightly different notation is employed with the basic formula

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \frac{B_\nu}{(2\nu)!} x^{2\nu}.)$$

They are rational, since in their formation only rational operations are used; as we easily verify, they vanish for all odd indices other than $\nu = 1$. The first few Bernoulli numbers are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42},$$

$$B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \ldots$$

We must content ourselves with a brief hint as to how these numbers are involved in the power series under consideration. Firstly, the transformation

$$1 + \frac{B_2}{2!}x^2 + \ldots = \frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1} = \frac{x}{2} \cdot \frac{e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}}{e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}}$$

yields

$$\frac{x}{2} \coth \frac{x}{2} = \sum_{\nu=0}^{\infty} \frac{B_{2\nu}}{(2\nu)!} x^{2\nu}.$$

If we replace $x$ by $2x$, we obtain

$$x \coth x = \sum_{\nu=0}^{\infty} \frac{2^{2\nu} B_{2\nu}}{(2\nu)!} x^{2\nu},$$

for $|x| < \pi$, after replacing $x$ by $-x$,

$$x \cot x = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{2^{2\nu} B_{2\nu}}{(2\nu)!} x^{2\nu}, \qquad |x| < \pi.$$

Now, using the equation $2 \cot 2x = \cot x - \tan x$, we find the series

$$\tan x = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{2^{2\nu}(2^{2\nu}-1)}{(2\nu)!} B_{2\nu} x^{2\nu-1},$$

valid for $|x| < \pi/2$.

For further information, we refer the reader, for example, to the more detailed treatise of K. Knopp, Theory and Application of Infinite Series, p. 183, (Blackie & Son, Ltd., 1928.

**Exercises** 8.5:

1. Prove that the power series for $\sqrt{(1-x)}$ still converges when $x = 1$.

2. Prove that there is for every positive $\varepsilon$ a polynomial in $x$ which represents $\sqrt{(1-x)}$ in the interval $0 \leq x \leq 1$ with an error less than $\varepsilon$.

3. Prove that there is for every positive $\varepsilon$ a polynomial in $t$ which represents $|t|$ in the interval $-1 \leq t \leq 1$ with an error less than $\varepsilon$.

4. **Weierstrass' Approximation Theorem:** Prove that, if $f(x)$ is continuous in $a \leq x \leq b$, then there exists for every positive $\varepsilon$ a polynomial $P(x)$ such that $|f(x)-P(x)| < \varepsilon$ for all values of $x$ in the interval $a \leq x \leq b$.

5. Prove that the following infinite products converge:

$$\prod_{n=1}^{\infty}(1+(\tfrac{1}{2})^{2n}); \qquad \prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}; \qquad \prod_{n=1}^{\infty}\left(1-\frac{z^n}{n}\right), \text{ if } |z|<1.$$

6. Prove by the above methods that $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ diverges.

7. Use the identity

$$\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{i=1}^{\infty}\left(\frac{1}{1-p_i^{-s}}\right)$$

(where $p_i$ is the $i$-th prime number) to prove that the number of primes is infinite.

8. Prove the identity

$$\prod_{\nu=1}^{\infty}(1+x^{2^\nu})=\frac{1}{1-x}$$

for $|x| < 1$.

Answers and Hints

# Chapter IX

# Fourier Series

Besides the power series, there is another class of infinite series which has a particularly important role both in pure and applied mathematics. These are the Fourier series, in which the individual terms are trigonometric functions and the sum is a periodic function.

## 9.1 Periodic Functions

**9.1.1 General Remarks:** Periodic functions of the time, i.e., functions which repeat their course after a definite interval of time, are encountered in many .applications. In most **machines**, a periodic process occurs in rhythm

with the rotation of a flywheel, e.g., the alternating current developed by a dynamo. Periodic functions are also associated with all **vibration phenomena**.

A periodic function with period $2l$ is represented by the equation

$$f(x+2l)=f(x),$$

true for all values of $x$. We call especially attention to the fact that $2l$ is called the **period**.

In representing periodic functions, it is often convenient to denote the independent variable $x$ by a point on the circumference of a circle instead of the usual point on a straight line. If a function $f(x)$ has the period $2\pi$, i.e., if the equation

$$f(x+2\pi)=f(x)$$

holds for all values of $x$ and we denote by $x$ the angle at the centre of a circle of unit radius, which is included between an arbitrary initial radius and the radius to a variable point on the circumference, then the periodicity of the function $f(x)$ is expressed simply by the fact that there corresponds to each point on the circumference just one value of the function. For example, in the case of a machine, **periodicity** may be expressed in terms of the position of a point on the flywheel.

It is worth noting that in addition to the period $21$, the function $f\{x)$ necessarily has also the period $4l$, since $f(x+4l)$ $=f(x+2l)=f(x)$; similarly, $f(x)$ has the periods $6l$, $8l$, $\cdots$; and it is also possible (though not necessarily true) that $f(x)$ may have shorter periods such as $l$ or $l/5$. Graphically speaking, the graph of the function has in any two consecutive intervals of length $21$ exactly the same shape. In order to have available a second interpretation, which some readers may prefer, we may think of the variable $x$ as the time and write, accordingly, sometimes $t$ instead of $x$; the function $f(x)$ then represents a periodic process or, as we shall also say, a vibration (or oscillation). The period $2l = T$ is then called the period of vibration (or period of oscillation).

If any arbitrary function $f(x)$ is given in a definite interval, say. $-l \le x \le l$, it can always be extended as a periodic function; we have only to define $f(x)$ outside the interval by the equation $f(x + 2nl) = f(x)$, where $n$ is an arbitrary positive or negative integer. We must point out here that, if $f(x)$ is continuous in the interval $-l \le x \le l$, but $f(-l)$ $\neq f(l)$, our extended periodic function will be discontinuous at the points $\pm l, \pm 3l, \cdots$, (Figs. 7 and Fig. 8), where $l=\pi$. Moreover, in this case, the extension fails to yield a single-valued function $f(x)$ at the points $x = \pm l, \pm 3l, \cdots$, since, for example, we have defined $f(3l)$ as $f(l+2l)$, which gives $f(3l) = f(l)$, and we also have defined it as $/(-l+4l)$, which yields $f(3l) = f(-l)$. We avoid this difficulty by not extending the function as defined for $-l \le x \le l$, but as defined either for $-l < x \le l$ or $-l \le x < l$, i.e., we either discard the original value $f(-l)$ or the original value $f(+l)$.

We should point out here a general fact,, which relates to periodic functions and is expressed by the equation

$$\int_{-l-a}^{l-a} f(x)\,dx = \int_{-l}^{l} f(x)\,dx,$$

or, in words: The integral of a periodic function over an interval, the length of which is one period $T == 2l$, has always the same value, no matter where the interval lies.

In order to prove this, we need only note that, by virtue of the equation $f(\xi - 2l) = f(\xi)$, the substitution $x = \xi - 2l$ yields

$$\int_{\alpha}^{\beta} f(x)\,dx = \int_{\alpha+2l}^{\beta+2l} f(\xi)\,d\xi = \int_{\alpha+2l}^{\beta+2l} f(x)\,dx.$$

In particular, for $\alpha = -l - \alpha$ and $\beta = -l$, it follows that

$$\int_{-l-a}^{-l} f(x)\,dx = \int_{l-a}^{l} f(x)\,dx,$$

whence

$$\int_{-l-a}^{l-a} f(x)\,dx = \int_{-l-a}^{-l} f(x)\,dx + \int_{-l}^{l-a} f(x)\,dx$$

$$= \int_{l-a}^{l} f(x)\,dx + \int_{-l}^{l-a} f(x)\,dx = \int_{-l}^{l} f(x)\,dx,$$

which proves our statement. If we recall the geometrical meaning of the integral, the statement is made obvious by Fig. 1.

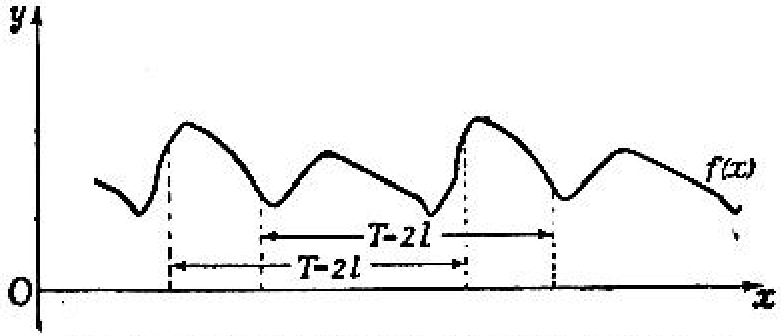



Fig. 1.—To illustrate the integral over a whole period

The simplest periodic functions, from which we shall later compose the most general periodic functions, are $a \sin\omega x$ and $a \cos\omega x$, or, more generally, $a\sin\omega(x - \xi)$ and $a\cos\omega(x - \xi)$, where $a(\geq 0)$, $\omega(\geq 0)$ and $\xi$ are constants. We call the processes represented by these functions* **sinusoidal vibrations** or **simple harmonic vibrations** (or **oscillations**). The period of vibration is $T=2\pi/\omega$. The number $\omega$ is called the **circular frequency** of the vibration**; since $1/T$ is the number of vibrations in unit time or **frequency**, $\omega$ is the number of vibrations in time $2\pi$. The number $a$ is called the **amplitude** of the vibration; it represents the maximum value of the function $a\sin\omega(x - \xi)$ or $a\cos\omega(x - \xi)$, since both sine and cosine have the maximum value 1. The number $\omega(x - \xi)$ is called the **phase** and the number $\omega\xi$ the **phase displacement**.

* Either of these formulae taken by itself (for all values of $a$ and $\xi$) represents the class of all sinusoidal vibrations; the two formulae are equivalent to one another, since $a\sin\omega(x - \xi) = a\cos\omega\{x - (\xi + \pi/2\omega)\}$.

**The reader should take care to distinguish between the terms frequency and **circular frequency**!

We obtain these functions graphically by stretching the sine curve in the ratios $1 : \omega$ along the $x$-axis and $a : 1$ along the $y$-axis and then translating the curve by a distance $\xi$ along the $x$-axis (Fig. 2) in the positive direction.

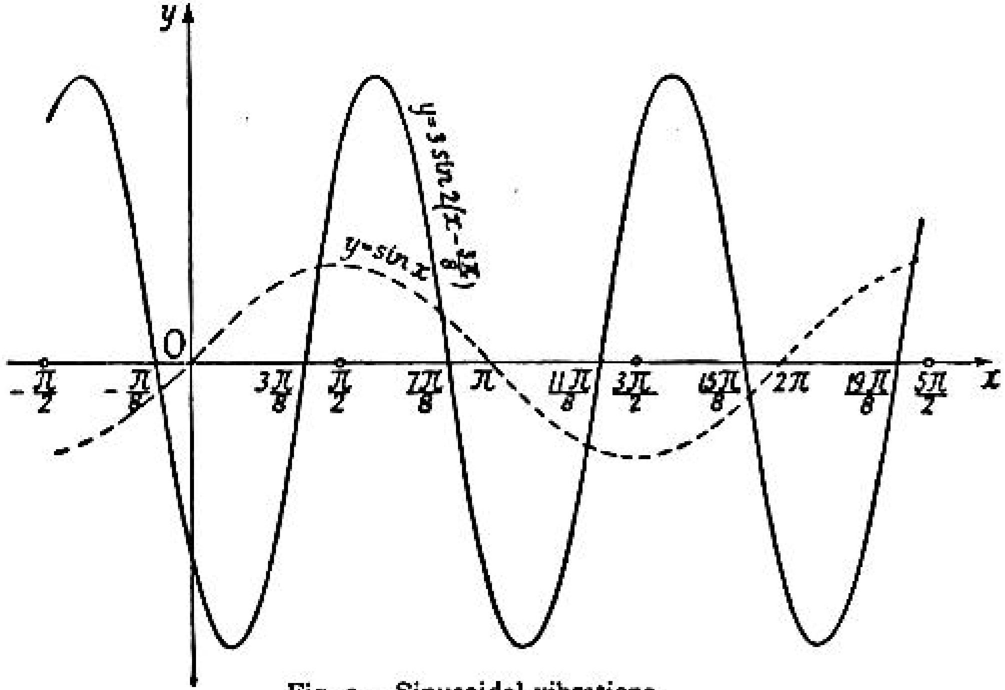



Fig. 2.—Sinusoidal vibrations

We can also repress sinusoidal vibrations by the addition formula for the trigonometric functions in the form

$$\alpha \cos \omega x + \beta \sin \omega x \quad \text{and} \quad \beta \cos \omega x - \alpha \sin \omega x$$

respectively, where $\alpha = -a\sin\omega\xi$ and $\beta = a\cos\omega\xi$. Conversely, every function of the form

$$\alpha \cos \omega x + \beta \sin \omega x$$

represents a sinusoidal vibration $a\sin\omega(x - \xi)$ with the amplitude $a = \sqrt{(\alpha^2 + \beta^2)}$ and the phase displacement $\omega\xi$ given by the equations $\alpha = -a\sin\omega\xi$ and $\beta = a\cos\omega\xi$. By using the expression $\alpha\sin\omega\xi + \beta\cos\omega\xi$, we see that the sum of two or more such functions with the same circular frequency $\omega$ always represent another sinusoidal vibration with the circular frequency $\omega$.

**9.1.2 Superposition of Sinusoidal Vibrations. Harmonics. Beats:**. Although many vibrations are found to be sinusoidal ( 5.4.4), it is nevertheless true that most periodic motions have a more complicated character, being obtained by superposition of sinusoidal vibrations. Mathematically speaking, this simply means that the motion,

e.g., the distance of a point from its initial position as a function of the time, is given by a function which is the sum of a number of pure periodic functions of the above type. The sine waves of the function are then piled up on top of one another (that is, their ordinates are added), or, as we say, they are **superposed**. In this superposition, we assume that all the circular frequencies (and, of course, the periods) of the superposed vibrations differ, because the superposition of two sinusoidal vibrations with the same circular frequency gives us another sinusoidal vibration with the same circular frequency (but with a different amplitude and phase displacement), as shown above.

If we consider the simplest case, the superposition of two sinusoidal vibrations with the circular frequencies $\omega_1$ and $\omega_2$, we find that there are two fundamentally different cases, depending on whether the two circular frequencies have a rational ratio or not, or, as we say, whether they are **commensurable** or **incommensurable**. We begin with the first case and, by way of an example, take the second circular frequency to be twice the first: $\omega_2=2\omega_1$. The period of the second vibration will then be half the period of the first vibration, $2\pi/2\omega_1 = T_2 = T_1/2$, whence it will necessarily have not only the period $T_2$, but also the doubled period $T_1$, since the function repeats itself after this double period; and the function after their superposition will also have the period $T_1$. The second vibration with twice the circular frequency and half the period of the first vibration is called the first **harmonic** of the first vibration (the **fundamental**).

Corresponding statements hold, if we introduce a further vibration with the circular frequency $\omega_3 = 3\omega_1$. Here again, the vibration function $\sin 3\omega_1 x$ will necessarily repeat itself with the period $2\pi/\omega_1 = T_1$. Such a vibration is called a second harmonic of the given vibration. Similarly, we can consider third, fourth, $\cdots$, $(n — 1)$-th harmonics with the circular frequencies $\omega_4 = 4\omega_1$, $\omega_5 = 5\omega_1$, $\cdots$, $\omega_n = n\omega_1$, and, moreover, with any phase displacements of your choice. Every such harmonic will necessarily repeat itself after the period $T_1=2\pi/\omega_1$ and, consequently, every function obtained by superposition of a number of vibrations, each of which is a harmonic of a given fundamental circular frequency $\omega_1$, will itself be a periodic function with the period $2\pi/\omega_1=T_1$. By superposing vibrations with circular frequencies ranging from that of the fundamental to that of the $(n - 1)$-th harmonic, we obtain a periodic function of the form

$$S(x) = a + \sum_{\nu=1}^{n} (a_\nu \cos \nu\omega x + b_\nu \sin \nu\omega x).$$

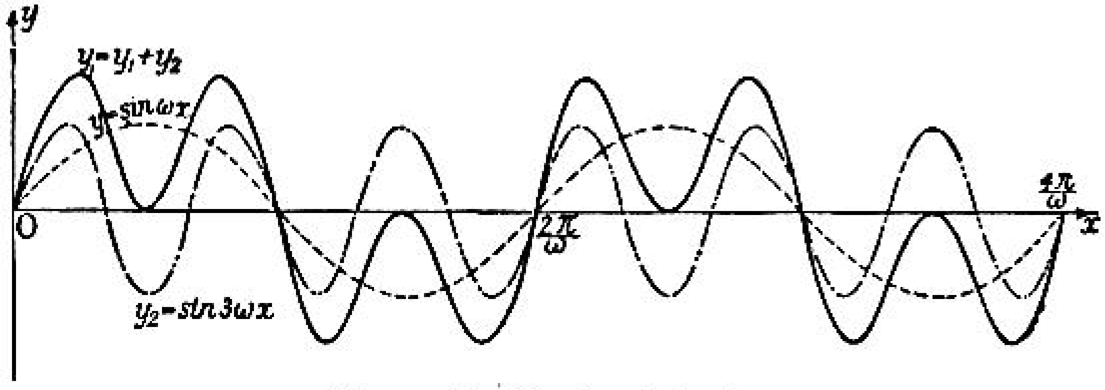



Fig. 3 .—Combination of vibrations

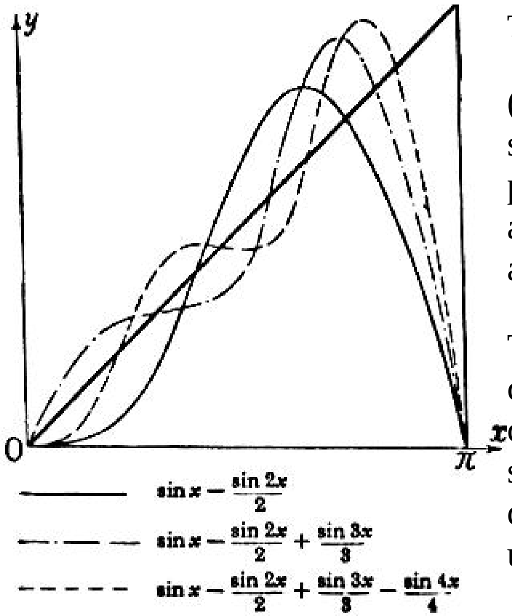



Fig. 4.—Combination of vibrations

The proportions of Fig.3 correspond to the assumption $\omega = 1$.

(The constant *a*, which we have introduced here in order to make the formula slightly more general, does not affect the periodicity, since it is periodic for any period.) Since this function contains 2*n* + 1 constants, which we can choose arbitrarily, we are thus able to generate very complicated curves which are not at all like the original curves. Figs. 3-5 display this situation.

The term harmonic originated in **acoustics**, where we find that there corresponds to a fundamental vibration with circular frequency a note of a certain pitch, then the first, second, third, etc., harmonics correspond to the sequence of harmonics of the fundamental, i.e., to the octave, octave + fifth, double octave, etc. (In acoustics, also the terms **overtone**, **(upper) partial** are used.

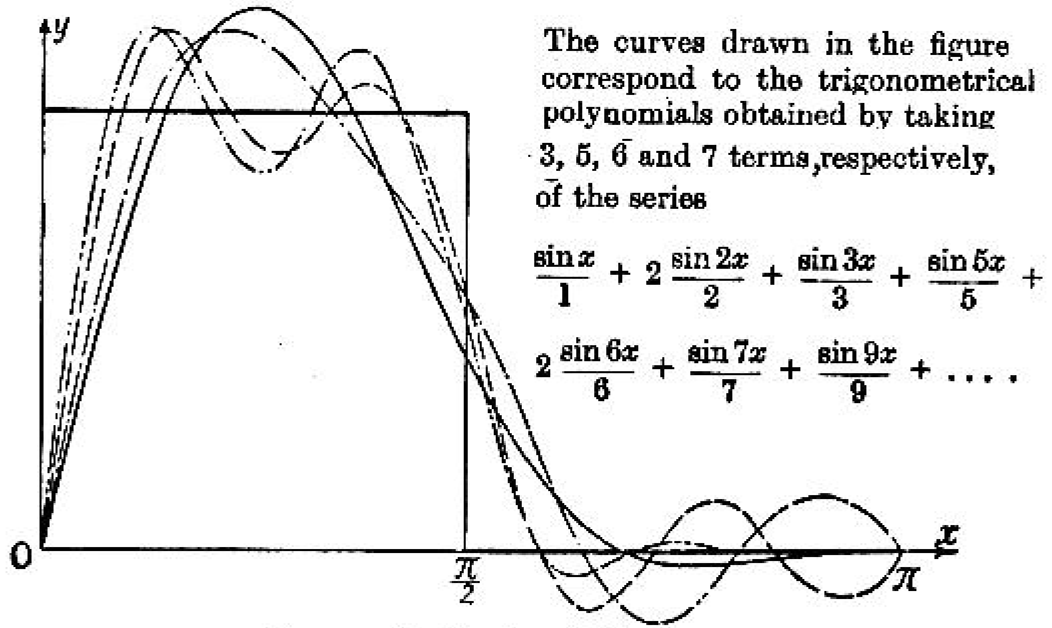



Fig. 5 .—Combination of vibrations

In general, in the case of superposition of vibrations in which the circular frequencies have rational ratios, these circular frequencies can all be represented as integral multiples of a common fundamental circular frequency. However, the superposition of two vibrations with incommensurable circular frequencies $\omega_1$ and $\omega_2$ represents an intrinsically different type of phenomenon. In that case, the process resulting from the superposition of sinusoidal vibrations will no longer be periodic. We cannot go here into the mathematical discussions that arise from this, but will merely remark that such functions always have an approximately periodic character or, as we say, are **almost periodic**. In 1935, such functions had just been studied in great detail.

A final remark on the superposition of sinusoidal vibrations concerns the phenomenon of **beats**. If we superpose two vibrations, both with unit amplitude but different circular frequencies $\omega_1$ and $\omega_2$, and, for the sake of simplicity, take the same value of $\xi$ for both (the generalization to arbitrary phase can be left to the reader), then we are merely concerned with the behaviour of the function

$$y = \sin \omega_1 x + \sin \omega_2 x \qquad (\omega_1 > \omega_2 > 0).$$

A well-known trigonometrical formula yields

$$y = 2\cos\tfrac{1}{2}(\omega_1 - \omega_2)x \sin\tfrac{1}{2}(\omega_1 + \omega_2)x.$$

This equation represents a phenomenon which we may think of as follows: We have a vibration with the circular frequency ½($\omega_1$ + $\omega_2$) and the period $4\pi/(\omega_1+\omega_2)$. However, this vibration does not have a constant amplitude; on the contrary, its amplitude is given by 2cos½($\omega_1+\omega_2$)$x$, which varies with the longer period $4\pi/(\omega_1-\omega_2)$). This point of view is particularly useful and easy to interpret when the two circular frequencies $\omega_1$and $\omega_2$ are relatively large, while their difference $\omega_1$-$\omega_2$ is small in comparison with them. Then the amplitude

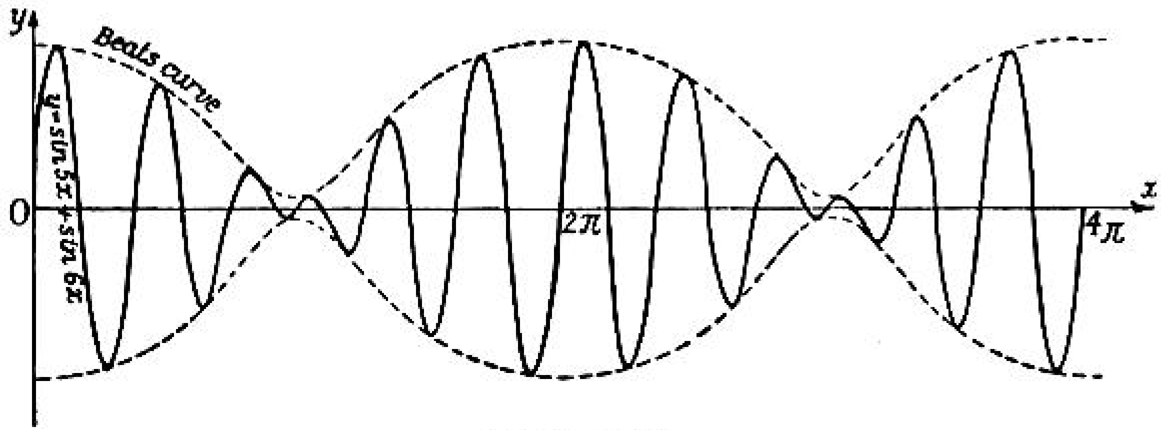



Fig. 6.—Beats

of the vibration $2\cos\frac{1}{2}(\omega_1 - \omega_2)x$ with period $4\pi/(\omega_1 + \omega_2)$ will vary only slowly compared with the period of vibration and this change of amplitude will repeat itself periodically with the long period $4\pi/(\omega_1 - \omega_2)$. These rhythmic changes of amplitude are called **beats**. Everyone is acquainted with this phenomenon in acoustics, and perhaps also in wireless telegraphy, in which the circular frequencies $\omega_1$and $\omega_2$ are, as a rule, far above those which the ear can detect, while the difference $\omega_1$- $\omega_2$ falls in the range of audible notes. The beats can then be heard, while the original vibrations remain imperceptible to the ear. Fig. 6 illustrates the phenomenon.

## 9.2 Use of complex notation

**9.2.1 General Remarks:** An investigation of vibration phenomena and periodic functions gains in formal simplicity if we employ complex numbers, combining each pair of trigonometric functions cos$\omega x$ and sin$\omega x$ to form an expression of the type cos$\omega x$ + $i$sin$\omega x$ = $e^{i\omega x}$(8.7.1). We must here bear in mind that one equation

between complex quantities is equivalent to two equations between real quantities and that our results must always be interpreted and made intelligible in the real domain.

If we replace everywhere the trigonometric functions by exponential functions in accordance with the formulae

$$2\cos\theta = e^{i\theta} + e^{-i\theta}, \quad 2i\sin\theta = e^{i\theta} - e^{-i\theta},$$

we express sinusoidal vibrations in terms of the complex quantities $e^{i\omega x}$, $e^{-i\omega x}$ or

$$ae^{i\omega(x-\xi)}, \quad ae^{-i\omega(x-\xi)},$$

respectively, where $a$, $\omega$ and $\omega\xi$ are the real quantities: Amplitude, circular frequency and phase displacement. The real vibrations are obtained from this complex expression simply by taking real and imaginary parts.

The convenience of this mode of representation for many purposes is due to the fact that the derivatives of the real vibrations with respect to the time are obtained by differentiating the complex exponential function just as if it were a real constant, as is expressed by the formula

$$\begin{aligned}\frac{d}{dx} a\{\cos\omega(x-\xi) + i\sin\omega(x-\xi)\} \\ &= a\omega\{-\sin\omega(x-\xi) + i\cos\omega(x-\xi)\} \\ &= ia\omega\{\cos\omega(x-\xi) + i\sin\omega(x-\xi)\},\end{aligned}$$

or

$$\frac{d}{dx} ae^{i\omega(x-\xi)} = ia\omega e^{i\omega(x-\xi)}.$$

**9.2.2 Application to the Study of Alternating Currents:** We shall now illustrate these matters by means of an important example and denote here the independent variable, the time, by $t$ instead of $x$.

We consider an electric circuit with resistance $R$ and inductance $L$, on which an external electromotive force (voltage) $E$ is impressed. In the case of a direct current, $E$ is constant and the current $I$ is given by Ohm's law $E = RI$.

However, if we are dealing with an alternating current, $E$ is a function of the time $t$, and consequently so is $I$; Ohm's law then becomes (cf. 3.7.6)

$$E - L\frac{dI}{dt} = RI.$$

In the simplest case, to which we will restrict ourselves here, the **external electromotive force** $E$ is sinusoidal with circular frequency $\omega$. Now, instead of taking this oscillation in the form $a\cos t$ or $a\sin\omega t$, we combine both possibilities formally in the complex form

$$E = \varepsilon e^{i\omega t} = \varepsilon\cos\omega t + i\varepsilon\sin\omega t,$$

where $\varepsilon$ ( >0) represents the amplitude. We shall operate with this **complex voltage** as if it were a real parameter and thus obtain a complex current $I$. Then, the significance of the relation thus found between the complex qnantities $E$ and $I$ is that the current corresponding to an electromotive force $\varepsilon\cos\omega t$ is the real part of $I$, while the current corresponding to an electromotive force $\varepsilon\sin\omega t$ is the imaginary part of $I$. The complex current can be calculated immediately, if we write down for $I$ an expression of the form $I = \alpha e^{i\omega t} = \alpha(\cos\omega t + i\sin\omega t)$, i.e., if we assume that $I$ is also sinusoidal with circular frequency $\omega$. The derivative of $I$ is then given formally by the expression

$$\frac{dI}{dt} = i\alpha\omega e^{i\omega t} = \alpha\omega(-\sin\omega t + i\cos\omega t).$$

On substitution of these quantities into the generalized form of Ohm's law and dividing by the factor $e^{i\omega t}$, we obtain the equation $\varepsilon - \alpha Lic\omega = R\alpha$, or

$$\alpha = \frac{\varepsilon}{R + i\omega L},$$

so that $E = (R + i\omega L)I = WI.$

We may regard this last equation as **Ohm's law for alternating currents** in complex form, if we call the quantity $W = R + i\omega L$ the **complex resistance of the circuit**. Ohm's law is then the same as for direct current: The current is equal to the voltage divided by the resistance.

If we write the complex resistance in the form

$$W = we^{i\delta} = w\cos\delta + iw\sin\delta,$$

where

$$w = \sqrt{(R^2 + L^2\omega^2)}, \quad \tan\delta = \frac{\omega L}{R},$$

we obtain $I = \frac{\varepsilon}{w} e^{i(\omega t - \delta)}.$

According to this formula, the current has the same period (and circular frequency) as the voltage; the amplitude $a$ of the current is connected with the amplitude $\varepsilon$ of the electromotive force by the equation

$$a = \frac{\varepsilon}{w},$$

and, in addition, there is a phase difference between the current and the voltage. The current does not reach its maximum at the same time as the voltage, but at a time $\delta/\omega$ later and the same is of course true for the minimum. In electrical engineering, the quantity $w = \sqrt{(R^2 + L^2\omega^2)}$ is frequently called the **impedance** or the **alternating current resistance** of the circuit for the circular frequency $\omega$; the phase displacement, usually stated in degrees, is called the **lag**.

**9.2.3 Complex Representation of the Superposition of Sinusoidal Vibrations:**. So far, the complex notation has been used to denote the combination of two sinusoidal vibrations. But a single vibration or a compound vibration of the type

$$S(x) = a + \sum_{\nu=1}^{n} (a_\nu \cos\nu x + b_\nu \sin\nu x)$$

(for the sake of simplicity, we have taken $\omega = 1$) can also be reduced to complex form by the substitution

$$\cos \nu x = \frac{1}{2}(e^{i\nu x} + e^{-i\nu x}), \quad \sin \nu x = \frac{1}{2i}(e^{i\nu x} - e^{-i\nu x}).$$

The above expression then assumes the form

$$S(x) = \sum_{\nu=-n}^{n} \alpha_\nu e^{i\nu x},$$

where the complex numbers $\alpha_\nu$ are linked to the real numbers $a$, $a_\nu$ and $b_\nu$ by the equations

$$a_\nu = \alpha_\nu + \alpha_{-\nu}, \quad a = \alpha_0, \quad b_\nu = i(\alpha_\nu - \alpha_{-\nu}).$$

In order that the equation $a_\nu = \alpha_\nu + \alpha_{-\nu}$ shall formally include the case $\nu = 0$, we often put $a = \alpha_0 = a_0/2$.

Conversely, we may regard any arbitrary expression of the form

$$\sum_{\nu=-n}^{n} \alpha_\nu e^{i\nu x}$$

as a function which represents the superposition of vibrations written in complex form. In order that the result of this superposition may be real, it is only necessary that $\alpha_\nu + \alpha_{-\nu}$ should be real and $\alpha_\nu - \alpha_{-\nu}$ pure imaginary, i.e., that $\alpha_\nu$ and $\alpha_{-\nu}$ are conjugate complex numbers.

**9.2.4 Deduction of a trigonometric Formula:** By using complex notation, we obtain a very simple proof of a formula which we shall require below. This is the **trigonometric summation formula**

$$\sigma_n(\alpha) = \tfrac{1}{2} + \cos\alpha + \cos 2\alpha + \ldots + \cos n\alpha = \frac{\sin(n+\frac{1}{2})\alpha}{2\sin\frac{1}{2}\alpha},$$

which holds for all values of $\alpha$ except the values $0, \pm 2\pi, \pm 4\pi, \cdots$. In order to prove this, we replace the cosine function by its exponential expression and thus write the sum $\sigma_n(\alpha)$ in the form

$$\sigma_n(\alpha) = \frac{1}{2} \sum_{\nu=-n}^{n} e^{i\nu\alpha}.$$

On the right hand side, we have a geometric progression with the common ratio $q = e^{i\alpha} \neq 1$. Using the ordinary formula for the sum, we have

$$\sigma_n(\alpha) = \frac{1}{2} e^{-in\alpha} \cdot \frac{1 - q^{2n+1}}{1 - q} = \frac{1}{2} \frac{e^{-in\alpha} - e^{(n+1)i\alpha}}{1 - e^{i\alpha}}.$$

On multiplying the numerator and denominator by $e^{-i\alpha/2}$, we obtain

$$\sigma_n(\alpha) = \frac{\sin(n + \frac{1}{2})\alpha}{2 \sin \frac{1}{2}\alpha},$$

as has been stated above.

**Exercises 9.1:**

1. Sketch the curve $y = \sum_{n=1}^{N} \frac{\sin nx}{n}$ for $N$ = 3, 5 6.

2. Sketch the curve $y = \sum_{n=1}^{N} \frac{\cos t}{n^4}$ for $N$ = 3, 6, 8.

3. Evaluate the sum $\sin\alpha + \sin 2\alpha + \ldots + \sin n\alpha.$

4. If

$$s_m(\alpha) = \frac{\sigma_0(\alpha) + \sigma_1(\alpha) + \ldots + \sigma_m(\alpha)}{m+1}, \sigma_n(\alpha) = \tfrac{1}{2} + \cos\alpha + \cos 2\alpha + \ldots + \cos n\alpha,$$

prove that

$$s_m(\alpha) = \frac{1}{m+1}\left[\frac{\sin\frac{(m+1)\alpha}{2}}{\sin\frac{\alpha}{2}}\right]^2.$$

(The expression $s_m$ is called the **Fejér kernel** and is of great importance in the more advanced study of Fonrier series.)

5. Show that $\frac{1}{\pi}\int_{-\pi}^{\pi} s_m(\alpha)\,d\alpha = 1,$ where $s_m(\alpha)$ is the Fejér kernel of Exercise 4.

Answers and Hints

## 9.3 Fourier Series

The function

$$S(x) = a + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x),$$

resulting from the superposition of sinusoidal vibrations, contains $2n + 1$ arbitrary constants $a$, $a_\nu$, $b_\nu$. There arises now the question whether these constants can be chosen such that in the interval $-\pi \le x \le +\pi$ the sum $S(x)$ shall approximate to a given function $f(x)$, and if so, how they are to be found. More precisely, we ask whether the given function $f(x\}$ can be expanded in an infinite series

$$f(x) = a + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

If we assume for the moment that this expansion of the function $f(x)$ is actually possible and that the series converges uniformly in the interval $-\pi \le x \le +\pi$, we readily obtain a simple relation between the function $f(x)$ and the coefficients $\alpha = ½a_0$, $a_\nu$ and $b_\nu$. (We shall soon see that the notation $\alpha = ½a_0$ is justified by its convenience.) We multiply the above hypothetical expansion by $\cos \nu x$ and integrate term by term, which is permissible on account of its uniform convergence. By virtue of the orthogonality relations

$$\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & \text{if } m \neq n, \\ \pi, & \text{if } m = n \neq 0. \end{cases}$$

$$\int_{-\pi}^{+\pi} \sin mx \cos nx \, dx = 0,$$

$$\int_{-\pi}^{+\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & \text{if } m \neq n, \\ \pi, & \text{if } m = n, \end{cases}$$

proved at the end of 4.3, we obtain at once for the coefficients the formula

$$a_\nu = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos \nu x \, dx.$$

Similarly, multiplying the series by sin $\nu x$ and integrating, we find

$$b_\nu = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin \nu x \, dx.$$

These formulae assign a definite sequence of coefficients $a_\nu$ and $b_\nu$, usually called **Fourier coefficients**, to every function $f(x)$ which is defined and continuous in the interval $-\pi \leq x \leq +\pi$, or has only a finite number of jump discontinuities there. If the function $f(x)$ is given, we can employ these quantities $a_\nu$, $b_\nu$ to form the **Fourier partial sum**

$$S_n(x) = \tfrac{1}{2} a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x),$$

and we may also write down formally the corresponding infinite Fourier series. Our problem is now to distinguish simple classes of functions $f(x)$ for which these Fourier series actually converge and represent them.

In order to formulate the result which we wish to prove, we introduce the definition:

A function $f(x)$ is said to be **sectionally continuous** in an interval, if it is itself sectionally continuous (i.e., is continuous in the interval except for a finite number of jump discontinuities) and if, in addition, its first derivative $f(x)$ is sectionally continuous.

We shall imagine that the function $f(x)$, originally defined in the interval $-\pi \leq x \leq \pi$, is periodically extended. At each point at which the function $f(x)$ has a jump discontinuity, we shall, if necessary, alter the function and assign to it the value which is the **arithmetic mean** of the left hand and right-hand limits of $f(x)$, i.e., we shall write

$$f(x) = \tfrac{1}{2}(f(x-0) + f(x+0)),$$

where $f(x - 0)$ and $f(x + 0)$ are simply the limits of $f(x)$ as $x$ approaches from the left and from the right hand side, respectively. This equation is obviously true for every point $x$ at which $f(x)$ is continuous.

Our goal is now the theorem: If the function $f(x)$ if sectionally smooth and satisfies the above equation, then its Fourier series converges at any point $x$ and represents the function.

Note that this theorem can be proved for more general classes of functions. However, the result formulated here is sufficient for all applications.

Moreover, we shall prove the theorem: In every closed interval in which the function $f(x)$ (imagined to be periodically extended) is continuous as well as sectionally smooth, the Fourier series converges uniformly.

Finally: If the function $f(x)$ is sectionally smooth and has no discontinuities, the Fourier series converges absolutely.

The proofs of these theorems will be postponed (9.5). We merely wish to emphasize here that the functions, which can be expanded according to these theorems, have a very high degree of arbitrariness; it is by no means necessary that the function should be given by a single analytical expression.

In the next section, we shall display the extraordinary fertility of Fourier expansions by discussing a number of examples.

## 9.4. Examples of Fourier Series

**9.4.1 Preliminary Remarks:** We shall assume that our functions $f(x)$ have the period $2\pi$ and are defined in the interval $-\pi \le x \le +\pi$, . Beyond this interval, to the left and right hand sides, they are to be extended periodically as in 8.1.1.

Before going into details, we note that if $f(x)$ is an **even** function, then clearly $f(a)$ sin $\nu x$ is **odd** and $f(x)$ cos $\nu x$ is even, so that

$$b_\nu = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin\nu x\,dx = 0; \quad a_\nu = \frac{2}{\pi}\int_0^\pi f(x)\cos\nu x\,dx.$$

We thus obtain a cosine series. On the other hand, if $f(x)$ is an odd function, then

$$a_\nu = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos\nu x\,dx = 0; \quad b_\nu = \frac{2}{\pi}\int_0^\pi f(x)\sin\nu x\,dx,$$

and we obtain a sine series.

Consequently, if the function $f(x)$ is initially given only in the interval $0 < x < \pi$, we can extend it in the interval $-\pi < x < 0$ either as an odd function or as an even function, and, correspondingly, expand it in the interval $0 < x < \pi$ in a sine series or in a cosine series.

**9.4.2 Expansion of the Functions $\psi(x) = x$ and $\varphi(x) = x^2$:** For the odd function $x$, we have

$$b_\nu = \frac{2}{\pi}\int_0^\pi x\sin\nu x\,dx,$$

and, on integration by parts,

$$\frac{\pi}{2}b_\nu = \frac{-x\cos\nu x}{\nu}\Big|_0^\pi + \frac{1}{\nu}\int_0^\pi \cos\nu x\,dx = (-1)^{\nu+1}\frac{\pi}{\nu}.$$

Hence, we obtain for the periodic function $\psi(x)$, which is equal to $x$ in the interval $-\pi < x < \pi$ (Fig. 7 below)

$$\psi(x) = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - + \cdots\right).$$

If we set $x = \pi/2$ , we obtain Gregory's series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - + \dots,$$

with which we are already familiar. The function $\psi(x)$, represented by this series, is not a continuous function; on the contrary, it jumps by $2\pi$ at the points $x = k\pi$, $k = \pm 1, \pm 3. \pm 5, \cdots$. At these points of discontinuity, that is, at the points $x = k\pi$, $k = \pm 1, \pm 3, \pm 5, \cdots$, each term of the series is zero, whence the function itself is zero. Hence, at the points of discontinuity, the series represents the arithmetic mean of the left hand and right hand limits.

If $\xi$ is any fixed number between $-\pi$ and $\pi$ and we replace $x$ in the above series by $(x - \xi)$, we obtain the series

$$\begin{aligned}\psi(x-\xi) &= 2\left(\frac{\sin(x-\xi)}{1} - \frac{\sin 2(x-\xi)}{2} + \frac{\sin 3(x-\xi)}{3} - + \dots\right) \\ &= -\frac{2}{1}\sin\xi\cos x + \frac{2}{1}\cos\xi\sin x + \frac{2}{2}\sin 2\xi\cos 2x \\ &\quad - \frac{2}{2}\cos 2\xi\sin 2x - \frac{2}{3}\sin 3\xi\cos 3x + \frac{2}{3}\cos 3\xi\sin 3x + \dots .\end{aligned}$$

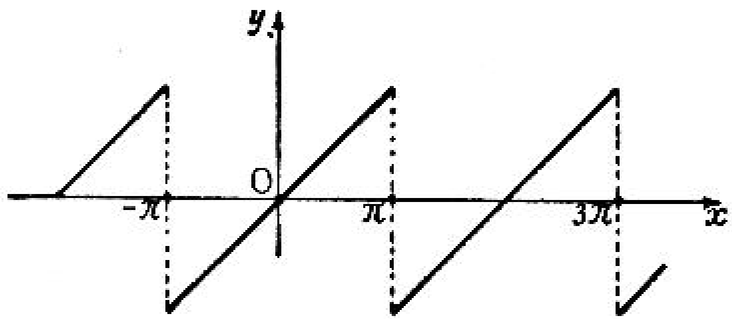


Fig. 7

This series may also be written in the form of a Fourier series with the coefficients

$$a_0 = 0, \quad a_n = 2\frac{(-1)^n}{n}\sin n\xi, \quad b_n = 2\frac{(-1)^{n-1}}{n}\cos n\xi,$$

which tend to zero as $n$ increases; this series represents a function with the discontinuities, described above, at the points $x = \xi \pm \pi$, $x = \xi \pm 3\pi, \cdots$.

For the even function $\varphi(x) = x^2$, on integrating twice by parts, we find

$$a_\nu = \frac{2}{\pi}\int_0^\pi x^2 \cos \nu x \, dx = (-1)^\nu \frac{4}{\nu^2} \qquad (\nu > 0), \quad a_0 = \frac{2\pi^2}{3},$$

Differentiating this series term by term and dividing by 2, we recover formally the series for $\psi(x) = x$.

**9.4.3 Expansion of the Function *x*cos *x*:**. For this odd function, we find

$$a_\nu = 0, \quad b_\nu = \frac{2}{\pi}\int_0^\pi x \cos x \sin \nu x \, dx.$$

Using the formula

$$\int_0^\pi x \sin \mu x \, dx = (-1)^{\mu+1}\frac{\pi}{\mu} \qquad (\mu = 1, 2, \ldots)$$

above, we find

$$b_\nu = \frac{2}{\pi}\int_0^\pi x\cos x \sin \nu x\,dx = \frac{1}{\pi}\int_0^\pi x(\sin(\nu+1)x + \sin(\nu-1)x)\,dx$$

$$= \frac{(-1)^{\nu+2}}{\nu+1} + \frac{(-1)^\nu}{\nu-1} = (-1)^\nu \frac{2\nu}{\nu^2-1} \quad (\nu = 2, 3, \ldots),$$

$$b_1 = -\frac{1}{2}.$$

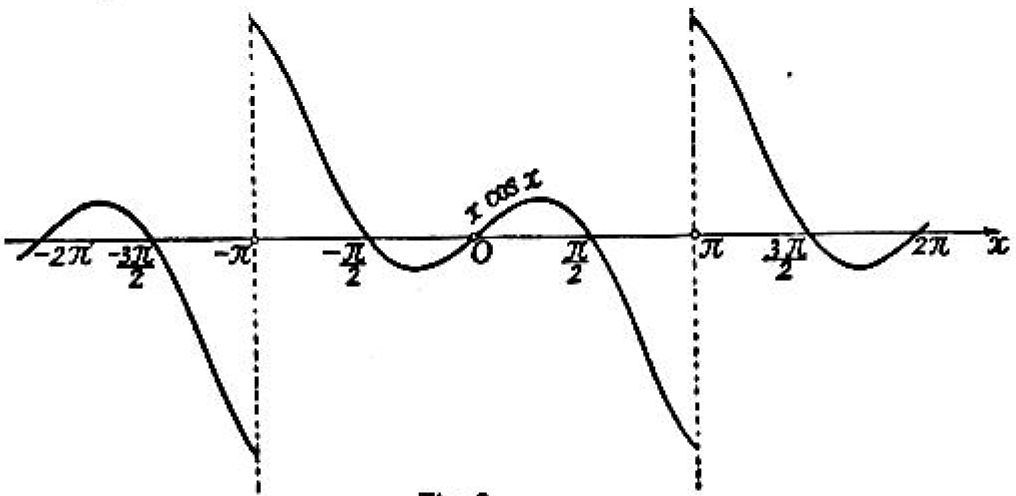



Fig. 8

Hence we obtain the series

$$x\cos x = -\frac{1}{2}\sin x + 2\sum_{\nu=2}^{\infty}\frac{(-1)^\nu \nu}{\nu^2-1}\sin \nu x,$$

and, if we add the series in 9.4.2, the series

$$x\cos x = -\frac{1}{2}\sin x + 2\sum_{\nu=2}^{\infty}\frac{(-1)^\nu \nu}{\nu^2-1}\sin \nu x,$$

When the function, which is equal to $x\cos x$ in the interval $-\pi < x < \pi$, is extended periodically beyond this interval, the same discontinuities (Fig. 8, below) appear as are exhibited by the function $\psi(x)$ considered in 9.4.2. On the other head, if the function $x(1 + \cos x)$ is periodically extended, it remains continuous at the end-points of the

interval and, in fact, its derivative also remains continuous, since the discontinuities are eliminated by the factor 1+cos$x$ which vanishes together with its derivative at the end-points.

**9.4.4 The Function $f(x) = |x|$:** This function is even, whence $b_\nu = 0$ and

$$a_\nu = \frac{2}{\pi}\int_0^\pi x \cos \nu x \, dx,$$

and, on integration by parts, we obtain

$$\int_0^\pi x \cos \nu x \, dx = \frac{1}{\nu} x \sin \nu x \Big|_0^\pi - \frac{1}{\nu}\int_0^\pi \sin \nu x \, dx$$
$$= \begin{cases} 0, \text{ if } \nu \text{ is even and } \neq 0, \\ -\dfrac{2}{\nu^2}, \text{ if } \nu \text{ is odd,} \end{cases}$$

whence

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right).$$

Setting $x = 0$, we obtain the remarkable formula

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots .$$

**9.4.5 The Step Function:** The function defined by the equations

$$f(x) = \begin{cases} -1, \text{ for } -\pi < x < 0, \\ 0, \text{ for } x = 0, \\ +1, \text{ for } 0 < x < \pi, \end{cases}$$

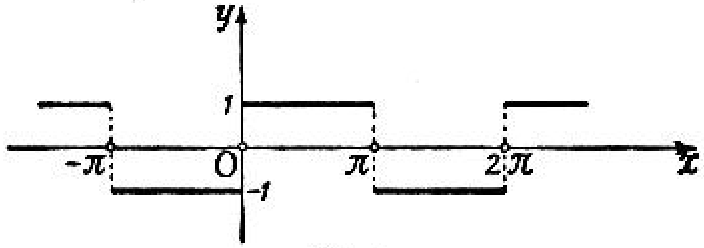



Fig. 9

as, indicated by Fig. 9, is an odd function, whence $a_\nu = 0$ and

$$b_\nu = \frac{2}{\pi}\int_0^\pi \sin \nu x \, dx = \begin{cases} 0 \text{ if } \nu \text{ is even,} \\ \dfrac{4}{\pi\nu} \text{ if } \nu \text{ is odd,} \end{cases}$$

so that its Fourier series is

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \dots\right).$$

For $x = \pi/2$, in particular, this again yields Gregory's series.

Note that this series can be derived *formally* from that for $|x|$ by term by term differentiation.

**9.4.6. The Function $f(x) = |\sin x|$:** The even function $f(x) = |\sin x|$ can be expanded in a sine series with the coefficients $a_\nu$ given by

$$\frac{\pi}{2} a_\nu = \int_0^\pi \sin x \cos \nu x \, dx = \frac{1}{2}\int_0^\pi \{\sin(\nu+1)x - \sin(\nu-1)x\}\, dx$$

$$= \begin{cases} 0 \text{ if } \nu \text{ is odd,} \\ \dfrac{-2}{\nu^2 - 1} \text{ if } \nu \text{ is even.} \end{cases}$$

We thus obtain

$$f(x) = |\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{\mu=1}^{\infty} \frac{\cos 2\mu x}{4\mu^2 - 1}.$$

**9.4.7 Expansion of the Function cos $\mu x$. Resolution of cotan into Partial Fractions. The Infinite Product for sin $x$:**. Let $f(x) = \cos \mu x$ for $-\pi < x < \pi$, where $\mu$ is not an integer. Since $f(x)$ is even, we obtain $b_\nu = 0$, while

$$\frac{\pi}{2} a_\nu = \int_0^\pi \cos \mu x \cos \nu x \, dx = \frac{1}{2}\int_0^\pi \{\cos(\mu+\nu)x + \cos(\mu-\nu)x\}\, dx$$

$$= \frac{1}{2}\left\{\frac{\sin(\mu+\nu)\pi}{\mu+\nu} + \frac{\sin(\mu-\nu)\pi}{\mu-\nu}\right\}$$

$$= \frac{\mu(-1)^\nu}{\mu^2 - \nu^2} \sin \mu\pi.$$

We thus find

$$\cos \mu x = \frac{2\mu \sin \mu\pi}{\pi}\left(\frac{1}{2\mu^2} - \frac{\cos x}{\mu^2 - 1^2} + \frac{\cos 2x}{\mu^2 - 2^2} - + \ldots\right).$$

This function remains continuous at the points $x = \pm\,\pi$. If we set $x = \pi$, divide both sides of the equation by sin $\mu\, x$ and then write $x$ instead of $\mu$, we obtain

$$\cot \pi x = \frac{2x}{\pi}\left(\frac{1}{2x^2} + \frac{1}{x^2 - 1^2} + \frac{1}{x^2 - 2^2} + \ldots\right).$$

This is the so-called resolution of cotan $x$ into partial fractions, a very important formula which is frequently discussed in analysis. We now rewrite this series in the form

$$\cot \pi x - \frac{1}{\pi x} = -\frac{2x}{\pi}\left\{\frac{1}{1^2 - x^2} + \frac{1}{2^2 - x^2} + \dots\right\}.$$

If $x$ lies in an interval $0 \leq x \leq q < 1$, the absolute value of the $n$-th term on the right hand side is less than $\frac{2}{\pi}\frac{1}{n^2 - q^2}$, whence the series converges uniformly in this interval and can be integrated term by term. We thus obtain

$$\pi \int_0^x \left(\cot \pi t - \frac{1}{\pi t}\right) dt = \log \frac{\sin \pi x}{\pi x} - \lim_{a \to 0} \log \frac{\sin \pi a}{\pi a} = \log \frac{\sin \pi x}{\pi x}$$

on the left hand side and

$$\log\left(1 - \frac{x^2}{1^2}\right) + \log\left(1 - \frac{x^2}{2^2}\right) + \dots = \lim_{n \to \infty} \sum_{\nu=1}^{n} \log\left(1 - \frac{x^2}{\nu^2}\right)$$

on the right hand side after multiplying both sides by $\pi$. If we step over from the logarithm to the exponential function, we find

$$\frac{\sin \pi x}{\pi x} = e^{\lim\limits_{n \to \infty} \sum\limits_{\nu=1}^{n} \log(1 - x^2/\nu^2)} = \lim_{n \to \infty} e^{\sum\limits_{\nu=1}^{n} \log(1 - x^2/\nu^2)}$$
$$= \lim_{n \to \infty} \prod_{\nu=1}^{n} \left(1 - \frac{x^2}{\nu^2}\right).$$

Hence

$$\sin \pi x = \pi x \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right) \dots .$$

Thus, we have obtained the famous expression for sin $x$ as an infinite product. Setting $x = ½$, we obtain Wallis' product

$$\frac{\pi}{2} = \prod_{\nu=1}^{\infty} \frac{2\nu}{2\nu-1} \cdot \frac{2\nu}{2\nu+1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots,$$

The formula for sin $\pi x$ is particularly interesting, because it shows directly that the function sin $\pi x$ vanishes at the points $x = 0, \pm 1, \pm 2, \cdots$ . In this respect, it corresponds to the factorization of a polynomial when its zeroes are known.

**9.4.8 More Examples:** Brief calculations, similar to the preceding ones, yield the following examples of expansions in series.

The function, defined by the equation $f(x) = \sin \mu x$ for $-\pi < x < \pi$ can be expanded in the series

$$f(x) = \sin \mu x = -\frac{2 \sin \mu\pi}{\pi} \left( \frac{\sin x}{\mu^2 - 1^2} - \frac{2 \sin 2x}{\mu^2 - 2^2} + \frac{3 \sin 3x}{\mu^2 - 3^2} - + \dots \right).$$

Setting $x$ $\pi/2$ and using the relation $\sin \mu\pi = 2 \sin \frac{\mu\pi}{2} \cos \frac{\mu\pi}{2}$, we obtain the resolution of the **secant** into partial fractions, i.e., of the function $\frac{1}{\cos \mu \frac{\pi}{2}}$; this expansion is

$$\pi \sec \pi x = \frac{\pi}{\cos \pi x} = 4 \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu} (2\nu - 1)}{4x^2 - (2\nu - 1)^2},$$

where we have replaced $\mu/2$ by $x$.

The series for the hyperbolic functions cosh $\mu x$ and sinh $\mu x$ ($-\pi < x < \pi$) are

$$\cosh \mu x = \frac{2\mu}{\pi} \sinh \mu\pi \left( \frac{1}{2\mu^2} - \frac{\cos x}{\mu^2 + 1^2} + \frac{\cos 2x}{\mu^2 + 2^2} - \frac{\cos 3x}{\mu^2 + 3^2} + - \dots \right),$$

$$\sinh \mu x = \frac{2}{\pi} \sinh \mu\pi \left( \frac{\sin x}{\mu^2 + 1^2} - \frac{2 \sin 2x}{\mu^2 + 2^2} + \frac{3 \sin 3x}{\mu^2 + 3^2} - + \dots \right).$$

**Exercises 9.2:**

1. Find the Fourier expansions for the functions which are periodic with period $2\pi$ and which are defined in $-\pi < x < \pi$ by:

$$(a)\ e^{ax}. \qquad (b)\ (x^2 - \pi^2)^2. \qquad (c)\ \sin ax(1 + \cos x).$$
$$(d)\ f(x) = 1\ (a \leqq x \leqq b), \quad f(x) = 0\ (-\pi < x < a), \quad f(x) = 0\ (b < x \leqq \pi).$$

2. The function $f(t)$ is periodic with period 1 and is given in $0 < x < 1$ by $f(t)=t$. Prove that
$$f(t) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi t}{n}.$$

3. The **Bernoulli polynomials** $B_n(t)$ are defined by the relations:

$$(a)\ B_1(t) = t - \tfrac{1}{2}; \quad (b)\ B_n'(t) = nB_{n-1}(t); \quad (c)\ \int_0^1 B_n(t)\,dt = 0.$$

Find $B_2(t)$, $B_3(t)$, $B_4t)$. (Note.—The numbers $B_n(0)$ are rational and , in fact, are the same as Bemoulli's numhers $B_n$ (cf. A8.4)

4. Verify the Fourier expansions for the Bernoulli polynomials:

$$B_1(t) = -\frac{1}{\pi}\left\{\sum_{n=1}^{\infty} \frac{\sin 2n\pi t}{n}\right\}, \qquad B_3(t) = \frac{3}{2\pi^3}\left\{\sum_{n=1}^{\infty} \frac{\sin 2n\pi t}{n^3}\right\},$$
$$B_2(t) = \frac{1}{\pi^2}\left\{\sum_{n=1}^{\infty} \frac{\cos 2n\pi t}{n^2}\right\}, \qquad B_4(t) = -\frac{3}{\pi^4}\left\{\sum_{n=1}^{\infty} \frac{\cos 2n\pi t}{n^4}\right\}.$$

5. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

6. Prove that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + - \dots = \frac{\pi^3}{32}.$$

7. Prove that

$$(a)\ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}.$$

$$(b)\ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}.$$

$$(c)\ 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \dots = \frac{7\pi^4}{720}.$$

8. Obtain the infinite product for the cosine from the relation

$$\cos \pi x = \frac{\sin 2\pi x}{2 \sin \pi x}.$$

Answers and Hints

## 9.5 The Convergence of Fourier Series

We now proceed to establish rigorously the theorems which were stated in 9.3 and illustrated in 9.4.

**9.5.1 The Convergence of the Fourier Series of a Sectionally smooth Function:**. We first recall that, if $f(x)$ is any function which is defined and sectionally continuous (i.e., continuous, except at most at a finite number of jump discontinuities) in the interval $-\pi \le x \le \pi$, we can form its Fourier coefficients according to the formulae

$$a_\nu = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t) \cos \nu t\, dt, \quad b_\nu = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t) \sin \nu t\, dt,$$

and write down formally the series

$$\tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

This series is called the **Fourier series** corresponding to $f(x)$, irrespectively of whether it converges or not. We will now find the conditions, which must be imposed on $f(x)$, in order to ensure that the Fourier series corresponding to $f(x)$ converges and represents $f(x)$. We will assume that $f(x)$ is extended periodically beyond the interval $-\pi < x \leq \pi$.

We shall now prove the theorem: If the function $f(x)$ is sectionally smooth, i.e. $f(x)$ and its derivative $f'(x)$ are sectionally continuous, and satisfies at each point of discontinuity (*s*) the condition $f(x)=½\{f(x-0)+f(x+0)\}$, then the Fourier series corresponding to $f(x)$ converges at every point and represents the function $f(x)$.

In order to prove this theorem, we consider the partial sums

$$S_n(x) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

If we substitute for the coefficients the above integral expressions and then interchange the order of integration and summation, we obtain

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\left\{\frac{1}{2} + \sum_{\nu=1}^{n} (\cos \nu t \cos \nu x + \sin \nu t \sin \nu x)\right\} dt,$$

or, by the addition theorem for the cosine,

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\left\{\frac{1}{2} + \sum_{\nu=1}^{n} \cos \nu (t-x)\right\} dt.$$

If we now apply the summation formula obtained above, this becomes

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)\,\frac{\sin (n+\frac{1}{2})(t-x)}{\sin \frac{1}{2}(t-x)}\, dt.$$

Finally, applying the transformation $\tau = (t - x)$ and noting the periodicity of the integrand, we obtain

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x+\tau)\,\frac{\sin (n+\frac{1}{2})\tau}{\sin \frac{1}{2}\tau}\, d\tau.$$

Starting with this form of the partial sum $S_n(x)$, we can prove that it tends to $f(x)$, by means of the lemma:

If the function $s(x)$ is sectionally continuous in the interval $a \leq x \leq b$, then the integral

$$I = \int_a^b s(t) \sin \lambda t \, dt$$

tends to 0 as $\lambda$ increases.

In the proof, we may assume that $s(x)$ is continuous in the entire interval, since otherwise we merely need carry out the argument for each sub-interval in which $s(x)$ is continuous

As in the similar argument in A8.2, we note that, if $\lambda$ is positive, the function $\sin \lambda t$ is alternately positive and negative in successive intervals of length $\pi/\lambda$. For large values of $\lambda$, the contributions to the integral from adjacent intervals almost cancel one another, since, on account of the continuity, the values of $s(x)$ in two such adjacent intervals differ only slightly from each another. We make use of this circumstance by transforming the integral $I$ by the substitution $t = \tau + h$, where $h = \pi/\lambda$; then, since $\sin \lambda t = - \sin \lambda\tau$, we obtain

$$I = -\int_{a-h}^{b-h} s(\tau + h) \sin \lambda\tau \, d\tau.$$

If we again replace the letter $\tau$ by $t$ and then add the two expressions for $I$, we find

$$2I = -\int_{a-h}^{a} s(t+h) \sin \lambda t \, dt + \int_a^{b-h} \{s(t) - s(t+h)\} \sin \lambda t \, dt$$
$$+ \int_{b-h}^{b} s(t) \sin \lambda t \, dt.$$

If $M$ is an upper bound for the absolute value of $s(x)$, i.e., if for all values of $x$ in the interval under consideration $|s(x)| \leq M$, then there follows at once from this expression for $I$ the inequality

$$2\,|\,I\,| \leq 2Mh + \int_a^{b-h} |\,s(t) - s(t+h)\,|\,dt.$$

Now, let $\varepsilon$ be any positive number; if we choose $\lambda$ so large that in the entire interval $a \le t \le b - h$ the expression $|s(t) - s(t + h)|$ remains less than $\varepsilon/(b - a)$ and also $Mh = M\pi/\lambda < \varepsilon/2$, then $|I|] < \varepsilon$; consequently, since $\varepsilon$ can be chosen as small as we please,

$$\lim_{\lambda \to \infty} I = 0.$$

If we assume that $s(x)$, besides being continuous, has a sectionally continuous derivative $s'(x)$, the proof of this lemma follows simply on integration by parts. In fact,

$$\int_a^b s(t) \sin \lambda t \, dt = \frac{1}{\lambda} \left\{ s(a) \cos \lambda a - s(b) \cos \lambda b + \int_a^b s'(t) \cos \lambda t \, dt \right\}.$$

We see here at once that, as $\lambda$ increases, the right hand side tends to zero.

Besides this lemma, we need the integration formula

$$\int_0^\pi \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \, dt = \frac{\pi}{2},$$

which is true for every positive integer $n$. We establish this readily by using our summation formula for the cosine, since

$$\int_0^\pi \frac{\sin(n + \frac{1}{2})\,t}{2 \sin \frac{1}{2}t} \, dt = \int_0^\pi \left(\tfrac{1}{2} + \sum_1^n \cos \nu t\right) dt = \frac{\pi}{2}.$$

**Proof of the Main Theorem:** By means of the lemma, the main theorem is readily proved, i.e., the formula

$$\lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x + t) \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \, dt = f(x).$$

We begin by subdividing the interval of integration at the origin. For fixed values of $x$, the function

$$s(t) = \frac{f(x+t) - f(x+0)}{2 \sin \frac{1}{2}t}$$

is sectionally continuous in the interval $0 \le t \le \pi$. In fact, this is obvious when $0 \le t \le \pi$, while the continuity at $t = 0$ follows from the assumed existence of the right hand derivative

$$\lim_{t \to 0,\, t > 0} \frac{f(x+t) - f(x+0)}{t} = \lim_{t \to +0} \frac{f(x+t) - f(x+0)}{2 \sin \frac{1}{2}t} \cdot \frac{2 \sin \frac{1}{2}t}{t}$$
$$= \lim_{t \to +0} \frac{f(x+t) - f(x+0)}{2 \sin \frac{1}{2}t}.$$

Hence, as $\lambda = n + \frac{1}{2}$ increases, the integral

$$\frac{1}{\pi} \int_0^\pi s(t) \sin \lambda t \, dt$$
$$= \frac{1}{2\pi} \int_0^\pi f(x+t) \frac{\sin \lambda t}{\sin \frac{1}{2}t} dt - \frac{1}{2\pi} \int_0^\pi f(x+0) \frac{\sin \lambda t}{\sin \frac{1}{2}t} dt$$

tends to zero.

However, since the factor $f(x + 0)$ can be taken out of the second integral on the right hand side and, for $\lambda = n + \frac{1}{2}$, the integral $\int_0^\pi \frac{\sin \lambda t}{2 \sin \frac{1}{2}t} dt$ is equal to $\pi/2$, we obtain immediately

$$\lim_{\lambda \to \infty} \frac{1}{2\pi} \int_0^\pi f(x+t) \frac{\sin \lambda t}{\sin \frac{1}{2}t} dt = \frac{1}{2} f(x+0).$$

By setting in this equation $x = 0$, $f(t) = (\sin \frac{1}{2}t)/t$ and then replacing $t$ by $u/\lambda$, we obtain the important relation $\lim_{\lambda \to \infty} \int_0^{\lambda\pi} \frac{\sin u}{u} du = \frac{\pi}{2}$ (cf. 4.8.5).

In the same way, we obtain for the interval $-\pi \leq t \leq 0$

$$\lim_{\lambda \to \infty} \frac{1}{2\pi} \int_{-\pi}^{0} f(x+t) \frac{\sin \lambda t}{\sin \frac{1}{2}t} \, dt = \frac{1}{2} f(x-0),$$

and by addition

$$\lim_{\lambda \to \infty} \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x+t) \frac{\sin \lambda t}{\sin \frac{1}{2}t} \, dt = f(x).$$

**9.5.2. Further Investigation of Convergence:** In the neighbourhood of those points, where the function $f(x)$ is discontinuous, the Fourier series does not converge uniformly; in fact, by 8.4.3, a uniformly convergent series of continuous functions possesses a continuous sum. Nevertheless, we have the important theorem:

If a sectionally smooth periodic function has no discontinuities, its Fourier series converges absolutely and uniformly. The convergence of the Fourier series for any sectionally smooth function whatsoever is uniform in every closed interval which contains no point of discontinuity of the function.

In order to prove this theorem, we start from a fundamental inequality satisfied by the Fourier coefficients of any function $f(x)$ which is sectionally continuous (note that $f(x)$ is not assumed to be sectionally smooth). This so-called **Bessel inequality** states that for all values of $n$

$$\frac{1}{2} a_0{}^2 + \sum_{\nu=1}^{n} (a_\nu{}^2 + b_\nu{}^2) \leqq \frac{1}{\pi} \int_{-\pi}^{+\pi} \{f(x)\}^2 \, dx.$$

The proof follows from the fact that the expression

$$\int_{-\pi}^{+\pi} \left\{ f(x) - \tfrac{1}{2} a_0 - \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) \right\}^2 dx$$

is always positive or zero. If we evaluate the integral by expanding the bracket under the integral sign and recall the orthogonality relations and definitions of the Fourier coefficients, we obtain at once Bessel's inequality in the form

$$\int_{-\pi}^{+\pi}\{f(x)\}^2\,dx-\pi\left\{\frac{1}{2}a_0^2+\sum_{\nu=1}^{n}(a_\nu^2+b_\nu^2)\right\}\geqq 0.$$

In addition to Bessel's inequality, we employ Schwarz's inequality : If $u_1,u_2,\cdots,u_n$ and $v_1,v_2,\cdots,v_n$ are arbitrary real numbers, it is always true that

$$\left(\sum_{\nu=1}^{n}u_\nu v_\nu\right)^2\leqq\sum_{\nu=1}^{n}u_\nu^2\cdot\sum_{\nu=1}^{n}v_\nu^2,$$

the equality sign occurring only when the sequence $u$ is proportional to the sequence $v$.

We now assume that the periodic function $f(x)$ is sectionally smooth as well as continuous. The derivative $g(x)=f'(x)$ is sectionally continuous and we easily show that $c_\nu$ and $d_\nu$, the Fourier coefficients of $g(x)$, satisfy the relations

$$\begin{aligned}&c_0=0,\\&\left.\begin{aligned}c_\nu&=\nu b_\nu\\d_\nu&=-\nu a_\nu\end{aligned}\right\}(\nu\geqq 1);\end{aligned}$$

in fact, on integration by parts, we have

$$\begin{aligned}c_\nu&=\frac{1}{\pi}\int_{-\pi}^{+\pi}g(x)\cos\nu x\,dx\\&=\frac{1}{\pi}f(x)\cos\nu x\Big|_{-\pi}^{+\pi}+\frac{\nu}{\pi}\int_{-\pi}^{+\pi}f(x)\sin\nu x\,dx=\nu b_\nu,\end{aligned}$$

similar proofs holding for the other statements.

Hence, Bessel's inequality applied to the function $g(x)$ yields

$$\sum_{\nu=1}^{n}\nu^2(a_\nu^2+b_\nu^2)=\sum_{\nu=1}^{n}(c_\nu^2+d_\nu^2)\leqq\frac{1}{\pi}\int_{-\pi}^{+\pi}\{g(x)\}^2\,dx.$$

If, for the sake of brevity, we denote the right hand side of this inequality by $M^2$ and apply Schwarz's inequality, we find that for $m > n$

$$\sum_{\nu=n+1}^{m} |a_\nu \cos \nu x + b_\nu \sin \nu x| \leqq \sum_{\nu=n+1}^{m} \sqrt{(a_\nu^2 + b_\nu^2)}$$
$$= \sum_{\nu=n+1}^{m} \left\{\frac{1}{\nu}(\nu\sqrt{(a_\nu^2 + b_\nu^2)})\right\} \leqq M \sqrt{\left(\sum_{\nu=n+1}^{m} \frac{1}{\nu^2}\right)},$$

since $\sqrt{(a_\nu^2 + b_\nu^2)}$ is the amplitude of the periodic function $a_\nu \cos \nu x + b_\nu \sin \nu x.$

However, owing to the convergence of $\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}$, the right hand side, which is independent of $x$, can be made as small as we please by choosing $n$ and $m$ large enough, which proves the absolute and uniform convergence of the series.

Incidentally, the same considerations show that for periodic functions with continuous derivatives of the ($h$- l)-th order and derivatives of the ($h$-1)-th order, which are at least sectionally continuous, the sum $\sum^{n} \nu^{2h}(a_\nu^2 + b_\nu^2)$ remains below a fixed bound. This gives us a definite statement about the order to which the Fourier coefficients vanish. For such a function, the Fourier series of the derivatives up to the order ($h$ - 1) converge absolutely and uniformly.

In order to prove the above theorem for sectionally smooth functions which are discontinuous, we first consider a **special function** $\psi(x)$ of this type.

In the interval $-\pi < x < \pi$, we define $\psi(x)$ as equal to $x$, outside this interval, $\psi(x)$ is extended periodically. According to 9.4.1, its Fourier series is

$$2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - + \dots\right).$$

This series cannot be uniformly convergent, because its sum is the discontinuous function $\psi(x)$. However, we shall show that the convergence is uniform in every interval $-l \leq x \leq l$ for which $0 < l < \pi$.

The proof is based on a special artifice*. We observe that in the interval $-l \le x \le l$ the function $\cos x/2$ is never less than the positive quantity $\cos l/2 = \kappa$. If we multiply the absolute value of the difference between the $m$-th and $n$-th partial sums of the above series ($m > n$), i.e., the expression

$$|S_m(x) - S_n(x)| = 2\left|\frac{\sin(n+1)x}{n+1} - \frac{\sin(n+2)x}{n+2} + - \dots \pm \frac{\sin mx}{m}\right|,$$

by the function $\cos x/2$, then, in accordance with the well-known trigonometric formula

$$2\sin u \cos v = \sin(u+v) + \sin(u-v),$$

we obtain the absolute value of the expression

$$\begin{aligned} 2\cos\frac{x}{2}&\left\{\frac{\sin(n+1)x}{n+1} - \frac{\sin(n+2)x}{n+2} + - \dots \pm \frac{\sin mx}{mx}\right\} \\ &= \frac{\sin(n+\frac{3}{2})x}{n+1} - \frac{\sin(n+\frac{5}{2})x}{n+2} + - \dots \pm \frac{\sin(m+\frac{1}{2})x}{m} \\ &+ \frac{\sin(n+\frac{1}{2})x}{n+1} - \frac{\sin(n+\frac{3}{2})x}{n+2} + \frac{\sin(n+\frac{5}{2})x}{n+3} - + \dots . \end{aligned}$$

* We are led to this artifice naturally by observing that the function $2y\cos y$, when extended periodically beyond the interval $-\pi/2 \le y \le \pi/2$ remains continuous, so that, according to the first part of the theorem, its Fourier series must converge uniformly and must represent the function. However, this series is obtained, if we multiply the Fourier series for $2y$ by $\cos y$. If we now put $y = x/2$, this multiplication leads to the steps in the text.

If we combine the terms on the right hand side with the same numerators, we obtain

$$\begin{aligned} &\frac{\sin(n+\frac{1}{2})x}{n+1} \pm \frac{\sin(m+\frac{1}{2})x}{m} \\ &+ \frac{\sin(n+\frac{3}{2})x}{(n+1)(n+2)} - \frac{\sin(n+\frac{5}{2})x}{(n+2)(n+3)} + - \dots \mp \frac{\sin(m-\frac{1}{2})x}{(m-1)m}, \end{aligned}$$

and, since $\cos x/2 \geq \kappa$ and $|\sin u| \leq 1$, the estimate

$$|S_m(x) - S_n(x)| \leqq \frac{1}{\kappa}\left[\frac{1}{n+1} + \frac{1}{m} + \frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(m-1)m}\right]$$

But the expression on the right hand side does not depend on $x$ and, by virtue of the convergence of the series $\sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu+1)}$, it can be made as small as we please by choosing $n$ and $m$ large enough. This implies the uniform convergence of the Fourier series, as we have asserted.

Now that we have obtained the expression for a particular discontinuous function, we can (cf. 9.4.2) transfer the discontinuity to any arbitrary point in the interval by translation of the curve or of the co-ordinate system. In fact, the function

$$\psi(x-\xi) = 2\left(\frac{\sin(x-\xi)}{1} - \frac{\sin 2(x-\xi)}{2} + \frac{\sin 3(x-\xi)}{3} - + \ldots\right)$$

is continuous except at the points $(2k+1)\pi + \xi$, where $k$ is an integer. However, on passing these points, the function jumps by an amount $-2\pi$ from the value $\pi$ to the value $-\pi$, while at these points themselves the value of the function is zero.

If now $f(x)$ is any sectionally smooth function, which in the interval $-\pi \leq x \leq \pi$ is discontinuous only at the points $\xi_1$, $\xi_2, \cdots, \xi_m$ and if on passing these points from the left to the right hand side the function jumps by the amounts $\delta_1, \delta_2, \cdots, \delta_m$, respectively, then the function

$$f(x) + \frac{\delta_1}{2\pi}\psi(x+\pi-\xi_1) + \frac{\delta_2}{2\pi}\psi(x+\pi-\xi_2) + \ldots + \frac{\delta_m}{2\pi}\psi(x+\pi-\xi_m)$$

will be continuous and sectionally smooth, whence, by the previous proof, it can be expanded in a uniformly convergent Fourier series. We now obtain the Fourier series of the function $f(x)$ by adding term by term the finite number of Fourier series corresponding to the functions

$$-\frac{\delta_1}{2\pi}\psi(x+\pi-\xi_1), \ldots, -\frac{\delta_m}{2\pi}\psi(x+\pi-\xi_m).$$

Hence the theorem is proved.

This result is quite adequate for most mathematical investigations and applications. However, we point out that the investigation of Fourier series has been advanced much further. The conditions for an expansion in Fourier series, which we have found here to be sufficient, are by no means necessary. Functions with far fewer continuity properties than those discussed here can be represented by Fourier series. There is an extensive literature devoted to these questions and to the general problem of the expandability of a function in a Fourier series. As a remarkable result of such investigations, we mention the fact that there are continuous functions, the Fourier series of which do not converge in any interval, no matter how small. Such a result does not in any impugnway reduce the usefulness of Fourier series; on the contrary, it must be regarded as evidence that the concept of a continuous function involves fairly complicated possibilities, as has already been shown by the example of continuous functions which nowhere have a derivative.

# Appendix to Chapter IX

## A9 Integration of Fourier Series

One of the remarkable properties of Fourier series is their term by term integrability. In general, a series can be integrated term by term, if it is uniformly convergent; otherwise, term by term integration may lead to false results. In contrast to this, for Fourier eeries, we have the theorem:

If $f(x)$ is sectionally continuous in $-\pi \le x \le \pi$ and if the Fourier series

$$\tfrac{1}{2}a_0 + \Sigma(a_\nu \cos x + b_\nu \sin x)$$

corresponds to $f(x)$, then this series can be integrated term by term between any two limits $\xi$ and $x$ lying in the interval $-\pi \le x \le \pi$, i.e., in symbols,

$$\int_\xi^x f(x)\,dx = \int_\xi^x \tfrac{1}{2}a_0\,dx + \sum_1^\infty \left(\int_\xi^x a_\nu \cos \nu x\,dx + \int_\xi^x b_\nu \sin \nu x\,dx\right).$$

Moreover, for every fixed value of $\xi$, the series on the right hand side converges uniformly in $x$.

The remarkable feature of this theorem is that not only do we not require that the Fourier series for $f(x)$ shall be uniformly convergent, but we need not even assume that it converges at all.

In order to prove this, let the function $F(x)$ be defined by the equation

$$F(x) = \int_{-\pi}^{x} \{f(x) - \tfrac{1}{2}a_0\}\,dx.$$

This function is sectionally smooth and, by the definition of $a_0$, we have $F(\pi) = F(-\pi) = 0$, so that $f(x)$ can be extended periodically and continuously. Hence the Fourier series

$$\tfrac{1}{2}A_0 + \sum_{\nu=1}^{\infty} (A_\nu \cos \nu x + B_\nu \sin \nu x)$$

of the function $F(x)$ converges uniformly to $F(x)$. We will now investigate the coefficients $A_\nu$ and $B_\nu$. By integration by parts, as in 9.5.1, we find that, for $\nu>0$, $A_\nu = b_\nu / \nu$ and $B_\nu = a_\nu / \nu$. Hence for any values $\xi$ and $x$ in the interval $-\pi \leq x \leq \pi$, we have

$$\begin{aligned} F(x) - F(\xi) &= \sum_{\nu=1}^{\infty} \{A_\nu (\cos \nu x - \cos \nu\xi) + B_\nu (\sin \nu x - \sin \nu\xi)\} \\ &= \sum_{\nu=1}^{\infty} \left\{\frac{a_\nu}{\nu} (\sin \nu x - \sin \nu\xi) - \frac{b_\nu}{\nu} (\cos \nu x - \cos \nu\xi)\right\}, \end{aligned}$$

converging uniformly in $x$. If we replace $F(x)$ by its definition, this becomes

$$\int_{\xi}^{x} f(x)\,dx - \tfrac{1}{2}a_0 \int_{\xi}^{x} dx = \sum_{\nu=1}^{\infty} \left(a_\nu \int_{\xi}^{x} \cos \nu x\,dx + b_\nu \int_{\xi}^{x} \sin \nu x\,dx\right),$$

as was to be proved.

It is easy to see that, if *f*(*x*) is periodic and sectionally continuous, the term by term integration can be performed over any interval whatever.

## Chapter X

# A Sketch of the Theory of Functions of Several Variables

Up to this point, we have been concerned exclusively with functions of a single independent variable. We must now go on to consider functions of several independent variables. Even the applications of the calculus force us to take this step. In almost all the relationBhips which occur in nature, in fact, the functions in question do not depend on a single independent variable; on the contrary, the dependent variable is usually determined by two, three, or more independent variables. Thus, for example, the volume of an ideal gas is a function of a single variable, the pressure, if we keep the temperature constant, but not otherwise. As a rule, the temperature also varies and the volume depends upon a pair of values, namely, the value of the pressure and that of the temperature, whence it is a function of two independent variables.

Also from the point of view of pure mathematics, the need for a detailed study of functions of several independent variables is urgent. Here we shall be able to take advantage of what we have learned previously, so that in many cases we have only to make simple extensions of our arguments.

It is usually sufficient to consider the case of only two independent variables $x$ and $y$, as long as no essentially new considerations are required for an extension to functions of three or more variables. Hence, in order to keep our statements and notation simple, we shall, as a rule, consider only two independent variables.

A systematic presentation of the differential and integral calculus for functions of several variables is impossible within the compass of this volume, but will be given in **Volume II** of this treatise. All which can be done here is to give the reader a preliminary view of some of the most important new concepts and operations. We shall frequently rely on intuitive plausibility, the full proofs to be developed subsequently in **Volume II**.

### 10.1 The Concept of Function in the Case of Several Variables

**10.1.1 Functions and their Ranges of Definition:** Equations of the form

$$u = x^2 + y^2, \quad u = x - y, \quad u = xy, \quad \text{or} \quad u = \sqrt{(1 - x^2 - y^2)}$$

assign a functional value $u$ to each pair of values $(x, y)$. In the first three of our example, this correspondence holds for every system of values $(x, y)$, while in the last case the correspondence has a meaning only for those pairs of values $(x, y)$ for which the inequality $x^2 + y^2 \leq 1$ is true.

In these cases we say that $u$ is a function of the independent variables $x$ and $y$. In general, we use this expression whenever some law assigns a value of $u$ as **dependent variable**, corresponding to each pair of values $(x,y)$ belonging to a certain specified set. The relation between $x$, $y$ and $u$ may be stated in terms of a **functional equation**, as above, or by means of a verbal description such as $u$ is the area of the rectangle with sides $x$ and $y$, or it may follow from physical observations as for instance in the case of the magnetic declination at different latitudes and longitudes. The essential thing is that there exists a correspondence. Similarly, $u$ is said to be a function of the three independent variables $x$, $y$, $z$, if for each triad of values $(x, y, u)$ of a certain set there exists a corresponding value of $u$ given by some definite law; it is similar for the general case of functions of $n$ independent variables $x_1, x_2, \cdots, x_n$.

The set of values which the pair $(x, y)$ can assume is called the **range of definition** of the function $u = f(x, y)$. For the purposes of this chapter, we shall restrict our attention to the simplest types of range of definition. We shall consider that $(x, y)$ is limited either to a **so-called rectangular region** (**domain**)

$$a \leqq x \leqq b, \quad c \leqq y \leqq d,$$

or else to a circle, determined by an inequality of the form

$$(x - a)^2 + (y - b)^2 \leqq r^2.$$

In the case of functions of three variables $x$, $y$, $z$, we shall again consider only rectangular regions

$$a \leqq x \leqq b, \quad c \leqq y \leqq d, \quad e \leqq z \leqq f$$

and spherical regions

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leqq r^2.$$

In dealing with more than three independent variables, geometrical intuition fails us, but it is often convenient to also extend geometrical terminology to this case. Thus, for functions of $n$ variables $x_1, x_2, \cdots, x_n$, we shall consider regions

$$a_1 \leqq x_1 \leqq b_1, \quad a_2 \leqq x_2 \leqq b_2, \ldots, \quad a_n \leqq x_n \leqq b_n$$

and also regions

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \ldots + (x_n - a_n)^2 \leqq r^2,$$

which we will call rectangular and spherical regions, respectively.

**10.1.2 The Simplest Types of Functions**. Just as in the case of functions of one variable, the simplest functions are the **rational integral functions** or **polynomials**. The most general polynomial of the first degree (**linear function**) has the form

$$u = ax + by + c,$$

where $a$, $b$ and $c$ are constants. The **general polynomial** of the second degree has the form

$$u = ax^2 + bxy + cy^2 + dx + ey + f.$$

The general polynomial is a sum of terms of the form $a_{mn}x^m y^n$, where the quantities $a_{mn}$ are arbitrary constants.

**Rational fractional functions** are quotients of polynomials; for example, there belongs to this class the linear fractional function

$$u = \frac{ax + by + c}{a'x + b'y + c'}.$$

By extraction of roots, we pass on from the rational functions to certain **algebraic functions**, for example,

$$u = \sqrt{\left(\frac{x - y}{x + y}\right)} + \sqrt[3]{\left\{\frac{(x + y)^2}{x^3 + xy}\right\}}.$$

An accurate definition of the term algebraic function is given in at the end of 10.5.1.

In the construction of more complicated functions of several variables, we almost always fall back on the well-known functions of a single variable, for example (10.4.1),

$$u = \sin(xy) \quad \text{or} \quad u = \log(y^2 + \cos \tfrac{1}{2}x).$$

**10.1.3 Geometrical Representation of Functions:** Just as we represent functions of one variable by means of curves, we seek to represent functions of two variables geometrically by means of **surfaces**; we shall consider hereafter only those functions which can actually be represented in this way. We achieve this representation very simply by considering a rectangular co-ordinate system in space with co-ordinates $x$, $y$ and $u$, and marking off above each point $(x, y)$ of the **range ($R$) of definition** of the function the point $P$ with the third co-ordinate $u = f(x, y)$. As the point $(x, y)$ ranges over the region $R$, the point $P$ describes a surface in space. We take this surface as the geometrical representation of the function.

Conversely, in **analytical geometry**, surfaces in space are represented by functions of two variables, so that there is between such surfaces and functions of two variables a reciprocal relationship.

For example, there corresponds to the function

$$u = \sqrt{(1 - x^2 - y^2)}$$

the **hemi-sphere** above the $x$, $y$ plane with unit radius and centre at the origin, to the function

$$u = x^2 + y^2,$$

the so-called **parabaloid of revolution**, obtained by rotating the parabola $w=x^2$ about the $u$-axis (Fig. 1), to the functions

$$u = x^2 - y^2 \text{ and } u = xy$$

the so-called **hyperbolic paraboloids** (Fig. 2). The linear function

$$u = ax + by + c$$

has for its graph a **plane in space**.

If in the function $u = f(x, y)$ one of the independent variables, say $y$, does not occur, so that $u$ depends on $x$ only, say $u = g(x)$, the function is represented in the $xyu$-space by a **cylindrical surface**, obtained by erecting the perpendiculars to the $ux$-plane at the points of the curve $u=g(x)$.

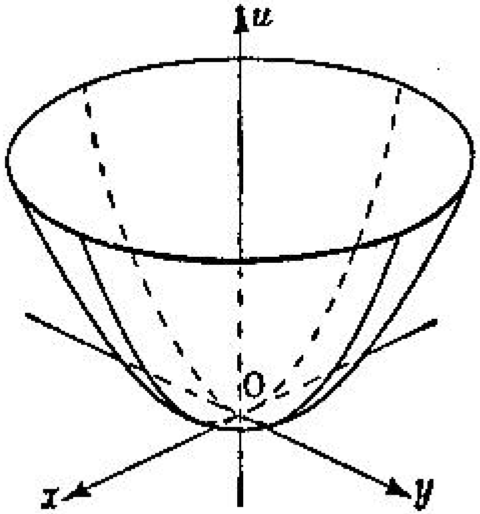



Fig. 1.— $u = x^2 + y^2$

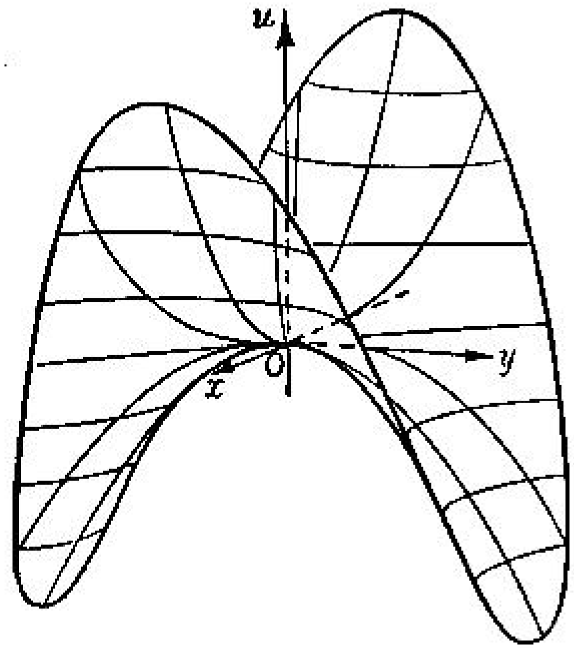



Fig. 2.— $u = x^2 - y^2$

However, this representation by means of rectangular co-ordinates has two disadvantages. Firstly, intuition fails us whenever we have to deal with three or more independent variables. Secondly, even in the case of two independent variables, it is often more convenient to confine the discussion to the $xy$-plane, since in the plane we can sketch and construct geometrically without difficulty. From this point of view, another geometrical representation of the function by means of **contour lines** is to be preferred. In the $xy$-plane, we take all the points for which $u = f(x, y)$ has a constant value, say $u = k$. These points will usually lie on a curve or curves, the so-called contour line for the given constant value of the function. We can also obtain these curves by cutting the surface $u = f(x, y)$ by the plane $u = k$ parallel to the $xy$-plane and projecting the curves of intersection perpendicularly onto the $xy$-plane. The system of these contour lines, marked with the corresponding values $k_1, k_2, \cdots$ of the height $k$, gives us a representation of the function. As a rule, $k$ is assigned values in arithmetic progression, say $k = \nu h$, where $\nu = 1, 2, \cdots$. The distance between the contour lines then gives us a measure of the steepness of the surface $u=f(x,y)$; in fact,

between every two neighbouring lines, the value of the function changes by the same amount. Where the contour lines are close together, the function rises or falls steeply, where they are far apart, the surface is flattish. This is the principle on which contour maps such as those of the Ordnance Surrey and the U.S. Geological Survey are constructed.

In this method, the linear function $u = ax + by + c$ is represented by a system of parallel straight lines $ax + by + c = k$. The function $u = x^2 + y^2$ is represented by a system of concentric circles (Fig. 3). The function $u = x^2 - y^2$, the surface of which has a saddle point at the origin (Fig. 2) is represented by the system of hyperbolas shown in Fig. 4.

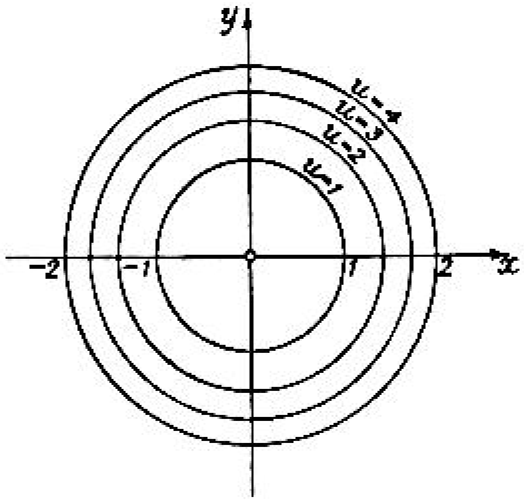



Fig. 3.—Contour lines of $u = x^2 + y^2$

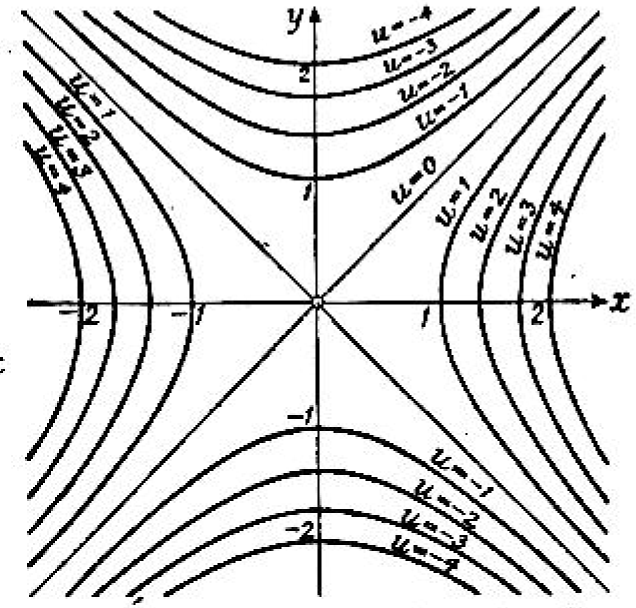



Fig. 4.—Contour lines of $u = x^2 - y^2$

The method of representing the function $u = f(x,y)$ by contour lines has the advantage of it being capable of extension to functions of three independent variables. Instead of contour lines, we then have level surfaces $f(x,y,z) = k$, where $k$ is a constant to which we can assign any suitable sequence of values. For example, the level surfaces for the function $u = x^2 + y^2 + z^2$ are concentric spheres about the origin of the co-ordinate system.

**Exercises 10.1:**

1. For each of the following functions, sketch the contour lines corresponding to a $z = -2, -1, 0, 1, 2, 3$:

(a) $z = x^2 y$.

(b) $z = x^2 + y^2 - 1$.

(c) $z = x^2 - y^2$.

(d) $z = y^2$.

(e) $z = y\left(1 - \dfrac{1}{x^2 + y^2}\right)$.

No Answers and no Hints

## 10.2 Continuity

**10.2.1 Definition:**. As in the case of functions of a single variable, the basic requirement, by which functions should he capable of being represented geometrically leads to the analytic condition of continuity. Here again, the concept of continuity is given by the definition:

A function $u = f(x, y)$, defined in a region $R$, is said to be continuous at a point $(\xi, \eta)$ of $R$, if for all points $(x,y)$ near $(\xi, \eta)$ the value of the function $f(x, y)$ differs but little from $f(\xi, \eta)$, the difference being arbitrarily small if only $(x, y)$ is near enough to $(\xi, \eta)$.

More precisely: The function $f(x, y)$, defined in the region $R$, is continuous at the point $(\xi, \eta)$ of $R$, provided that it is possible for every positive number $\varepsilon$ to find a positive distance $\delta = \delta(\varepsilon)$ (in general, depending on $\varepsilon$ and tending to 0 with $\varepsilon$) such that for all points of the region, the distance of which from $(\xi, \eta)$ is less than $\delta$, that is, for which the inequality

$$(x - \xi)^2 + (y - \eta)^2 \leqq \delta^2$$

holds, there is satisfied the relation

$$|f(x, y) - f(\xi, \eta)| \leqq \epsilon,$$

or, in other words, the relation

$$|f(\xi + h, \eta + k) - f(\xi, \eta)| \leqq \epsilon$$

is to hold for all pairs of values $(h, k)$ such that $h^2 + k^2 \leq \delta^2$ and $(\xi + h, \eta + k)$ belongs to the region $R$.

If a function is continuous at every point of a region *B*, we say that it is **continuous** in *R*.

In the definition of continuity, we can replace the distance condition $h^2+k^2\leq\delta^2$ by the equivalent condition:

There shall correspond to every $\varepsilon > 0$ two positive numbers $\delta_1$ and $\delta_2$ such that

$$|f(\xi+h, \eta+k)-f(\xi, \eta)| \leq \epsilon$$

whenever

$$|h| \leq \delta_1 \quad and \quad |k| \leq \delta_2.$$

The two conditions are equivalent. In fact, if the original condition is fulfilled, so is the second one, if we take $\delta_1 = \delta_2 = \delta/2$; and conversely, if the second condition is fulfilled, so is the first, if we take for $\delta$ the smaller one of the two numbers $\delta_1$ and $\delta_2$.

The following facts are almost obvious:

The sum, difference and product of continuous functions are also continuous. The quotient of continuous functions is continuous except when the denominator vanishes. Continuous functions of continuous functions are themselves continuous (10.4.1). In particular, all polynomials are continuous and so are all rational fractional functions except when the denominator vanishes.

Another obvious fact, which, however, is worth stating, is:

If a function $f(x, y)$ is continuous in a region *R* and differs from zero at an interior point P of the region, it is possible to mark off about *P* a neighbourhood, say a circle, belonging entirely to *R*, in which $f(x,y)$ is nowhere equal to zero. In fact, if the value of the function at *P* is *a*, we can mark off about *P* a circle so small that the value of the function within the circle differs from *a* by less than $a/2$, and therefore it is certainly not zero.

**10.2.2 Examples of Discontinuities:** In the case of functions of one variable, we encounter three kinds of discontinuities: Infinite and jump discontinuities, and discontinuities at which no limit is approached from one or both sides. With functions of two or more variables, no such simple classification is possible. In particular, the situation is made more complicated by the fact that discontinuities may occur not merely at isolated points, but also along entire curves.

Thus, for the function $u = 1/(x - y)$, the line $x = y$ is a line of infinite discontinuity. As we approach the line from one side or the other, the values of $u$ increase numerically beyond all bounds through positive or negative values. The function $u = 1/(x - y)^2$ has the same line of discontinuity, but tends to $+\infty$ as we approach the line from either side. The function $u = 1/(x^2 + y^2)$ has the single point of discontinuity $x = 0$, $y = 0$. The function $u = \sin \frac{1}{\sqrt{(x^2 + y^2)}}$ tends to no limit as we approach the origin; the surface which it represents is obtained by rotating the graph of the function $u = \sin 1/x$ about the $u$-axis.

Another instructive example of a discontinuous function is given by the rational function $u = 2xy/(x^2 + y^2)$. In the first instance, the function is undefined at $x = 0$, $y = 0$, and we supplement the definition by assuming that $u(0,0) = 0$. This function has a peculiar type of discontinuity at the origin. If we put $x = 0$, i.e., if we move along the y-axis, the function becomes $u(0,y) = 0$, which has the constant value 0 for all values of $y$. Along the $x$-axis, we likewise have $u(x, 0) = 0$. Thus, at the origin, the function $u(x,y)$ is continuous in $x$, if we keep $y$ at the constant value 0, and is continuous in $y$, if we keep $x$ at the constant value 0. Nevertheless, the function is discontinuous when considered as a function of the two variables $x$ and $y$. In fact, at every point of the line y=$x$, we find that $u = 1$, so that arbitrarily near to the origin we can find points at which $u$ assumes the value 1. The function is therefore discontinuous at the origin and cannot be defined at the origin in such a way as to make it continuous.

More generally, on the straight line $y = x \tan \alpha$, inclined at the angle $\alpha$ to the $x$-axis, we have $u = 2\tan\alpha/(1+\tan^2\alpha) = 2\sin\alpha\cos\alpha = \sin 2\alpha$. The surface corresponding to the $u = 2\, xy/(x^2+y^2)$ is therefore formed by rotating a straight line at right angle to the $u$-axis about that axis until it coincides with the $x$-axis and simultaneously raising or lowering it so that the height $\sin 2\alpha$ is associated with the angle $\alpha$. As $\alpha$ increases to $45^o$, the straight line rises to the height 1, and consequently falls to the level of the $y$-axis and below to the depth 1, thereafter rising again to the level of the x-axis. The surface enveloped by the moving straight line is known as the **cylindroid** and is important in mechanics.

The above example shows that a function can he continuous in $x$ for every fixed value of $y$ and continuous in $y$ for every fixed value of $x$ and yet be discontinuous when considered as a function of the two variables. The essential point in this definition of continuity is that the value of the function at a point $P$ must be arbitrarily close to the value of the function at a point $Q$, provided only that $Q$ is near enough to $P$; it is not permissible to restrict the position of $Q$ relative to $P$ in any other way.

**Exercises 10.2:**

1. Examine the continuity of the function $z = \dfrac{x^2 + y}{\sqrt{x^2 + y^2}}$. Sketch the level lines $z=k$ ($k$ = -4, -2, 0, 2, 4). Exhibit (on the same graph) the behaviour of $z$ as a function of $x$ alone for $y$ = -2, -1, 0, 1, 2, Similarly, exhibit the behaviour of $z$ as a function of $y$ alone for $x = 0, \pm 1, \pm 2$. Finally, exhibit the behaviour of $z$ as a function of $r$ alone when $\theta$ is constant ($r$, $\theta$ being polar co-ordinates).

2. Show that the following functions are continuous:

(a) $\sin(x^2 + y)$.

(b) $\dfrac{\sin xy}{\sqrt{x^2 + y^2}}$.

(c) $\dfrac{x^3 + y^3}{x^2 + y^2}$.

(d) $x^2 \log(x^2 + y^2)$.

3. Decide whether or not the following functions are continuous and, they are not, where they are discontinuous:

(a) $\sin(x^2 + y)$.

(b) $\dfrac{\sin xy}{\sqrt{x^2 + y^2}}$.

(c) $\dfrac{x^3 + y^3}{x^2 + y^2}$.

(d) $x^2 \log(x^2 + y^2)$.

Answers and Hints

## 10.3 The Derivatives of a Function of Several Variables

**10.3.1 Definition. Geometrical Representation:** If we assign in a function of several variables definite numerical values to all but one of the variables and allow only that one variable, say $x$, to vary, the function becomes a function of one variable. For

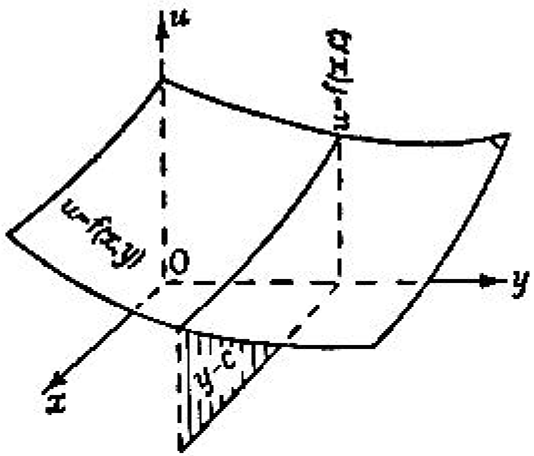



Fig. 5

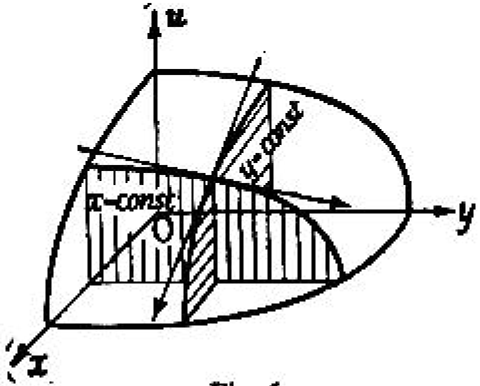



Fig. 6

Sections of $u = f(x, y)$

example, consider a function $u = f(x,y)$ of the two variables $x$ and $y$ and give $y$ the definite fixed value $y = y_0 = c$. The function $u = f(x, y_0)$ of the single variable $x$, which is thus formed, may be simply represented geometrically by letting the surface $u = f(x, y)$ be cut by the plane $y = y_0$ (Figs. 5 and 6). The curve of intersection thus formed in the plane is represented by the equation $u=f(x,y_0)$. If we differentiate this function in the usual way at the point $x = x_0$ (assuming that the derivative exists), we obtain the **partial derivative of $f(x,y)$ with respect to $x$** at the point $(x_0, y_0)$. According to the ordinary definition of the derivative, this is the limit

$$\lim_{h \to 0} \frac{f(x_0 + h,\ y_0) - f(x_0,\ y_0)}{h}.$$

If $(x_0, y_0)$ is a point on the boundary of the region of definition, we make the restriction that in the passage to the limit the point $(x + h, y_0)$ must always remain in the region.

Geometrically speaking, this partial derivative denotes the tangent of the angle between a parallel to the $x$-axis and the tangent line to the curve $u = (x,y_0)$. It is therefore the **slope of the surface u = $f(x,y)$ in the direction of the $x$-axis.**

In order to represent these partial derivatives, **several different notations** are in use of which we mention:

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = f_x(x_0, y_0) = u_x(x_0, y_0).$$

If we wish to emphasize -that the partial derivative is the limit of a difference quotient, we denote it by

$$\frac{\partial f}{\partial x} \text{ or } \frac{\partial}{\partial x} f.$$

We use here a special curved letter $\partial$ instead of the ordinary *d* used in the differentiation of functions of one variable, in order to show that we are dealing with a function of several variables and differentiating with respect to one of them.

Sometimes. it is convenient to use Cauchy's symbol *D* and to write

$$\frac{\partial f}{\partial x} = D_x f;$$

however, we shall rarely use this symbol.

In exactly the same way, we define the partial derivative of $f(x, y)$ with respect to $y$ at the point $(x_0, y_0)$ by the relation

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k} = f_y(x_0, y_0) = \frac{\partial f}{\partial y} = D_y f(x_0, y_0).$$

It represents the slope of the curve of intersection of the surface $u = f(x, y)$ with the plane $x = x_0$, perpendicular to the $x$-axis.

Now, let us think of the point $(x_0, y_0)$, hitherto considered to be fixed, as variable and accordingly omit the suffices. In other word, we think of the differentiation as being carried out at any point $(x, y)$ of the region of definition of $f(x, y)$. Then the two derivatives are themselves functions of $x$ and $y$:

$$u_x(x, y) = f_x(x, y) = \frac{\partial f(x, y)}{\partial x} \text{ and } u_y(x, y) = f_y(x, y) = \frac{\partial f(x, y)}{\partial y}.$$

For example, the function $u = x^2 + y^2$ has the partial derivative $u_x = 2x$ (in differentiating with respect to $x$, the term $y^2$ is considered to be a constant, whence is has the derivative 0) and $u_y = 2y$. The partial derivatives of $u = x^2y$ are $u_x = 3x^2y$ and $u_y = x^3$.

Similarly, we make the definition for any number ($n$) of independent variables:

$$\begin{aligned} &\frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_1} \\ &\quad = \lim_{h \to 0} \frac{f(x_1 + h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h} \\ &\quad = f_{x_1}(x_1, x_2, \ldots, x_n) = D_{x_1} f(x_1, x_2, \ldots, x_n), \end{aligned}$$

it being assumed that the limit exists.

Of course, we can also form higher partial derivatives of $f(x, y)$ by again differentiating the partial, first order derivatives $f_x(x, y)$ and $f_y(x, y)$ with respect to one of the variables, and repeating this process. We indicate the order of differentiation by the order of the suffixes or by the order of the symbols $\partial x$ and $\partial y$ in the denominator from the right to the left hand side, and use for the second partial derivatives the:

On the other hand, in Continental usage, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ becomes $\frac{\partial^2 f}{\partial y \partial x}$.

We likewise denote the third partial derivatives by

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3} = f_{xxx},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial y\,\partial x^2} = f_{yxx},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\ \partial y}\right) = \frac{\partial^3 f}{\partial x^2\,\partial y} = f_{xxy}, \text{ etc.;}$$

and, in general, the *n*-th derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right) = \frac{\partial^n f}{\partial x^n} = f_{x^n},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right) = \frac{\partial^n f}{\partial y\,\partial x^{n-1}} = f_{yx^{n-1}}, \text{ \&c.}$$

Finally, we shall study a few examples of the actual calculation of partial derivatives. According to the definition, all the independent variables are to be kept constant except the one with respect to which we are differentiating. We therefore have merely to regard the other variables as constants and carry out out the differentiation according to the rules by which we differentiate functions of a single independent variable.

For example, we have:

**1. Function** $f(x, y) = xy;$

**first derivatives,** $f_x = y,\ f_y = x;$

**second derivatives,** $f_{xx} = 0,\ f_{xy} = f_{yx} = 1,\ f_{yy} = 0.$

**2. Function** $f(x, y) = \sqrt{(x^2 + y^2)};$

**first derivatives,** $f_x = \frac{x}{\sqrt{(x^2 + y^2)}},\quad f_y = \frac{y}{\sqrt{(x^2 + y^2)}}.$

(Thus for the radius vector $r = \sqrt{(x^2 + y^2)}$ from the origin to the point ($x$, y), the partial derivatives with respect to $x$ and $y$ are given by the cosine, $\cos\varphi = x/r$, and the sine, $\sin\varphi = y/r$ of the angle $\varphi$, which the radius vector makes with the positive direction of the $x$-axis.)

Second derivatives

$$f_{xx} = \frac{\sqrt{(x^2 + y^2)} - \dfrac{x^2}{\sqrt{(x^2 + y^2)}}}{x^2 + y^2} = \frac{y^2}{\sqrt{(x^2 + y^2)^3}} = \frac{\sin^2\varphi}{r},$$

$$f_{xy} = f_{yx} = -\frac{xy}{\sqrt{(x^2 + y^2)^3}} = -\frac{\sin\varphi\cos\varphi}{r},$$

$$f_{yy} = \frac{\sqrt{(x^2 + y^2)} - \dfrac{y^2}{\sqrt{(x^2 + y^2)}}}{x^2 + y^2} = \frac{x^2}{\sqrt{(x^2 + y^2)^3}} = \frac{\cos^2\varphi}{r}.$$

3. **Reciprocal of the radius vector in three dimensions:**

$$f(x, y, z) = \frac{1}{\sqrt{(x^2 + y^2 + z^2)}} = \frac{1}{r};$$

**first derivatives,**

$$f_x = -\frac{x}{\sqrt{(x^2 + y^2 + z^2)^3}} = -\frac{x}{r^3},$$

$$f_y = -\frac{y}{\sqrt{(x^2 + y^2 + z^2)^3}} = -\frac{y}{r^3},$$

$$f_z = -\frac{z}{\sqrt{(x^2 + y^2 + z^2)^3}} = -\frac{z}{r^3};$$

**second derivatives,**

$$f_{xx} = -\frac{1}{r^3} + \frac{3x^2}{r^5}, \quad f_{yy} = -\frac{1}{r^3} + \frac{3y^2}{r^5}, \quad f_{zz} = -\frac{1}{r^3} + \frac{3z^2}{r^5},$$

$$f_{xy} = f_{yx} = \frac{3xy}{r^5}, \quad f_{yz} = f_{zy} = \frac{3yz}{r^5}, \quad f_{zx} = f_{xz} = \frac{3zx}{r^5}.$$

We see from this that there holds for the function $f = \frac{1}{\sqrt{(x^2 + y^2 + z^2)}}$ the equation

$$f_{xx} + f_{yy} + f_{zz} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = 0,$$

for all values of *x*, *y*, *z* except 0,0,0; as we may say, the equation

$$f_{xx} + f_{yy} + f_{zz} = 0$$

is satisfied identically in x,y,z by the function f(x,y,*z*) = l/*r*.

**4. Function**
$$f(x, y) = \frac{1}{\sqrt{y}} e^{-(x-a)^2/4y};$$

**first derivatives,**

$$f_x = \frac{1}{\sqrt{y}} \frac{-(x-a)}{2y} e^{-(x-a)^2/4y},$$

$$f_y = \left(\frac{-1}{2y^{3/2}} + \frac{(x-a)^2}{4y^{5/2}}\right) e^{-(x-a)^2/4y};$$

**second derivatives,**

$$f_{xx} = \left(\frac{-1}{2y^{3/2}} + \frac{(x-a)^2}{4y^{5/2}}\right) e^{-(x-a)^2/4y},$$

$$f_{xy} = f_{yx} = \left(\frac{3}{4} \frac{x-a}{y^{5/2}} - \frac{(x-a)^3}{8y^{7/2}}\right) e^{-(x-a)^2/4y},$$

$$f_{yy} = \left(\frac{3}{4} \frac{1}{y^{5/2}} - \frac{3}{4} \frac{(x-a)^2}{y^{7/2}} + \frac{(x-a)^4}{16y^{9/2}}\right) e^{-(x-a)^2/4y}.$$

The equation

$$f_{xx} - f_y = 0$$

is therefore satisfied identically in *x* and *y*.

Just as in the case of one independent variable, the possession of derivatives is a special property of a function. (Note that the expression differentiable implies more than that the partial derivatives with respect to *x* and *y* exist (cf. Vol. II!) All the same, this property is possessed by all functions of practical importance except perhaps at isolated exceptional points.

In contrast to functions of one variable, the possession of derivatives does not imply the continuity of the function. This is clearly shown by the example $u = \dfrac{2xy}{x^2 + y^2}$ already considered. because the partial derivatives exist for it everywhere and yet the function is discontinuous at the origin. But, as is stated by the following theorem, the possession of **bounded derivatives does imply continuity**:

If a function $f(x,y)$ has partial derivatives $f_x$ and $f_y$ everywhere in a region $R$, and these derivatives satisfy everywhere the in equalities

$$|f_x(x, y)| < M, \quad |f_y(x, y)| < M,$$

where $M$ is independent of $x$ and $y$, then $f(x, y)$ is continuous everywhere in $R$.

In particular, if $f_x$ and $f_y$ are continuous, they are necessarily bounded, so that $f(x,y)$ is also continuous.

We shall leave the proof of this theorem for Volume II. The reader will have noted that in all our examples the equation $f_{xy} = f_{yx}$ is satisfied. In other words, it made no difference whether we differentiated first with respect to $x$ and then with respect to $y$ or *vice versa*. This is no accidental occurrence. In fact, we have the theorem:

If the **mixed partial derivatives** $f_{xy}$ and $f_{yx}$ of a function $f(x,y)$ are continuous in a region $R$, then the equation

$$f_{yx} = f_{xy}$$

holds everywhere in the interior of this region, i.e., the order of differentiation with respect to $x$ and $y$ is immaterial.

Applying this theorem to $f_x$ and $f_y$, then to $f_{xx}$, $f_{xy}$, $f_{yy}$, etc., we find that

$$\begin{gathered} f_{xxy} = f_{xyx} = f_{yxx}, \\ f_{xyy} = f_{yxy} = f_{yyx}, \\ f_{xxyy} = f_{xyxy} = f_{xyyx} = f_{yxxy} = f_{yxyx} = f_{yyxx}, \text{ \&c.,} \end{gathered}$$

and, in general, we have the result:

In repeated differentiation of a function of two variables, the order of differentiation can be changed arbitrarily, provided only that the derivatives in question are continuous functions.

For the proof of this theorem, we again refer the reader to Volume II.

## Exercises 10.2:

1. Find the first partial derivatives of the functions:

(a) $\sqrt[3]{x^2+y^2}$.

(b) $\sin(x^2-y)$.

(c) $e^{x-y}$.

(d) $\dfrac{1}{\sqrt{(1+x+y^2+z^2)}}$.

(e) $y\sin(xz)$.

(f) $\log\sqrt{(1+x^2+y^2)}$.

2.. Find all the first and second partial derivatives of the functions:

(a) $xy$.

(b) $\log xy$.

(c) $\tan(\operatorname{arc\,tan} x + \operatorname{arc\,tan} y)$.

(d) $x^y$.

(e) $e^{(x^y)}$.

3.* Find a function $f(x, y)$ which is a function of $(x^2 + y^2)$ and is also a product of the form $\psi(x)\psi(y)$, i.e., solve the equations

$$f(x, y) = \varphi(x^2 + y^2) = \psi(x)\psi(y)$$

Answers and Hints

### 10.4 The Chain Rule and the Differentiation of Inverse Functions

**10.4.1 Functions of Functions (Compound Functions):**. It often happens that a function $u$ of the independent variables $x$, $y$ is stated in the form

$$u = f(\xi, \eta, \ldots),$$

where the arguments $\xi$, $\eta$, ··· of the function $f$ are themselves functions of $x$ and $y$:

$$\xi = \phi(x, y), \quad \eta = \psi(x, y), \ldots .$$

We then say that

$$u = f(\xi, \eta, \dots) = f(\phi(x, y), \psi(x, y), \dots) = F(x, y)$$

is given as a compound function of $x$ and $y$.

For example, the function

$$u = e^{x^2 y}(x + y)^3$$

may be written as a compound function by means of the relations

$$u = e^{\xi}\eta^3 = f(\xi, \eta); \quad \xi = x^2 y, \eta = x + y.$$

Similarly, the function

$$u = \log(x + 1) \,.\, \operatorname{arc}\cos\sqrt{(4 - x^2 - y^2)}$$

can be expressed in the form

$$u = \eta \operatorname{arc}\cos\xi = f(\xi, \eta); \quad \xi = \sqrt{(4 - x^2 - y^2)}, \quad \eta = \log(x + 1).$$

In order to make this concept more precise, we assume to begin with that the functions $\xi = \phi(x, y), \eta = \psi(x, y), \dots$ are defined in a certain region $R$ of the independent variables $x$, $y$. Then there corresponds to every point $(x,y)$ of $R$ a point $(\xi,\eta, \cdots)$. As the point $(x,y)$ ranges over $R$, the point $(\xi,\eta, \cdots)$ will range over a certain set of values. We assume that the point $(\xi,\eta, \cdots)$ always lies within a region $S$ in which $f(\xi,\eta, \cdots)$ is defined. The function

$$u = f(\phi(x, y), \psi(x, y), \dots) = F(x, y)$$

is then defined in the region $R$.

Referring to our examples, we find in the first one of them that $\xi$ and $\eta$ are defined for every $x$, $y$ and $f(\xi,\eta)$ is defined for every $\xi,\eta$, so that our region $R$ can be taken to be the entire $xy$-plane. However, in the second example, the region $S$ is restricted by the inequality $|\xi| \leq 1$, since for $|\xi| > 1$ the function

arcos $\xi$ is undefined. Secondly, the region $R$ is restricted by the inequalities $x+1>0$ and $x^2 + y^2 \leq 4$, since for other values both $\xi$ and $\eta$ are not defined. Thirdly, the region $R$ must be further limited by the inequality $3 \leq x^2 + y^2$ in order that the point with co-ordinates $\xi, \eta$ shall fall into $S$, i.e., the restriction $|\xi| \leq 1$ implies that $x^2 + y^2 \geq 3$. Hence $R$ consists of the part of the ring $3 \leq x^2+y^2 \leq 4$ lying to the right of the line $x = -1$.

The following theorem on compound functions is an immediate consequence of the definitions:

If the function $u = f(\xi, \eta, \cdots)$ is continuous in $S$ and the functions $\xi = \phi(x, y)$, $\eta = \psi(x, y))$, $\cdots$ are continuous in $R$, then the compound function $u = F(x, y)$ is continuous in $R$. $\xi = \phi(x, y)$

The reader should be able to prove this statement

**10.4.2 The Chain Rule:** We now turn our attention to compound functions of the type $u = f(\xi, \eta, \cdots)$, where $\xi, \eta, \cdots$ depend on the single variable $x$:

$$\xi = \phi(x), \quad \eta = \psi(x), \ldots .$$

For such functions, we have the important theorem known as the **chain rule**:

If the function $u = f(\xi, \eta, \cdots)$ has continuous partial derivatives of the first order in $S$ and the functions $\xi = \phi(x)$, $\eta = \psi(x)$, $\cdots$ have continuous first derivatives in the interval $R$, $a \leq x \leq b$, then $u = f(\phi(x), \psi(x), \cdots) = Fxi)$ has a continuous derivative in $R$ and

$$F'(x) = f_\xi \phi'(x) + f_\eta \eta'(x) + \ldots .$$

The right-hand side of this equation is an abbreviation for

$$f_\xi\{\phi(x), \psi(x), \ldots\}\phi'(x) + \ldots .$$

In order to simplify the notation, we shall assume that $f$ is a function of the three arguments $\xi, \eta, \zeta$. We shall denote by $x_0$ an arbitrary fixed point of the interval $a \leq x \leq b$, by $\xi_0, \eta_0, \zeta_0$ the corresponding values $\xi_0 = \phi(x_0)$, $\eta_0 = \psi(x_0)$, $\xi_0 = \chi(x_0)$ and by $\xi, \eta, \zeta$ the values $\phi(x)$, $\psi(x)$, $\chi(x)$, corresponding to a variable point $x = x_0 + h$. We first write down the identity

$$
\begin{aligned}
F(x) - F(x_0) \\
&= f(\xi, \eta, \zeta) - f(\xi_0, \eta_0, \zeta_0) \\
&= \{f(\xi, \eta, \zeta) - f(\xi_0, \eta, \zeta)\} + \{f(\xi_0, \eta, \zeta) - f(\xi_0, \eta_0, \zeta)\} \\
&\qquad + \{f(\xi_0, \eta_0, \zeta) - f(\xi_0, \eta_0, \zeta_0)\}.
\end{aligned}
$$

In each bracket on the right hand side, we observe that only one of the independent variables changes its value. Hence we can apply to each bracket the mean value theorem for functions of a single variable and obtain

$$
\begin{aligned}
F(x) - F(x_0) \\
&= (\xi - \xi_0) f_\xi(\bar{\xi}, \eta, \zeta) + (\eta - \eta_0) f_\eta(\xi_0, \bar{\eta}, \zeta) + (\zeta - \zeta_0) f_\zeta(\xi_0, \eta_0, \bar{\zeta}),
\end{aligned}
$$

where $\bar{\xi}$ lies between $\xi$ and $\xi_0$, $\bar{\eta}$ between $\eta_0$ and $\eta$, and $\bar{\zeta}$ between $\zeta_0$ and $\zeta$. Moreover, by the mean value theorem, we have

$$
\begin{aligned}
\xi - \xi_0 &= \phi(x) - \phi(x_0) = (x - x_0)\,\phi'(x_1), \\
\eta - \eta_0 &= \psi(x) - \psi(x_0) = (x - x_0)\,\psi'(x_2), \\
\zeta - \zeta_0 &= \chi(x) - \chi(x_0) = (x - x_0)\,\chi'(x_3),
\end{aligned}
$$

where all $x_1$, $x_2$, and $x_3$ lie between $x_0$ and $x$. Substituting these values in the last equation and dividing by $x - x_0$, we have

$$
\begin{aligned}
\frac{F(x) - F(x_0)}{x - x_0} \\
&= f_\xi(\bar{\xi}, \eta, \zeta)\,\phi'(x_1) + f_\eta(\xi_0, \bar{\eta}, \zeta)\,\psi'(x_2) + f_\zeta(\xi_0, \eta_0, \bar{\zeta})\,\chi'(x_3).
\end{aligned}
$$

We now let $x$ tend to $x_0$. Owing to the continuity of $\phi(x)$, $\psi(x)$, $\chi(x)$, the quantities $\xi, \eta, \zeta$ tend to $\xi_0, \eta_0, \zeta_0$, respectively, and, *a fortiori*, so do $\bar{\xi}, \bar{\eta}, \bar{\zeta}$. Also $x_1$, $x_2$ and $x_3$ tend to $x_0$. Since all the functions on the right hand side are continuous, we have

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = F'(x_0)$$
$$= f_\xi(\xi_0, \eta_0, \zeta_0)\,\phi'(x_0) + f_\eta(\xi_0, \eta_0, \zeta_0)\,\psi'(x_0) + f_\zeta(\xi_0, \eta_0, \zeta_0)\,\chi'(x_0),$$

thus establishing the formula for $F'(x)$.

The continuity of $F'(x)$ follows immediately from the formula, since, by assumption, $\phi'$, $\psi'$, $\chi'$, are continuous $f_\xi$, $f_\eta$ and $f_\zeta$ are are continuous functions of continuous functions.

This theorem may be extended to compound functions of two or more variables, as follows:

If the funding $u = f(\xi, \eta, \cdots)$ has continuous partial derivatives of the first order in the region $S$ and the functions $\xi = \phi(x,y)$, $\eta = \psi(x,y)$, $x,y$ have continuous partial derivatives of the first order in $R$, then $u = F(x, y) = f\{\phi(x,y), \psi(x,y), \cdots$ has continuous partial derivatives of the first order in $R$ and their derivatives are given by

$$F_x = f_\xi \phi_x + f_\eta \psi_x + \cdots,$$
$$F_y = f_\xi \phi_y + f_\eta \psi_y + \cdots.$$

These formulae are often written in the abbreviated form

$$u_x = u_\xi \xi_x + u_\eta \eta_x + \cdots,$$
$$u_y = u_\xi \xi_y + u_\eta \eta_y + \cdots.$$

In order to derive them, we temporarily introduce the notation $g(\mathrm{x}) = \phi(x,y_0)$, $h(\mathrm{x}) = \psi(x,y_0)$, $\cdots$, where $y_0$ is a fixed value of $y$. By the definition of the partial derivatives, it follows that $g'(\mathrm{x}) = \phi_x(x,y_0)$, $h'(\mathrm{x}) = \psi_x(x,y_0)$, $\cdots$. Similarly, if we writs $H(\mathrm{x}) = F(x, y_0)$, we have $H'(x) = F_x(x, y_0)$. We now apply the theorem just proved to the function $u = H(x) = f(\xi, \eta, \cdots) = f\{g(x), h(x), \cdots\}$ and obtain

$$H'(x_0) = f_\xi g'(x_0) + f_\eta h'(x_0) + \cdots.$$

Returning to the original symbols, we have

$$F_x(x_0, y_0) = f_\xi \phi_x(x_0, y_0) + f_\eta \psi_x(x_0, y_0) + \dots .$$

The other formula is proved in a similar manner.

If we wish to calculate higher order derivatives, we need only differentiate again the right hand side of our formulae with respect to $x$ and y, regarding $f_\xi$, $f_\eta$, ⋯ as compound functions. Thus, for u = $f(\xi,\eta) = f\{\phi(x,y),\ \psi(x,y)\}$, we have

$$\begin{aligned}
u_{xx} &= f_{\xi\xi}\phi_x^2 + 2f_{\xi\eta}\phi_x\psi_x + f_{\eta\eta}\psi_x^2 + f_\xi\phi_{xx} + f_\eta\psi_{xx},\\
u_{xy} &= f_{\xi\xi}\phi_x\phi_y + f_{\xi\eta}(\phi_x\psi_y + \phi_y\psi_x) + f_{\eta\eta}\psi_x\psi_y + f_\xi\phi_{xy} + f_\eta\psi_{xy},\\
u_{yy} &= f_{\xi\xi}\phi_y^2 + 2f_{\xi\eta}\phi_y\psi_y + f_{\eta\eta}\psi_y^2 + f_\xi\phi_{yy} + f_\eta\psi_{yy}.
\end{aligned}$$

### 10.4.3 Examples:

We emphasize that the following differentiations can also be carried out directly without using the chain rule.

1. $u = e^{x\tan y + y\cos x}$.

Here we put $\xi = x\tan y$, $\eta = y\cos x$, so that $\xi_x = \tan y$, $\xi_y = \dfrac{x}{\cos^2 y}$, $\eta_x = -y\sin x$, $\eta_y = \cos x$. Since $u = e^{\xi+\eta}$, $u_\xi = u_\eta = e^{\xi+\eta}$, and

$$u_x = e^{x\tan y + y\cos x}(\tan y - y\sin x),$$

$$u_y = e^{x\tan y + y\cos x}\left(\frac{x}{\cos^2 y} + \cos x\right).$$

2. An example of a compound function of a single variable is

$$u = \{g(x)\}^{h(x)} = \xi^\eta = f(\xi, \eta),$$

where we put $\xi = g(x)$, $\eta = h(x)$. We immediately obtain

$$\frac{du}{dx} = f_\xi\xi' + f_\eta\eta' = \eta\xi^{\eta-1}\xi' + \xi^\eta\log\xi \,.\, \eta'$$

$$= \{g(x)\}^{h(x)}\left\{h(x)\frac{g'(x)}{g(x)} + h'(x)\log g(x)\right\}.$$

We have already dealt with a special case of this by rather artificial methods (cf. 2.3.3).

**10.4.4 Change of the Independent Variables:**. A particularly important type of compound function occurs in the process of changing the **independent variables**. For example, let $u = f(\xi,\eta)$ be a function of $\xi$ and $\eta$, which we interpret as rectangular co-ordinates in the $\xi\eta$-plane. If we rotate the axes in the $\xi\eta$-planes through an angle $\theta$, we obtain a new system of co-ordinates $x,y$, related to the co-ordinates $(\xi,\eta)$ by the equations:

$$\xi = x\cos\theta - y\sin\theta, \qquad \eta = x\sin\theta + y\cos\theta, \quad \text{or}$$

$$x = \xi\cos\theta + \eta\sin\theta, \qquad y = -\xi\sin\theta + \eta\cos\theta.$$

The function $u = f(\xi,\eta)$ can then be expressed as a function of the new variables $x$, $y$:

$$u = f(\xi, \eta) = F(x, y).$$

The chain rule yields then immediately:

$$u_x = u_\xi\cos\theta + u_\eta\sin\theta, \quad u_y = -u_\xi\sin\theta + u_\eta\cos\theta.$$

Thus, the partial derivatives are transformed by the same formulae as the independent variables. This is true for rotation of the axes in space also. Another important type of change of co-ordinates is the change from rectangular co-ordinates $x$, $y$ to polar co-ordinates $r$, $\theta$. This is done by means of the equations

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$r = \sqrt{(x^2 + y^2)}, \qquad \theta = \arctan\frac{y}{x}.$$

We then find that for an arbitrary function $u = f(x,x)$ with continuous partial derivatives of the first order we have

$$u = f(x, y) = f(r\cos\theta, r\sin\theta) = F(r, \theta),$$

$$u_x = u_r r_x + u_\theta \theta_x = u_r \frac{x}{r} - u_\theta \frac{y}{r^2} = u_r \cos\theta - u_\theta \frac{\sin\theta}{r},$$

$$u_y = u_r r_y + u_\theta \theta_y = u_r \frac{y}{r} + u_\theta \frac{x}{r^2} = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

Whence we obtain the often useful equation

$$u_x^2 + u_y^2 = u_r^2 + \frac{1}{r^2} u_\theta^2.$$

In general, let us consider a pair of functions $\xi = \phi(x, y)$, $\eta = \psi(x, y)$ which are continuous and have continuous derivatives in a region $R$ of the $xy$-plane. These equations assign to each point $(x, y)$ in $R$ a point $\xi = \phi(x, y)$, $\eta = \psi(x, y)$ in the $\xi\eta$-plane. As $(x, y)$ ranges over R, the corresponding point $(\xi,\eta)$ will range over some set of values $S$ in the $\xi\eta$-plane. Naturally, it is possible that several distinct points $(x, y)$ will give the same values for $(\xi,\eta)$, so that there corre-spond to several points $(x, y)$ only one point $(\xi,\eta)$. We shall assume that this is not s, but instead that there corresponds to one point Q$(\xi,\eta)$ in $S$ exactly one point $P(x,y)$ in $R$. We may therefore look at the correspondence from either point of view - saying that $Q$ corresponds to $P$ or that $P$ corresponds to $Q$. The latter point of view can be expressed as follows: There corresponds to each point $(\xi,\eta)$ in S one $x$ and one $y$, namely, the co-ordinates of $P$, or, in , equations, there are two functions $x = g(\xi,\eta)$, $y = h(\xi,\eta)$, defined in $S$, which represent the corresponding inverse to $\xi = \phi(x,y)$, $\eta = \psi(x,y)$.

It happens often that the functions $g(\xi,\eta)$, $h(\xi,\eta)$ are by no means easy to calculate, even when they do exist, whence we shall now discover how to obtain the partial derivatives $g_\xi$, $g_\eta$, $h_\xi$, $h_\eta$ directly from the partial derivatives $\phi_x$, $\phi_y$, $\psi_x$, $\psi_y$ without at all calculating $g$ and $h$ themselves. For this purpose, we observe that, if we choose any point $Q(\xi, \eta)$, find the corresponding point P$\{g(\xi,\eta),h(\xi,\eta)\}$ in $R$ and then find the point in $S$ corresponding to $P$, which is $\phi\{g(\xi,\eta),h(\xi,\eta)\}$, $\psi\{g(\xi,\eta),h(\xi,\eta)\}$, we have simply returned to the point Q. In other words, the equations $\xi = \phi\{g(\xi,\eta),h(\xi,\eta)\}$, $\eta = \psi\{g(\xi,\eta),h(\xi,\eta)\}$ are identities in $\xi$ and $\eta$. We now differentiate both sides of both equations with respect to $\xi$ and $\eta$ and find

$$1 = \phi_x g_\xi + \phi_y h_\xi, \quad 0 = \phi_x g_\eta + \phi_y h_\eta,$$
$$0 = \psi_x g_\xi + \psi_y h_\xi, \quad 1 = \psi_x g_\eta + \psi_y h_\eta.$$

If an equation expresses an identical relationship, differentiation with respect to any independent variable in it yields an identity, as follows immediately from the definition.

By solving these equations, we find that

$$g_\xi = \frac{\psi_y}{D}, \quad g_\eta = -\frac{\phi_y}{D}, \quad h_\xi = -\frac{\psi_x}{D}, \quad h_\eta = \frac{\phi_x}{D},$$

or

$$x_\xi = \frac{\eta_y}{D}, \quad x_\eta = -\frac{\xi_y}{D}, \quad y_\xi = -\frac{\eta_x}{D}, \quad y_\eta = \frac{\xi_x}{D},$$

where by $D$ is the determinant

$$D = \xi_x \eta_y - \xi_y \eta_x = \begin{vmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{vmatrix},$$

which we assume to be non-zero.

This determinant $D$, called the **functional determinant** or **Jacobian of ($\xi$,$\eta$) with respect to ($x$,$y$)**, occurs so frequently that one uses frequently the special symbol

$$D = \frac{\partial(\xi, \eta)}{\partial(x, y)}.$$

**Exercises 10.3:**

1. Calculate the first order partial derivatives of

$$(a)\ f = \frac{1}{\sqrt{(x^2+y^2+2xy\cos z)}}. \qquad (c)\ f = x^2 + y\log(1+x^2+y^2+z^2).$$

$$(b)\ f = \arcsin\frac{x}{z+y^2}. \qquad (d)\ f = \arctan\sqrt{(x+yz)}.$$

2. Caculate the derivatives of

$$(a)\ f = x^{(x^x)},\ (b)\ f = \left(\left(\frac{1}{x}\right)^{1/x}\right)^{1/x}.$$

3. Prove that, if $f(x, y)$ satisfies the **Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, \quad \text{so does } \varphi(x, y) = f\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

4. Prove that the functions

$$(a)\ f(x, y) = \log\sqrt{(x^2+y^2)}. \qquad (b)\ g(x, y, z) = \frac{1}{\sqrt{(x^2+y^2+z^2)}}.$$

$$(c)\ h(x, y, z, w) = \frac{1}{x^2+y^2+z^2+w^2},$$

satisfy the respective Laplace equations:

$$(a)\ f_{xx} + f_{yy} = 0. \qquad (b)\ g_{xx} + g_{yy} + g_{zz} = 0.$$

$$(c)\ h_{xx} + h_{yy} + h_{zz} + h_{ww} = 0.$$

5. Given $z = r^2 \cos\theta$, where $r$ and $\theta$ are polar coordinates, find $z_x$ and $z_y$ at the point $\theta = \pi/4$, r = 2. Express $z_r$ and $z_\theta$ in terms of $z_x$ and $z_y$

6.By the transformation $\xi = a + \alpha x + \beta y$, $\eta = b - \beta x + y\alpha$, in which $a,b,\alpha, \beta$, are constants and $\alpha^2 + \beta^2 = 1$, the function $u(x, y]$ is transformed mto a function $U(\xi,\eta)$ of $\xi$ and $\eta$. Prove that

$$U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}{}^2 = u_{xx}u_{yy} - u_{xy}{}^2.$$

7. Find the Jacobians of the transformations:

$$(a)\ \xi = ax + by,\ \eta = cx + dy; \qquad (b)\ r = \sqrt{(x^2 + y^2)},\ \theta = \arc\tan\frac{y}{x};$$
$$(c)\ \xi = x^2,\ \eta = y^2.$$

8. If $x = (u,v)$, $y = y(u,v)$ and $u = u(\xi,\eta)$, $v = v(\xi,\eta)$, prove that

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(\xi, \eta)}.$$

9, As a corollary to 8., prove that

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\dfrac{\partial(u, v)}{\partial(x, y)}}.$$

10. Using 9., find the Jacobian of the transformations which are the inverses of those in Example 7.

Answers and Hints

## 10.5 Implicit Functions

In our study of functions of several variables, we have as yet had no analogue to the inverse function. We can regard the inverse function of $y = f(x)$ to be the function obtained by solving the equation $y - f(x) = 0$ for $x$. In this section, we shall seek more generally to solve equations $F(x, y) = 0$ for $x$ or $y$ and to discuss functions of several variables in a corresponding way.

Even in elementary analytical geometry, curves are frequently represented not by equations $y = f(x)$ or $x = \phi(y)$, but by an equation involving $x$ and y in the form $F(x, y) = 0$. For example, we have the circle $x^2 + y^2 - 1 = 0$, the ellipse $x^2/a^2 + y^2/b^2 - 1 = 0$ and. the lemniscate $(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0$. In order to obtain $y$ as a function of $x$ or $x$ as a function of $y$, we must solve the equation for $y$ or for $x$. We then say that the function $y = f(x)$ or $x = \phi(y)$ found in this way is defined implicitly by the equation $F(x, y) = 0$, and that the solution of this equation yields the function explicitly. In the examples cited and in many others, the solution can be carried out and the solutions stated explicitly in terms of the elementary functions. In other cases, the solution can be obtained in terms of an infinite

series or other limiting processes, i.e., we can approximate to the solution $y = f(x)$ or $x = \phi(y)$ as closely as we please.

However, for many purposes, it is more convenient to base the discussion on the implicit definition $F(x, y) = 0$, instead of resorting to an exact or approximate solution of the equation.

The idea that every function $F(x, y)$ yields a function $y = f(x)$ or $x = \phi(y)$ given implicitly by means of the equation $F(x, y) = 0$ is erroneous. On the contrary, it is easy to give examples of functions $F(x, y)$ which, when equated to zero, permit no solution in terms of functions of one variable. Thus, for example, the equation $x^2 + y^2 = 0$ is satisfied by the single pair of values $x = 0$, $y = 0$ only, while the equation $x^2 + y^2 + 1 = 0$ is not satisfied by any real values. It is therefore necessary to investigate this matter more closely, in order to find out whether an equation $F(x, y) = 0$ can actually be solved and what properties its solution has. We cannot undertake in detail such an investigation here, but will content ourselves with a geometrical interpretation which suggests the required results, the rigorous proofs being left to Volume II.

**10.5.1 Geometrical Interpretation of Implicit Functions:**. In order to discuss this problem geometrically, we represent the function $u = F(x, y)$ by a surface in three-dimensional space. Finding values $(x, y)$ which satisfy the equation $F(x, y) = 0$ is the same thing as finding values $(x, y)$ which satisfy two equations $F(x, y) = \underline{u}$, $u = 0$; in other words, we wish to find the intersection of the surface $u = F(x, y)$ and the plane $u = 0$ - the $xy$-plane. We then suppose that we have a definite point $(x_0,y_0)$ which satisfies the equation $F(x_0, y_0) = 0$), i.e., at $(x_0, y_0)$, the surface $u = F(x, y)$ has a point in common with the plane $u = 0$. (If there does not exist such a point, there is no intersection and the equation $F(x, y) = 0$ cannot be solved.) If the tangent plane to the surface $u = F(x, y)$ at the point $(x_0,y_0)$ is not horizontal, it cuts the plane $u = 0$ in a single straight line. Intuition then tells us that the surface $u = F(x, y)$, lying near the tangent plane, likewise cuts the plane $u = 0$ in a single well defined curve. How far this curve extends does not at present concern us. The tangent plane will be horizontal, if both the curves $u = F(x_0, y)$ and $u = F(x, y_0)$ have horizontal tangent lines at $(x_0, y_0)$, i.e., if $F_x(x_0, y_0) = 0$ and $F_y(x_0, y_0)) = 0$. Thus, if either $F_x(x_0, y_0) \neq 0$ or $F_y(x_0, y_0) \neq 0$, the tangent plane is not horizontal, and, as we ave just seen, we may expect that a solution in the form $y = f(x)$ or $x = \phi(y)$ hwill exist.

On the other hand, if both $F_x(x_0, y_0)$ and $F_y(x_0, y_0)$ have the value 0, we readily see that there is no guarantee that a solution is possible.

For example, for $F = 1 - \sqrt{(1 - x^2 - y^2)}$, the corresponding spherical surface is $u = 1 - \sqrt{(1 - x^2 - y^2)}$, has the point (0, 0) in connection with the $xy$-plane. The partial derivatives $F_x(0,0)$ and $F_y(0,0)$ are both zero; we find that no point other than (0,0) satisfies the equation $F = 0$. For the function $F(x,y)=xy$, we find that $F(0, 0) = 0$,

while $F_x(0, 0) = F_y(0, 0) = 0$. Here all the points on the $x$-axis and all the points on the $y$-axis satisfy the equation $F(x,y)=0$; in the neighbourhood of the origin, we have no unique solution $x=\phi(y)$ or $y=f(x)$. Thus, we see that when $F_x(x_0,y_0) = F_y(x_0,y_0) = 0$, we cannot be sure that a solution exists.

Accordingly, we return to the case in which one of the partial derivatives—say, $F_y(x_0,y_0)$ to be specific—is not zero, the graphical suggestion that a smooth surface should be cut by a non-tangent plane in a smooth curve leads us to expect that one has the theorem:

If the function $F(x,y)$ has continuous derivatives $F_x$ and $F_y$, and if at the point $(x_0,y_0)$ holds the equation $F(x_0,y_0) = 0$, while $F_y(x_0,y_0)$ is not zero, then we can mark off about the point $(x_0,y_0)$ a rectangle $x_1 \leq x \leq x_2$,, $y_1 \leq y \leq y_2$ such that for every $x$ in the interval $x_1 \leq x \leq x_2$ the equation $F(x, y) = 0$ determines just value $y=f(x)$, lying in the interval $y_1 \leq y \leq y_2$. This function $y = f(x)$ satisfies the equation $y = f(x_0)$, and the equation

$$F\{x, f(x)\} = 0$$

is satisfied for every $x$ in the interval. Moreover, the function $y = f(x)$ is continuous and has a continuous derivative.

This can actually be rigorously proved and will be proved in Volume 2. Assuming it to be true, we can add the following:

The derivative of the function $y = f(x)$ is given by the equation

$$y' = f'(x) = -\frac{F_x}{F_y}.$$

This follows immediately by using the chain rule. In fact,

$$\frac{d}{dx} F\{x, f(x)\} = F_x \frac{dx}{dx} + F_y \frac{df}{dx} = F_x + F_y f'.$$

However, since $F\{x,f(x)\}$identically zero, its derivative is also zero, whence $F_x + F_y f' = 0$, and the formula is established.

If we regard the right hand side of the formula as a compound function of $x$ and differentiate according to the chain rule, replacing $y'$ by $-F_x/F_y$, we have

$$y'' = -\frac{F_y(F_{xx} + F_{yx}y') - F_x(F_{xy} + F_{yy}y')}{F_y^2}$$
$$= -\frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{F_y^3}.$$

Continuing the process, we may calculate $y^{iii}$, $^{iv}$, etc.

By using this formula, we can usually find the derivative of a function given in implicit form much more easily than by solving first and then differentiating.

For example, for the circle

$$F(x, y) = x^2 + y^2 - 1 = 0,$$

we have

$$y' = -\frac{F_x}{F_y} = -\frac{x}{y}.$$

This is easily verified. In fact, solving the equation of the circle for $y$, we obtain two solutions, namely, $y = \sqrt{(1 - x^2)}$ and $y = -\sqrt{(1 - x^2)}$, giving the upper and lower semi-circles, respectively. For the upper part, we have

$$y' = \frac{-x}{\sqrt{(1 - x^2)}}, \text{ for the lower } \quad y' = \frac{+x}{\sqrt{(1 - x^2)}},$$

so that in either case

$$y' = -\frac{x}{y}.$$

Another example is $F(x,y) = e^{x+y} + y - x = 0$. We find $F_x(½,-½) = 0$, while $F_y(½,-½) = 2$. Thus, the equation has a solution $y = f(x)$; but its actual explicit calculation is not simple. Nevertheless, we have

$$y' = -\frac{F_x}{F_y} = -\frac{e^{x+y}-1}{e^{x+y}+1}.$$

In order that the function $f(x)$ may have a maximum or minimum, we must have $y' = 0$, i.e., $e^{x+y} - 1 = 0$, whence $y = -x$. Substitution of $y = -x$ into the equation $F(x, y) = 0$ yields $1 - 2x = 0$, whence $x = ½$, $y = -½$. If we calculate $f''(x)$ for $x = ½$, we find it to be negative so $-½$ is the maximum of $y$.

An extension of this theorem for implicit functions to functions of a larger number of independent variables readily suggests itself:

Let $F(x, y, \cdots, z, u)$ be a continuous function of the independent variables $x, y, \cdots, z, u$ with continuous partial derivatives $F_x, F_y, \cdots, F_z, F_u$. For the system of values $(x_0, y_0, \cdots, z_0, u_0)$, let $F(x_0,y_0,\cdots,z_0,u_0)=0$ and $F_u(x_0,y_0,\cdots,z_0,u_0)\neq 0$. Then we can mark off an interval $u_1 \leq u \leq u_2$ about $u_0$ and a region $R$ containing $(x_0,y_0,\cdots,z_0)$ such that for every $(x, y, \cdots, z)$ in $R$ the equation $F(x,y,\cdots,z,u)= 0$ is satisfied by just one value of $a$ in the interval $u_1 \leq u \leq u_2$. This value of $n$, which $u$ = we denote by $u = f(x, y, \cdots, z)$ is a continuous function of $x, y, \cdots, z$ and possesses continuous partial derivatives $f_x, f_y, \cdots, f_z$ and

$$u_0 = f(x_0, y_0, \ldots, z_0).$$

The derivatives of $f$ are given by the equations

$$\begin{aligned} F_x + F_u f_x &= 0, \\ F_y + F_u f_y &= 0, \\ \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \quad \cdot \\ F_z + F_u f_z &= 0. \end{aligned}$$

For the proof of the existence and continuity of $u$, we again refer the reader to Volume II. The formulae for $f_{xy}$, etc., follow immediately from the chain rule.

Incidentally, the concept of an implicit function enables us to give a general definition of the term **algebraic function**.

We say that $u = f(x, y, \cdots, z)$ is an algebraic function of the independent variables $x,y,\cdots,z$ if $u$ can be defined implicitly by an equation $F(x,y,\cdots,z,u)=0$, where $F$ *is* a polynomial in $x,y,\cdots,z,u$, that is, if $u$ satisfies an **algebraic equation**. Functions which do not satisfy any algebraic equation are called **transcendental equations**.

As an example of our differentiation formula, consider the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{u^2}{c^2} - 1 = 0.$$

We have for the partial derivatives

$$u_x = -\frac{2x}{a^2} \cdot \frac{c^2}{2u} = -\frac{c^2}{a^2} \cdot \frac{x}{u},$$

$$u_y = -\frac{2y}{b^2} \cdot \frac{c^2}{2u} = -\frac{c^2}{b^2} \cdot \frac{y}{u};$$

and by redifferentiation

$$u_{xx} = -\frac{c^2}{a^2} \cdot \frac{1}{u} + \frac{c^2}{a^2} \cdot \frac{x}{u^2} u_x = -\frac{c^2 a^2 u^2 + c^4 x^2}{a^4 u^3},$$

$$u_{xy} = +\frac{c^2}{a^2} \cdot \frac{x}{u^2} u_y = -\frac{c^4}{a^2 b^2} \frac{xy}{u^3},$$

$$u_{yy} = -\frac{c^2}{b^2} \cdot \frac{1}{u} + \frac{c^2}{b^2} \cdot \frac{y}{u^2} u_y = -\frac{c^2 b^2 u^2 + c^4 y^2}{b^4 u^3}.$$

**Exercises 10.4:**

1. Prove that the following equations have unique solutions for $y$ near the points indicated:

(a) $x^2 + xy + y^2 = 7$ $\quad (2, 1)$.

(b) $x \cos xy = 0$ $\quad \left(1, \frac{\pi}{2}\right)$.

(c) $xy + \log xy = 1$ $\quad (1, 1)$.

(d) $x^5 + y^6 + xy = 3$ $\quad (1, 1)$.

2. Find the first derivatives of the solutions in 1.

3. Find the second derivatives of the sulutions in 1.

4. Find the maxima and minima of the function $y = f(x)$ defined by the equation $x^2 + xy + y^2 = 27$.

6. Show that the equation $x + y + z = \sin xyz$ can be solved for $z$ near the point (0,0,0). Find the partial derivatives of the solution.

Answers and Hints

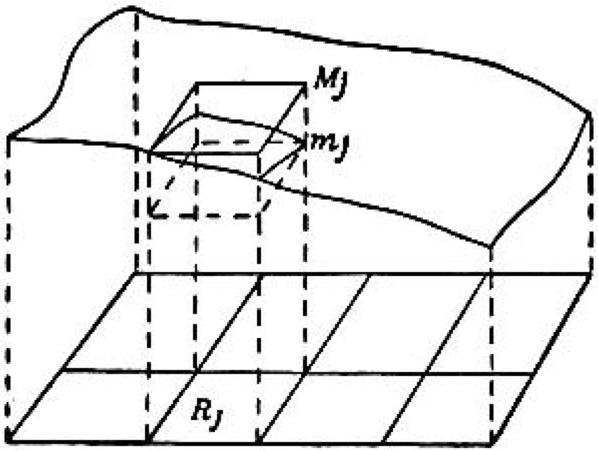


Fig. 7

## 10.6 Multiple and Repeated Integrals

**10.6.1 Multiple Integrals:** Consider a function $u = f(x, y)$ which is defined and continuous in the rectangle $R(a \leq x \leq b,\ c \leq y \leq d)$ and which assumes only positive values. We wish to assign a volume to the portion of three-dimensional space bounded by the rectangle $R$, the surface $u = f(x, y, z)$, and the four planes $x = a$, $x = b$, $y = c$, $y = d$ perpendicular to the $xy$-plane. Moreover, the volume should be defined so as to satisfy certain elementary conditions:
(1), if the three-dimensional region is a prism, i.e., if the function $u$ is a constant $k$, the volume should be the product of the base by the height, $V=(b-a)(d-c)k$ ;
(2) if we divide the rectangle $R$ into smaller rectangles $R_1$ and $R_2$ by drawing straight lines, then the volume over $R_1$ should be equal to the volume over $R_1$ plus the volume over $R_2$;
(3) if the three-dimensional region $R_1$ completely includes $R_2$, the volume of $R_1$ should be at least as large as that of $R_2$.

These considerations lead us to a method of defining $V$ which is an immediate extension of the method of defining area in 2.1.1. By constructing lines parallel to the sides, we subdivide the rectangle $R$ into smaller rectangles $R_1, R_2, \cdots, R_n$, the areas of which will be denoted by $\Delta R_1, \Delta R_2, \cdots, \Delta R_n$. In each rectangle $R_j$, the fonction has a least value

$m_j$ and a larger value $M_j$, whence a prism with base $R$ and height $M_j$ completely includes the portion of the region over $R_j$, while this portion of the region contains the prism with base $R_j$ and height $m_j$ (Fig. 7). Hence, we see that the vohume of the portion in question lies between $m_jR_j$ and $M_jR_j$. Thus, the total volume $V$ should be such that

$$\sum_{j=1}^{n} m_j \Delta R_j \leqq V \leqq \sum_{j=1}^{n} M_j \Delta R_j.$$

Now let the number $n$ of rectangles increase beyond all bounds in such a way that the length of the longest diagonal tends to zero. Intuition leads us to expect that both the sums $\Sigma m_j \Delta R_j$ and $\Sigma M_j \Delta R_j$ will converge and tend to the same limit, whence we call this limit the volume $V$.

The reader will have observed that we have carried out an immediate generalization of the discussion in 2.1.2. As in Chapter II, we call the common limit of the sums $\Sigma m_j \Delta R_j$ and $\Sigma M_j \Delta R_j$ the integral of the function $n = f(x, y)$ over the rectangle $R$ and denote it by the symbol

$$\iint_R f(x, y)\, dr.$$

It is at once clear that, if we choose in each rectangle $R_j$ a point $(\xi_j, \eta_j)$ and find the corresponding value of the function $f(\xi_j, \eta_j)$, then there must hold the limiting relation

$$\lim_{n \to \infty} \Sigma f(\xi_j, \eta_j) \Delta R_j = \iint_R f(x, y)\, dr;$$

in fact, the sum $\Sigma f(\xi_j, \eta_j) \Delta R_j$ lies between $\Sigma m_j \Delta R_j$ and $\Sigma M_j \Delta R_j$, both of which approach the integral as a limit.

As a particular method of subdividing $r$ into smaller rectangles, we may subdivide the side $a \leq x \leq b$ into $n$ intervals of length $\Delta x = (b - a)/n$ and the side $c \leq y \leq d$ into $m$ intervals of length $\Delta y = (d - c)/m$, and then draw parallels to the axes through the points of division thus marked. The area of each rectangle $R_j$ is then $\Delta R_j = \Delta x \Delta y$. Choosing a point $(\xi_j, \eta_j)$ arbitrarily in each rectangle $R_j$, we form the sum

$$\Sigma_j f(\xi_j, \eta_j) \Delta R_j = \Sigma_j f(\xi_j, \eta_j) \Delta x \Delta y.$$

As both $n$ and $m$ increase without limit, this sum approaches the integral as a limit. This type of subdivision suggests a second notation for the integral, which has been in common use since the time of Leibnitz, namely

$$\int\int_R f(x, y)\, dx\, dy.$$

The proof that such a limit exists if $u = f(x, y)$ is continuous can be carried out as in the A2.1). However. we shall assume without proof the even stronger statement:

The function $f(x, y)$ is continuous except along a finite number of **smooth** curves (curves with continuous derivatives) $y = f(x)$ or $x = \phi(y)$ along which $f(x,y)$ has jump discontinuities, then there exists double integral

$$\int\int_R f(x, y)\, dx\, dy.$$

We leave its proof to Volume II. It depends essentially on the fact that, as the number of rectangles increases, the total area of the rectangles having points in common with the curves of discontinmty tends to zero. Thus, even though $M_i$ and $m_i$ may differ considerably for such rectangles, they give rise to a little difference between the sums $\Sigma m_i \Delta R_i$ and $\Sigma M_i \Delta R_i$.

With this assumption, we can find the area under surfaces $u = f(x, y)$ for which $(x, y)$ ranges over quite complicated regions $R$. In fact, let the region $R$ be bounded by a finite number of curves $x = \phi(y)$ or $y = \psi(x)$ with continuous derivatives and $f(x, y)$ be continuous in $R$. We enclose $R$ in a rectangle $R'$ and assign the points of $R'$, which $\int\int_{R'} f(x, y)\, dr,$ taken over the region $R'$, as the volume under the surface $u = f(x, y)$, where $(x, y)$ is in $R$. This integral is usually denoted by $\int\int_R f(x, y)\, dx\, dy.$.

Certain simple, but important theorems relating to these double integrals follow directly from the definition. We shall simply state the theorems, as the reader will be able to prove them without any trouble.

If $f(x, y)$ and $g(x, y)$ are integrable over a rectangle, then so are $f \pm g$ and $cf$, where $c$ is a constant:

$$\iint_R \{f(x, y) \pm g(x, y)\} dr = \iint_R f(x, y)\, dr \pm \iint_R g(x, y)\, dr,$$

$$\iint_R c f(x, y)\, dr = c \iint_R f(x, y)\, dr.$$

If $f(x, y) \geq g(x, y)$ in $R$, then

$$\iint_R f(x, y)\, dr \geqq \iint_R g(x, y)\, dr.$$

If $I_j R$ is the sum of two regions $R_1$ and $R_2$, then

$$\iint_R f(x, y)\, dr = \iint_{R_1} f(x, y)\, dr + \iint_{R_2} f(x, y)\, dr.$$

### 10.6.2 Reduction of Double Integrals to Repeated Single Integrals:

We now have a definition of the double integral with its interpretation as a volume and with the many possibilities of usefulness which our experience with the single integral suggests; however, as yet, we do not possess a method for evaluating such integrals. In this section, we shall see how the calculation of a double integral can be reduced to that of two single integrals.

Let $u = f(x, y)$ be a function which is defined and continuous in a rectangle $R$,
$a \leq x \leq b$, $c \leq y \leq d$. If we fix upon any value $x_0$ in the interval $a \leq x \leq b$, the function $f(x_0, y)$ is a continuous function of the remaining variable $y$, whence there exists the integral

$$\int_c^d f(x_0, y)\, dy$$

and it can be evaluated by the methods of the earlier chapters. This integral has a definite value for each value of $x_0$ which we may select; in other words, the integral is a function $\phi(x_0)$ of the quantity $x_0$:

$$\int_c^d f(x, y)\, dy = \phi(x).$$

For example, let u = $f(x, y) = x^2y^3$, $0 \le x \le 1$, $0 \le y \le 3$. For each fixed $x$ in the interval $0 \le x \le 1$, the integral $\int_0^3 x^2y^3\,dy$ can be evaluated and, in fact, is $81x^2/4$, i.e., it is a function of $x$. Or if $f(x, x) = e^{xy}$, $1 \le x \le 2$, $1 \le y \le 4$, we have $\int_1^4 e^{xy}\,dy = \frac{1}{x}(e^{4x} - e^x)$.

Having thus found the function $\phi(x)$, we can prove that it is continuous; this is a simple consequence of the uniform continuity of $f(x, y)$.It is therefore possible to integrate $\phi(x)$ between the limits a and $b$, thus obtaining the repeated integral

$$\int_a^b \phi(x)\,dx = \int_a^b \left(\int_c^d f(x, y)\,dy\right) dx.$$

By reversing the order of the process, first calculating the function of $y$ defined by $\int_a^b f(x, y)\,dx$ and then integrating from $c$ to $d$, *we* obtain the other repeated integral

$$\int_c^d \left(\int_a^b f(x, y)\,dx\right) dy.$$

These integrals, as we have seen, are obtained by a double application of the ordinaly simple integration which we have studied before. Their importance lies in the following fact:

For continuous functions $f(x, y)$ and for functions $f(x, y)$ with at most jump discontinuities on a finite number of smooth curves, the repeated integrals are equal to the double integral:

$$\iint_R f(x, y)\,dr = \int_a^b \left(\int_c^d f(x, y)\,dy\right) dx$$
$$= \int_c^d \left(\int_a^b f(x, y)\,dx\right) dy.$$

We shall content ourselves with an intuitive discussion of the case where $f(x,y)$ is continuous. In our original discussion of the double integral regarded as the volume lying above the rectangle $a \le x \le b$, $c \le y \le d$ and below

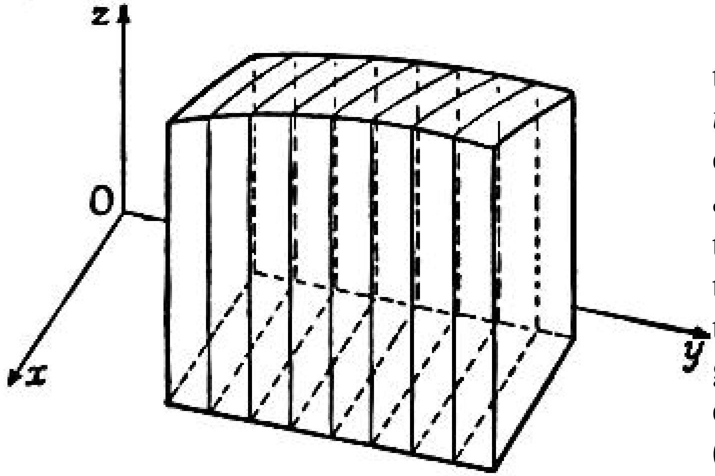


Fig. 8

the surface
$u = f(x, y)$, we obtained this volume by subdividing the solid into vertical colunms and then letting the diagonals of the bases of these columns approach zero. Instead of this, we can divide the solid into slices of thickness $k=(d-c)/n$ by drawing the lines $y = c + \nu k$ ($\nu = 0, 1, \cdots, n$) parallel to the $x$-axis and then constructing a plane perpendicular to the $xy$-plane through each line (Fig. 8). These planes cut the solid into $n$ slices which grow thinner as $n$ increases and the total volume of which is equal to the double integral. We now see that the volume of each slice is approximately (but, of course, not as a rule exactly) equal to the product of the thickness $k$ by the area of the left-hand face, i.e., equal to

$$k \int_a^b f(x, c + \nu k)\, dx.$$

Hence, if we write

$$\phi(y) = \int_a^b f(x, y)\, dx$$

the desired volume is represented approximately by

$$\sum_{\nu=0}^{n-1} k\phi(c + \nu k).$$

As $n \to \infty$, these sums tend to

$$\int_c^d \phi(y)\, dy.$$

It is therefore reasonable to expect that the volume or double integral is exactly equal to

$$\int_c^d \phi(y)\, dy = \int_c^d \left( \int_a^b f(x, y)\, dx \right) dy,$$

which is the statement made ahove. A similar discussion makes it equally plausible that the statement

$$\int_a^b \left(\int_c^d f(x, y)\, dy\right) dx = \int\int_R f(x, y)\, dr$$

is also true.

**10.6.3 Examples and Remarks:** A few examples will serve to illustrate how the last theorem may be used to evaluate double integrals. For the function $u=f(x,y)= x^2y$, $0 \le x \le 1$, $0 \le y \le 2$, we have

$$\int\int_R x^3 y\, dr = \int_0^1 \left(\int_0^2 x^3 y\, dy\right) dx = \int_0^1 \left(\frac{1}{2} x^3 y^2 \Big|_0^2\right) dx$$
$$= \int_0^1 2x^3\, dx = \frac{1}{2} x^4 \Big|_0^1 = \frac{1}{2}.$$

This example belongs to a general class of functions the integration of which is often simplified by the theorem:

If the function $u = f(x, y)$, $a \le x \le b$, $c \le y \le d$ can be represented as the product of a function of $x$ alone and a function of $y$ alone,

$$f(x, y) = \phi(x)\, \psi(y),$$

then its double integral is the product of two simple integrals:

$$\int\int_R f(x, y)\, dr = \left(\int_a^b \phi(x)\, dx\right) \left(\int_c^d \psi(y)\, dy\right).$$

In fact, on integration with respect to $y$, the function $\phi\,(x)$ can be treated like a constant and placed in front of the integral sign, while, on integration with respect to $x$, $\int_c^d \psi(y)\, dy$ is a constant, whence

$$\int_a^b \left(\int_c^d \phi(x)\,\psi(y)\,dy\right) dx = \int_a^b \left(\phi(x)\int_c^d \psi(y)\,dy\right) dx$$
$$= \left(\int_c^d \psi(y)\,dy\right)\left(\int_a^b \phi(x)\,dx\right).$$

For the function $u = \sin(x + x)$, $0 \le x \le \pi/2$, $0 \le y \le \pi/2$, we have

$$\iint_R \sin(x+y)\,dr = \int_0^{\pi/2}\left(\int_0^{\pi/2} \sin(x+y)\,dy\right) dx$$
$$= \int_0^{\pi/2}\left(-\cos\left(x+\frac{\pi}{2}\right) + \cos x\right) dx = \int_0^{\pi/2} (\sin x + \cos x)\,dx$$
$$= (-\cos x + \sin x)\Big|_0^{\pi/2} = 1 + 1 = 2.$$

Again, let us calculate the volume $V$ of the vertical prism the base of which - the $xy$-plane - is bounded by the co-ordinate axes and the line $x + y = 1$ and which lies below the plane $u = 2x + 3y$. We first extend the function $u = f(x, y)$ to the square $0 \le x \le 1$, $0 \le y \le 1$ by equating it to 0 outside the triangle (the base of the prism). Then, for each $x$ in the interval. the function $f(x, y)$ is different from 0 for $0 \le y \le 1$ only, whence

$$\int_0^1 f(x,y)\,dy = \int_0^{1-x} f(x,y)\,dy = \int_0^{1-x} (2x+3y)\,dy$$
$$= 2x(1-x) + \frac{3}{2}(1-x)^2 = -\frac{1}{2}x^2 - x + \frac{3}{2},$$

and

$$V = \iint_R f(x,y)\,dr = \int_0^1 \left(-\frac{1}{2}x^2 - x + \frac{3}{2}\right) dx = \frac{5}{6}.$$

The device just used can be extended to any function $u = f(x, y)$ which is defined in a region $R$, bounded above and below by curves $y=\psi(x)$ and $y=\phi(x)$. In fact, let $R$ be defined by the inequalities $a \le x \le b$, $\phi(x) \le y \le \psi(x)$. Mark off a rectangle $R'$, $a \le x \le b$, $c \le y \le d$, which contains completely $R$, and outside $R$ set $f = 0$. Then

$$\int_c^d f(x, y)\, dy = \int_{\phi(x)}^{\psi(x)} f(x, y)\, dy$$

for every $x$ in the interval $a \leq x \leq b$, so that

$$\int\int_R f(x, y)\, dr = \int\int_{R'} f(x, y)\, dr = \int_a^b \left(\int_c^d f(x, y)\, dy\right) dx$$
$$= \int_a^b \left(\int_{\phi(x)}^{\psi(x)} f(x, y)\, dy\right) dx.$$

Thus, in order to find the volume $V$ of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{u^2}{c^2} - 1 = 0,$ we note that $\frac{1}{2}V$ is the volume under $u = f(x, y) = c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)},$ this function $f(x, y)$ being defined only inside an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqq 1, \text{ or } -b\sqrt{\left(1 - \frac{x^2}{a^2}\right)} \leqq y \leqq b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}, -a \leqq x \leqq a.$$

In calculating the repeated integral, we first have

$$\int_{-b}^{b} f(x, y)\, dy = \int_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}} c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)} dy$$
$$= -\frac{1}{2}c\left(b - \frac{bx^2}{a^2}\right) \text{arc cos} \frac{y}{\sqrt{(b^2 - b^2x^2/a^2)}} + \frac{cy}{2}\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}\Bigg|_{-b\sqrt{(1-x^2/a^2)}}^{+b\sqrt{(1-x^2/a^2)}}$$
$$= -\frac{c}{2}\left(b - \frac{bx^2}{a^2}\right)(0 - \pi) + 0 = \frac{c\pi}{2} b\left(1 - \frac{x^2}{a^2}\right).$$

Proceeding with the integration, we find

$$\frac{1}{2}V=\int_{-a}^{a}\left(\int_{-b}^{b}f(x,y)\,dy\right)dx=\int_{-a}^{a}\frac{\pi}{2}cb\left(1-\frac{x^2}{a^2}\right)dx=\frac{\pi c}{2}b\left(x-\frac{x^3}{3a^2}\right)\Big|_{-a}^{a}$$

$$=\frac{\pi}{2}cb\left(\frac{4}{3}a\right)=\frac{2\pi}{3}abc,$$

whence

$$V=\frac{4}{3}\pi abc.$$

**10.6.4 Polar Co-ordinates:** In the definition of the double integral, the sub-division into rectangles was, of course, chosen simply because such a sub-division is most convenient in connection with rectangular co-ordinates. However, as we already know, there are many applications in which polar co-ordinates are much more suitable than rectangular ones. If we are considering a function $f(\rho, \phi)$, where $\rho$ and $\phi$ are polar co-ordinates, the most convenient sub-division is not into rectangles, but into regions bounded by arcs of circles $\rho$ = const. and radii $\phi$ = const. Now, let our function $f(\rho, \phi)$ be defined in a region $R$, specified by the inequalities $a \le \rho \le b$, $\alpha \le \phi \le \beta$. (If $f(\rho, \phi)$ is originally defined in a region $R$ not of this type, we enclose $R'$ in a larger region $R$ of the desired form and set $f(\rho, \phi) = 0$ outside $R'$.) Now, just as in 10.6, we can insert points of sub-division

$$\rho_0=a,\ \rho_1,\ \rho_2,\ \ldots,\ \rho_n=b,\ \phi_0=\alpha,\ \phi_1,\ \phi_2,\ \ldots,\ \phi_m=\beta$$

and construct the corresponding radii and arcs of circles, thus dividing $R$ into regions $R_{ij}$ of area $\Delta R_{ij}$. In each $R_{ij}$, we select a point $(\phi_{ij}, \phi_{ij})$ and form the sum $\sum f(\rho_{ij},\phi_{ij})\,\Delta R_{ij}$, and then let $m$ and $n$ increase without limit. Then, the sum will again tend to the volume under the surface $u = f(\rho, \phi)$ and we may denote this by the integral

$$\iint_R f(\rho,\phi)\,dr.$$

So far we have encountered nothing essentially new. The point of importance is to leam how to evaluate these integrals by reduction either to repeated integrals or to integrals in terms of rectangular co-ordinates. For this purpose, we mark off a pair of rectangular axes in a new plane, the $\rho\phi$-plane and call them the $\rho$-axis and the $\phi$-axis, respectively. Corresponding to the point in $R$ with polar co-ordinates $\rho$, $\phi$, we plot the point in the $\rho\phi$-plane with rectangular co-ordinates $\rho$, $\phi$. Thus, the region $R$, $a \le \rho \le b$, $\alpha \le \phi \le \beta$( is represented in the $\rho\phi$-plane by a

rectangle $R'$, $a \leq \rho \leq b$, $\alpha \leq \phi \leq \beta$, and each small region $R_{ij}$, $\rho_{i-1} \leq \rho \leq \rho_i$, $\phi_{i-1} \leq \phi \leq \phi_i$ is represented by a small rectangle $R_{ij}$. But the area $\Delta R_{ij}'$ of the rectangle $R_{ij}'$ is not the same as the area $\Delta R_{ij}$ of $R_{ij}$. The relationship between them is easily found. The area $\Delta R_{ij}'$ is simply

$$(\phi_j - \phi_{j-1})(\rho_i - \rho_{i-1}),$$

while the area $\Delta R_{ij}$ is

$$\begin{aligned}\Delta R_{ij} &= \tfrac{1}{2}(\phi_j - \phi_{j-1})(\rho_i^2 - \rho_{i-1}^2)\\ &= \tfrac{1}{2}(\rho_i + \rho_{i-1})(\phi_j - \phi_{j-1})(\rho_i - \rho_{i-1}) = \tfrac{1}{2}(\rho_i + \rho_{i-1})\Delta R_{ij}'.\end{aligned}$$

Now, in each region $R_{ij}$, select the point

$$\bar{\rho}_i = \tfrac{1}{2}(\rho_i + \rho_{i-1}),\ \bar{\phi}_j = \tfrac{1}{2}(\phi_j + \phi_{j-1}).$$

However,

$$\Sigma f(\bar{\rho}_i, \bar{\phi}_j)\Delta R_j = \Sigma f(\bar{\rho}_i, \bar{\phi}_j)\bar{\rho}_i \Delta R_{ij}',$$

and the latter expression is just the sum which we form in defining the double integral of the function $f(\rho, \phi)$ over the rectangle $R'$ in the $\rho\phi$-plane. Hence, as the size of the subdivision decreases, the sum approaches this integral and

$$\begin{aligned}\int\int_R f(\rho, \phi)\,dr &= \int\int_{R'} f(\rho, \phi)\rho\,dr' = \int\int_{R'} f(\rho, \phi)\rho\,d\rho\,d\phi\\ &= \int_a^b \left(\int_a^\beta f(\rho, \phi)\rho\,d\phi\right)d\rho = \int_\alpha^\beta \left(\int_a^b f(\rho, \phi)\rho\,d\rho\right)d\phi.\end{aligned}$$

An example, calculate the volume $V$ of a sphere of radius $a$. The upper hemisphere is given by the equation $u = \sqrt{(a^2 - \rho^2)}, 0 \leq \rho \leq a, 0 \leq \varphi \leq 2\pi.$ Thus,

$$\frac{1}{2}V = \int_0^{2\pi}\left(\int_0^a \sqrt{(a^2-\rho^2)}\,\rho\,d\rho\right)d\varphi = \int_0^{2\pi}\left(-\frac{1}{3}(a^2-\rho^2)^{3/2}\Big|_0^a\right)d\varphi$$

$$= \frac{1}{3}\int_0^{2\pi} a^3\,d\varphi = \frac{2\pi a^3}{3},$$

whence

$$V = \frac{4}{3}\pi a^3.$$

**10.6.5 Evaluation of the Probability Integral:** The formulae of the preceding subsection enable us to calculate the area under the curve $y = e^{-x^2},\ -\infty < x < \infty,$ which frequently occurs in the **theory of probability**. This integration is especially interesting in that we can evaluate the definite integral from -∞ to ∞ of a function for which we cannot find at all a primitive function or an indefinite integral.

Consider first the integral $I_a$ of the function $e^{-(x^2+y^2)} = e^{-\rho^2}$ over the circle $0 \le \rho \le a$, given by

$$I_a = \int_0^{2\pi}\left(\int_0^a e^{-\rho^2}\rho\,d\rho\right)d\phi = \int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}e^{-a^2}\right)d\phi = \pi(1-e^{-a^2}).$$

The square $-a \le x \le a$, $-a \le y \le a$ contains the circle $0 \le \rho \le a$ and is contained in the circle $0 \le \rho \le 2a$, and the integrand $e^{-x^2-y^2}$ is everywhere positive, whence

$$\pi(1-e^{-a^2}) = I_a \le \int_{-a}^{a}\left(\int_{-a}^{a} e^{-x^2-y^2}\,dy\right)dx \le I_{2a} = \pi(1-e^{-4a^2}).$$

This integral can be rewritten

$$\int_{-a}^{a} e^{-x^2}\left(\int_{-a}^{a} e^{-y^2}dy\right)dx = \left(\int_{-a}^{a} e^{-x^2}dx\right)^2,$$

whence

$$\pi(1-e^{-a^2}) \leq \left(\int_{-a}^{a} e^{-x^2}dx\right)^2 \leq \pi(1-e^{-4a^2}).$$

If we now let *a* increase without bound, we obtain the required result

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

### 10.6.6 Moments and centres of Mass; Moments of Inertia:

In 5.2.7, we have seen that the moments about the *x*-axis of a system of points $P_1, P_2, \cdots, P_n$ with co-ordinates $(x_1, y_1)(x_2, y_2), \cdots (x_n, y_n)$ and masses $m_1, m_2, \cdots, m_n$ is given by $\sum_{\nu=1}^{n} m_\nu y_\nu,$ and that the ordinate of its centre of mass is given by

$$\eta = \frac{1}{M}\sum_{\nu=1}^{n} m_\nu y_\nu, \quad M = \sum_{\nu=1}^{n} m_\nu$$

with analogous expressions for the moment about the *y*-axis and the abscissa of the centre of mass. We will now extend these ideas to masses distributed uniformly over a region *R*. We assume that a mass is distributed with density 1 over the region *R*, i.e., that each portion of *R* with area $\Delta R$ has also mass $\Delta R$. Then the total mass *M* of *R* is the same as the area of *R*,

$$M = \iint_R dr.$$

Now sub-divide R into portions $R_1, R_2, \cdots, R_n$ with areas $\Delta R_1, \Delta R_2, \cdots, \Delta R_n$ and in each portion $R_\nu$ choose a point $(\xi_\nu, \eta_\nu)$. If we imagine that the total mass $\Delta R_\nu$ of the portion $R_\nu$ is concentrated at the point $(\xi_\nu, \eta_\nu)$, the moment of the resulting system of points with respect to the *x*-axis will be $\Sigma \eta_\nu \Delta R_\nu,$ and the ordinate of the centre of mass will be

$$\frac{\Sigma \eta_\nu \Delta R_\nu}{\Sigma \Delta R_\nu} = \frac{\Sigma \eta_\nu \Delta R_\nu}{M}.$$

If we now let $n \to \infty$ and the diameter of the largest $R_\nu$ tend to 0, these sums tend to the integrals

$$T_x = \iint_R y\,dr, \qquad \eta = \frac{\iint_R y\,dr}{M},$$

respectively. We take these expressions as the definitions of the moment $T_x$, of $R$ about the $x$-axis and of the ordinate $\eta$ of its centre of mass. Similarly, the moment about the $y$-axis and the abscissa of the centre of mass, respectively, are given by

$$T_y = \iint_R x\,dr, \quad \xi = \frac{\iint_R x\,dr}{M}, \quad M = \iint_R dr.$$

For example, the moment of the semicircle $R$, $-\rho \leqq x \leqq \rho$, $0 \leqq y \leqq \sqrt{(\rho^2 - x^2)}$, about the $x$-axis is

$$T_x = \iint_R y\,dr = \int_{-\rho}^{\rho}\left(\int_0^{\sqrt{(\rho^2-x^2)}} y\,dy\right)dx$$

$$= \int_{-\rho}^{\rho} \frac{1}{2}(\rho^2 - x^2)\,dx = \frac{1}{2}\left(\rho^2 x - \frac{x^3}{3}\right)\Big|_{-\rho}^{\rho} = \frac{2}{3}\rho^3,$$

whence

$$M = \iint_R dr = \text{area } R = \frac{1}{2}\pi\rho^2, \quad \eta = \frac{2}{3}\rho^3 \div \left(\frac{1}{2}\pi\rho^2\right) = \frac{4}{3}\frac{\rho}{\pi}.$$

By a similar argument, starting from the definition of the moment of inertia $I_x$ of a system of particles,

$$I_x = \Sigma m_\nu y_\nu^2,$$

we arrive at the expression for the moment of inertia of the region $B$ about the $y$-axis

$$I_x = \iint_R y^2\, dr,$$

and, similarly, we obtain the moment of inertia with respect to the $y$-axis

$$I_y = \iint_R x^2\, dr.$$

Analogous formulae hold for three-dimensional integrals $\xi = \dfrac{\iiint_R x\, dr}{M}, \quad \eta = \dfrac{\iiint_R y\, dr}{M}, \quad \zeta = \dfrac{\iiint_R z\, dr}{M},$ sional regions $R$; the co-ordinates $\xi, \eta, \zeta$ of the centre of mass are given by

$$\xi = \frac{\iiint_R x\, dr}{M}, \quad \eta = \frac{\iiint_R y\, dr}{M}, \quad \zeta = \frac{\iiint_R z\, dr}{M},$$

where $M = \iiint_R 1\, dr = \text{volume of } R.$ In order to find the moments of inertia $I_x$, $I_y$, $I_z$, of $R$ about the $x$, $y$, $z$-axes, respectively, we must remember that the distance of the point $(x, y, z)$ from the $x$-axis is $\sqrt{(y^2 + z^2)}$, whence, for a system of particles, the moment of inertia about the $x$-axis is

$$\Sigma m_\nu \sqrt{(y_\nu^2 + z_\nu^2)}^2 = \Sigma m_\nu (y_\nu^2 + z_\nu^2),$$

and, on dividing $R$ into sub-regions and passing to the limit as before, we obtain

$$I_x = \iiint_R (y^2 + z^2)\, dr.$$

Similarly, we find

$$I_y = \iiint_R (x^2 + z^2)\, dr, \; I_z = \iiint_R (x^2 + y^2)\, dr.$$

Thus, the moment of inertia of the cube - $h \leq x \leq h$, $-h \leq y \leq h$, $-h \leq z \leq h$ about the $z$-axis is

$$
\begin{aligned}
I_z &= \int_{-h}^{h} \left\{ \int_{-h}^{h} \left( \int_{-h}^{h} (x^2 + y^2)\, dz \right) dy \right\} dx \\
&= \int_{-h}^{h} \left\{ \int_{-h}^{h} 2h(x^2 + y^2) dy \right\} dx = \int_{-h}^{h} 2h \left( 2x^2 h + \frac{2}{3} h^3 \right) dx \\
&= \frac{4h}{3} (x^3 h + x h^3) \Big|_{-h}^{h} = \frac{4h}{3} (4h^4) = \frac{16}{3} h^5.
\end{aligned}
$$

The significance of the moment of inertia, as we have already remarked in 5.2.9, lies in the fact that it has in rotatory motion the part taken by the mass in translatory motion. For example, if the region $R$ rotates about the x-axis with angular velocity $\omega$, its kinetic energy is $\frac{1}{2} I_x \omega^2$. However, this is not the only application of the concept of moment of inertia; for example, it is also important in **structnral engineering**, where it is found that the stiffness of a beam of a given material is proportional to the moment of inertia of the cross-section taken about a line through its centre of mass. The reader will find further information about this in any textbook on the **strength of materials**.

**10.6.7 Further Applications:** The student should not assume that the applications already discussed exhaust the possibilities of the double integral. For instance, we have not proved the important theorem that the area $A$ of the surface $z = f(x, y)$, where $(x, y)$ is in $R$, is given by the integral

$$
A = \iint_R \sqrt{\left( 1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right)}\, dr,
$$

provided $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous; moreover, we have left many other interesting fields untouched. However, these further developments are not within the scope of the present volume and must be left to Volume II.

**Exercises 10.4:**

1. Perform the Integrations:

(a) $\int_0^a \int_0^b xy(x^2 - y^2)\,dy\,dx.$

(b) $\int_0^\pi \int_0^\pi \cos(x + y)\,dy\,dx.$

(c) $\int_1^e \int_1^2 \frac{1}{xy}\,dy\,dx.$

(d) $\int_0^a \int_0^b xe^{xy}\,dy\,dx.$

(e) $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx.$

(f) $\int_0^2 \int_0^{2-x} y\,dy\,dx.$

2. Find the volume between the *xy*-plane and the paraboloid $z = 2 - x^2 - y^2$.

3. Find the volume common to the two cylinders

$$x^2 + z^2 = 1 \text{ and } y^2 + z^2 = 1.$$

4. Find by integration the volume of the smaller of the two portions into which a sphere of radius *r* is cut by a plane the perpendicular distance of which from its centre is *h* (< *r*).

5. For the following figures find the area, the centre of gravity, the moments about the *x*- and *y*-planes and the moments of inertia about the *x*- and *y*-axes:

(a) the semicircle $0 \leqq y \leqq \sqrt{(r^2 - x^2)}$;
(b) the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$;
(c) the rectangle $-a \leqq x \leqq a$, $-b \leqq y \leqq b$;
(d) the ellipse $|y| \leqq b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}$;
(e) the triangle with vertices $(0, 0)$, $(a, 0)$, $(0, b)$:

6. Find the centre of gravity, the moments of inertia about the *x*, *y* and *z*-axes and the volume for

(a) the parallelepiped $0 \leqq x \leqq a$, $0 \leqq y \leqq b$, $0 \leqq z \leqq c$;
(b) the hemisphere $0 \leqq z \leqq \sqrt{(a^2 - x^2 - y^2)}$;
(c) the triangular prism with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

[Answers and Hints]

Chapter XI

# The Differential Equations for the Simplest Types of Vibrations

On several occasions, we have already encountered differential equations, i.e., equations from which an unknown function is to be determined and which involve not only this function but also its derivatives.

The simplest problem of this type is that of finding the indefinite integral of a given function $f(x)$. This problem requires us to find a function $y = F(x)$ which satisfies the differential equation $y' - f(x) = 0$. Moreover, we solved a problem of the same type in 3.7, where we showed that an equation of the form $y' = \alpha y$ is satisfied by an exponential function $y = ce^{\alpha x}$. As we saw in 5.4, differential equations arise in connection with the problems of mechanics; indeed, many branches of pure mathematics and most of applied mathematics depend on differential equations. In this chapter, without going into the general theory, we shall consider the differential equations of the simplest types of vibrations. These are not only of theoretical value, but also extremely important in applied mathematics.

It will be convenient to bear the following general ideas and definitions in mind. By a **solution** of a differential equation, we mean a function which, when substituted into the differential equation, satisfies the equation for all values of the independent variable under consideration. Instead of solution, the term **integral** is often used: In the first place, because the problem is more or less a generalization of the ordinary problem of integration, and, in the second place, because it frequently happens that the solution is actually found by integration.

## 11.1 Vibration Problems of Mechanics and Physics

**11.1.1 The Simplest Mechanical Vibrations:**. The simplest type of mechanical vibration has already been considered in 5.4.3. We have considered there a particle of mass $m$ which is free to move on the $x$-axis and which is brought back to its initial position $x = 0$ by a **restoring force**. We assumed the magnitude of this restoring force to be proportional to the displacement $x$; in fact, we equated it to $-kx$, where $k$ is a positive constant and the negative sign expresses the fact that the force is always directed towards the origin. We shall now assume that there is also a **frictional force** present which is proportional to the velocity $dx/dt = \dot{x}$ of the particle and opposed to it. This force is then given by an expression of the form $-r\dot{x}$, with a positive **frictional constant** $r$. Finally, we shall assume that the particle is also acted on by an external force which is a function $f(t)$ of the time $t$. Then, by **Newton's fundamental law**, the product of the mass $m$ and the acceleration $\ddot{x}$ must be equal to the total force, that is, the elastic force plus the frictional force plus the external force. This is expressed by the equation

$$m\ddot{x} + r\dot{x} + kx = f(t).$$

This equation determines the motion of the particle. Recalling the previous examples of differential equations, such as the integration problem $\dot{x} = \frac{dx}{dt} = f(t)$ with its solution $x = \int f(t)\,dt + c,$ or the solution of the particular differential equation $m\ddot{x} + kx = 0$ in 5.4.3, we observe that these problems have an infinite number of different solutions. Here too. we shall find that there are an infinite number of solutions, which are expressed in the following way. It is possible to find a **general solution** or **complete integral** $x(t)$ of the differential equation, depending not only on the independent variable $t$, but also on two parameters $c_1$ and $c_2$, called the **constants of integration**. If we assign special values to these constants, we obtain a **particular solution**, and every solution can be found by assigning special values to these constants. The complete integral is then the totality of all particular solutions.

This fact is quite understandable (5.4.4). We cannot expect that the differential equation alone will determine the motion completely. On the contrary, it is plausible that at a given instant, say, at the time t = 0, we should be able to choose the initial position $x(0) = x_0$ and the initial velocity $\dot{x}(0) = \dot{x}_0$ (in short, the **initial state**) arbitrarily; in other words, at time $t = 0$, we should be able to start the particle from any initial position with any velocity. This being done, we may expect the rest of the motion to be definitely determined. The two arbitrary constants $c_1$ and $c_2$ in the general solution are just enough to enable us to select the particular solution which fits these initial conditions. In 11.2.1, we shall see that this can be done in one way only.

If there is no external force present, i.e., if $f(t) = 0$, the motion is called a **free motion**. The differential equation is then said to be **homogenous**. If $f(t)$ is not equal to zero for all values of $t$, we say that the motion is **forced** and that the differential equation is **non-homogeneous**. The term $f(t)$ is also occasionally referred to as the **perturbation term**.

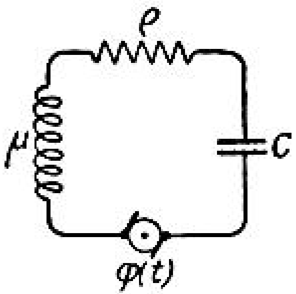


**Fig. 1.—Oscillatory electrical circuit**

**11.1.2 Electrical Oscillations:** A mechanical system of the simple type described can actually be realized only approximately. An approximation is offered by the pendulum, provided its oscillations are small. The oscillations of a **magnetic needle**, the oscillations of the centre of a **telephone** or **microphone** diaphragm and other mechanical vibrations can be represented to within a certain degree of accuracy by systems such as we have described. But there is another type of phenomenon which corresponds far more exactly to our differential equation. This is the oscillatory electrical circuit.

We will consider the circuit sketched in Fig. 1 with inductance $\mu$, resistance $\rho$ and capacity $C = 1/\kappa$. We also assume that the circuit is acted upon by an **external electromotive force** $\varphi(t)$ which is known as a function of

the time $t$, such as the voltage supplied by a dynamo or the voltage due to electric waves. In order to describe the process taking place in the circuit, we denote the voltage across the **condenser** by $E$ and the **charge** in the condenser by $Q$. These quantities are then connected by the equation $CE = E/\kappa = Q$. The current $I$, which like the voltage $E$ is a function of the time, is defined as the rate of change of the charge per unit time, i.e., as the rate at which the charge on the condenser drops: $I = -\dot{Q} = -dQ/dt = -\dot{E}/\kappa$. **Ohm's** law states that the product of the current and the resistance is equal to the electromotive force (voltage), i.e., it is equal to the condenser voltage $E$ minus the counter electromotive force due to self-induction plus the external electromotive force $\phi(t)$. We thus arrive at the equation

$$I\rho = E - \mu \dot{I} + \phi(t) \text{ or } -\frac{\rho}{\kappa}\dot{E} = E + \frac{\mu}{\kappa}\ddot{E} + \phi(t),$$

or

$$\mu\ddot{E} + \rho\dot{E} + \kappa E = -\kappa\phi(t),$$

which is satisfied by the voltage in the circuit. Hence, we see that we have obtained a differential equation of exactly the type considered in above in 10.1.1. Instead of the mass, we have the inductance, instead of the frictional force the resistance and instead of the elastic constant the reciprocal of the capacity, while the external electromotive force (apart from a constant factor) corresponds to the external force. If the electromotive force is zero, the differential equation is homogeneous.

If we multiply both sides of the differential equation by $-1/\kappa$ and differentiate with respect to the time, we obtain for the current $I$ the equation

$$\mu\ddot{I} + \rho\dot{I} + \kappa I = \dot{\phi}(t),$$

which differs from the equation for the voltage only on the right hand side and has for free oscillations ($\phi = 0$) the same form.

## 11.2 Solution of the Homogeneous Equation. Free Oscillations

**11.2.1 The Formal Solution:** We easily obtain a solution of the homogeneous equation $m\ddot{x} + r\dot{x} + kx = 0$ in 11.1.1 in the form of an exponential expression by seeking to determine a constant $\lambda$ in such a way that the

expression $e^{\lambda t} = x$ is a solution, If we substitute this expression and its derivatives $\dot{x} = \lambda e^{\lambda t}$, $\ddot{x} = \lambda^2 e^{\lambda t}$, in the differential equation and remove the common factor $e^{\lambda t}$, we obtain the quadratic equation

$$m\lambda^2 + r\lambda + k = 0$$

for $\lambda$. The roots of this equation are

$$\lambda_1 = -\frac{r}{2m} + \frac{1}{2m}\sqrt{(r^2 - 4mk)}, \quad \lambda_2 = -\frac{r}{2m} - \frac{1}{2m}\sqrt{(r^2 - 4mk)}.$$

Each of the two expressions $x = e^{\lambda_1 t}$ and $x = e^{\lambda_2 t}$ is, at least formally, a particular solution of the differential equation, as we see by carrying out the calculations in the reverse order. There can now occur three different cases:

**1. $r^2 - 4mk > 0$**: The two roots $\lambda_1$ and $\lambda_2$ are then real, negative and unequal; we have two solutions $x = u_1 = e^{\lambda_1 t}$ and $x = u_2 = e^{\lambda_2 t}$. With the help of these two solutions, we can at once construct a solution with two arbitrary constants. In fact, on differentiation, we see that

$$x = c_1 u_1 + c_2 u_2$$

is also a solution of the differential equation. We shall show in 10.2.3 that this expression is in fact the most general solution of the equation, i.e., that we can obtain every solution of the equation by substituting suitable numerical values for $c_1$ and $c_2$.

**2. $r^2 - 4mk = 0$**: The quadratic equation has a double root. Thus, to start with, we have,apart from a constant factor, only the single solution $x = w_1 = e^{-rt/2m}$. But we readily verify that in this case the function

$$x = w_2 = te^{-rt/2m}$$

is also a solution of the differential equation. In fact, we find that

$$\dot{x} = \left(1 - \frac{r}{2m}t\right)e^{-rt/2m}, \quad \ddot{x} = \left(\frac{r^2}{4m^2}t - \frac{r}{m}\right)e^{-rt/2m},$$

and substituting we see that the differential equation

$$m\ddot{x} + r\dot{x} + \frac{r^2}{4m}x = m\ddot{x} + r\ddot{x} + kx = 0$$

is satisfied. Then the expression

$$x = c_1 e^{-rt/2m} + c_2 t e^{-rt/2m}$$

yields a solution of the differential equation with two arbitrary integration constants of $c_1$ and $c_2$.

We are led to this solution naturally by the following limiting process:If $\lambda_1 \neq \lambda_2$, then the expression $(e^{\lambda_1 t} - e^{\lambda_2 t})/(\lambda_1 - \lambda_2)$also represents a solution. If we now let $\lambda_1$ tend to $\lambda_2$ and write $\lambda$ instead $\lambda_1$, $\lambda_2$, our expression becomes $\frac{d}{d\lambda}e^{\lambda t} = te^{\lambda t}$.

**3. $r^2 - 4mk < 0$:** We set $r^2 - 4mk = -4m^2\nu^2$and obtain two solutions of the differential equation in complex form, given by $x = u_1 = e^{-rt/2m + i\nu t}$ and $x = u_2 = e^{-rt/2m - i\nu t}$. Euler's formula $e^{\pm i\nu t} = \cos \nu t \pm i \sin \nu t$yields for the real and imaginary parts of the complex solution $u_1$, on the one hand, $v_1 = e^{-rt/2m}\cos \nu t, \quad v_2 = e^{-rt/2m}\sin \nu t,$ and, on the other hand, $v_1 = \frac{u_1 + u_2}{2}, \quad v_2 = \frac{u_1 - u_2}{2i}.$ We see from the second version that $v_1$ and $v_2$ are (real) solutions of the differential equation. We verify this directly by differentiation and substitution.

We can again form from the two particular solutions a general solution

$$x = c_1 v_1 + c_2 v_2 = (c_1 \cos \nu t + c_2 \sin \nu t)e^{-rt/2m}$$

with two arbitrary constants $c_1$ and $c_2$ . This result may also be written in the form

$$x = ae^{-rt/2m}\cos \nu(t - \delta),$$

where we have set $c_1 = a \cos \nu\delta$, $c_2 = a \sin \nu\delta$, and $a$, $\delta$ are two new constants.We recall that we have already come across this relation for the special case $r$=0 in 5.4.3.

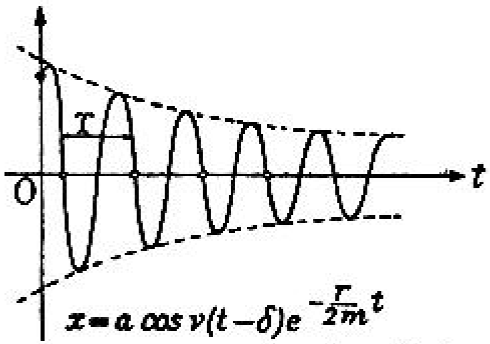


**Fig. 2.—Damped harmonic oscillations**

**11.2.2 Physical Interpretation of the Solution:** In the two cases $r > 2\sqrt{mk}$ and $r = 2\sqrt{mk}$ , the solution is given by the exponential curve or by the graph of the function $te^{-rt/2m}$, which for large values of $t$ resembles the exponential curve, or by the superposition of such curves. In these cases, the process is aperiodic, i.e., as the time increases, the distance $x$ approaches the value 0 asymptotically, without oscillating about the value x = 0. Hence, the motion is not oscillatory. The effect of friction or damping is so large that it prevents the elastic force from setting up oscillatory motions.

It is quite different in the case $r < 2\sqrt{mk}$, , where the damping is so small that one has complex roots $\lambda_1$, $\lambda_2$. The expression $x = a \cos \nu(t - \delta) e^{-rt/2m}$ now yields **damped harmonic oscillations**. These are oscillations which follow the sine law and have the circular frequency

$$\nu = \sqrt{\left(\frac{k}{m} - \frac{r^2}{4m^2}\right)},$$

but the amplitude of which, instead of being constant, is given by $ae^{-rt/2m}$, the i.e., the amplitude decreases exponentially; the larger is $r/2m$, the faster is the rate of decrease. In the physical literature, this damping factor is frequently called the **logarithmic decrement** of the damped oscillation, the term indicating that the logarithm of the amplitude decreases at the rate $r$/2m. A damped oscillation of this kind is shown in Fig. 2. As before, we call the quantity $T = 2\pi/\nu$ the period of the oscillation and the quantity $\nu\delta$ the phase displacement. For the special case $r$ = 0, we obtain again simple harmonic oscillations with the frequency $\nu_0 = \sqrt{k/m}$, the **natural frequency** of the undamped oscillatory system.

**11.2.3 Fulfilment of Given Initial Conditions. Uniqueness of the Solution:** We must still show that the solution with the two constants $c_1$ and $c_2$ can be made to fit any pre-assigned initial state, and also that it represents all the possible solutions of the equation. Suppose that we have to find a solution, which at time $t$ = 0 satisfies the initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, where the numbers $x_0$, and $\dot{x}_0$ can have any values. Then, in **Case 1** in 11.2.1, we must set

$$c_1 + c_2 = x_0,$$
$$c_1\lambda_1 + c_2\lambda_2 = \dot{x}_0.$$

Hence we have for the constants $c_1$ and $c_2$ have two linear equations with the unique solutions

$$c_1 = \frac{\dot{x}_0 - \lambda_2 x_0}{\lambda_1 - \lambda_2}, \quad c_2 = \frac{\dot{x}_0 - \lambda_1 x_0}{\lambda_2 - \lambda_1}.$$

In **Case 2** of 11.2.1, the same process yields the two linear equations

$$c_1 = x_0, \quad \lambda c_1 + c_2 = \dot{x}_0 \qquad \left(\lambda = -\frac{r}{2m}\right),$$

from which $c_1$ and $c_2$ can again be determined uniquely. Finally, in **Case 3** of 11.2.1, the equations determining the constants assume the form

$$a \cos \nu\delta = x_0, \; a\left(\nu \sin \nu\delta - \frac{r}{2m} \cos \nu\delta\right) = \dot{x}_0,$$

with the solutions

$$\delta = \frac{1}{\nu} \operatorname{arc} \cos \frac{x_0}{a}, \quad a = \frac{1}{\nu}\sqrt{\left\{\nu^2 x_0^2 + \left(\dot{x}_0 + \frac{r}{2m} x_0\right)^2\right\}}.$$

Thus, we have shown that the general solution can be made to fit any initial conditions. We have still to show that there is no other solution. For this purpose, we need only show that, for a given initial state, there can never be two different solutions.

If two such solutions $u(t)$ and $v(t)$ were to exist, for which

$$u(0) = x_0, \; \dot{u}(0) = \dot{x}_0 \text{ and } v(0) = x_0, \; \dot{v}(0) = \dot{x}_0,$$

then their difference $w = u - v$ would also be a solution of the differential equation and we should have $w(0) = 0, \; \dot{w}(0) = 0.$ Hence, this solution would correspond to an initial state of rest, i.e., to a state in which at time $t = 0$ the particle is in its position of rest and has zero velocity. We must show that it can never set itself into motion. In order to do this, we multiply both sides of the differential equation $m\ddot{w} + r\dot{w} + kw = 0$ by $2\dot{w}$ and recall that

$$2\dot{w}\ddot{w} = \frac{d}{dt}\dot{w}^2 \text{ and } 2w\dot{w} = \frac{d}{dt}w^2.$$

We thus obtain

$$\frac{d}{dt}(m\dot{w}^2) + \frac{d}{dt}(kw^2) + 2r\dot{w}^2 = 0.$$

If we integrate between the times $t = 0$ and $t = \tau$ and use the initial conditions

$$w(0) = 0, \ \dot{w}(0) = 0,$$

we find

$$m\dot{w}^2(\tau) + kw^2(\tau) + 2r\int_0^\tau \left(\frac{dw}{dt}\right)^2 dt = 0.$$

However, this equation would yield a contradiction, if at any time $t > 0$ the function $w$ differed from 0. In fact, the left hand side of the equation would be positive, since we have taken $m$, $k$ and $r$ to be positive, while the right hand side is zero. Hence $w = u = v$ is always equal to 0, which proves that the solution is unique.

**Exercises 11.1:**

Find the general solution and also the solution for which $x(0) = 0, \ \dot{x}(0) = 1$:

1. $\ddot{x} - 3\dot{x} + 2x = 0.$
2. $\ddot{x} + 3\dot{x} + 2x = 0.$
3. $2\ddot{x} + \dot{x} - x = 0.$
4. $\ddot{x} + 4\dot{x} + 4x = 0.$
5. $4\ddot{x} + 4\dot{x} + x = 0.$

6. Find the general solution, and also the solution, for which $x(0) = 0, \ \dot{x}(0) = 1$, of the equation

$$\ddot{x} + \dot{x} + x = 0.$$

Determine the frequency ($\nu$), the period ($T$), the amplitude ($a$), and the phase ($\delta$) of the solution.

7. Find the solution of

$$2\ddot{x} + 2\dot{x} + x = 0$$

for which $x(0) = 1, \dot{x}(0) = -1$. Calculate the amplitude ($a$), and the phase ($\delta$) of the solution.

Answers and Hints

## 11.3 The Non-homogeneous Equation. Forced Oscillations

**11.3.1 General Remarks:** Before proceeding to the solution of the problem when an external force $f(t)$ is present, i.e., to the solution of the non-homogeneous equation, we make the observations:

If $w$ and $v$ are two solutions of the non-homogeneous equation, their difference $u = w - v$ satisfies the homogeneous equation; we see this at once by substitution. Conversely, if $u$ is a solution of the homogeneous equation and $v$ a solution of the non-homogenous equation, then $w = u + v$ is also a solution of the non-homogeneous equation. Hence we obtain from one solution - the **particular integral** - of the non-homogeneous equation all its solutions by adding the complete integral of the homogeneous equation - the **complementary function**. Hence we must only find a single solution of the non-homogeneous equation. Physically speaking, this means that, if we have a forced oscillation due to an external force and superpose on it an arbitrary free oscillation, represented by a solution of the homogeneous equation, we obtain a phenomenon which satisfies the same non-homogeneous equation as the original forced oscillation. If a frictional force is present, the free motion in the case of oscillatory motion fades out as time goes on, due to the damping factor $e^{-rt/2m}$. Hence, for a given forced vibration with friction, it is immaterial what free vibration we superpose; the motion will always tend to the same final state as time goes on.

Secondly, we note that the effect of a force $f(t)$ can be split up in the same way as the force itself. We mean by this: If $f_1(t)$ and $f_2(t)$ and $f(t)$ are three functions such that

$$f_1(t) + f_2(t) = f(t),$$

and if $x_1 = x_1(t)$ is a solution of the differential equation $m\ddot{x} + r\dot{x} + kx = f_1(t)$ and $x_2 = x_2(t)$ is a solution of the equation $m\ddot{x} + r\dot{x} + kx = f_2(t)$, then $x(t)=x_1(t)+x_2(t)$ is a solution of the differential equation $m\ddot{x} + r\dot{x} + kx = f(t)$. Naturally, a corresponding statement holds if $f(t)$ consists of any number of terms. This simple but important fact is called the **principle of superposition**. The proof follows from a glance at the equation itself. By subdividing the function $f(t)$ into two or more terms, we can thus split the differential equation into several equations, which in certain circumstances may be easier to manipulate.

The most important case is that of a periodic external force $f(t)$. Such a periodic external force can be resolved into purely periodic components by expansion in a Fourier series, and, provided it is continuous and sectionally smooth - the only case of importance in physics -, it can be approximated as closely as we please by a sum of a finite number of purely periodic functions. Hence it is sufficient to find the solution of the differential equation subject to the assumption that the right hand side has the form

$$a \cos \omega t \quad \text{or} \quad b \sin \omega t,$$

where $a$, $b$ and $\omega$ are arbitrary constants.

Instead of working with these trigonometric functions, we can obtain the solution more simply and neatly by using complex notation. We set $f(t) = ce^{i\omega t}$ and the principle of superposition shows that we need only consider the differential equation

$$m\ddot{x} + r\dot{x} + kx = ce^{i\omega t},$$

where $c$ is an arbitrary real or complex constant. Such a differential equation represents actually two real differential equations. In fact, if we split the right hand side into two terms, e.g., if we take $c = 1$ and write $e^{i\omega t} = \cos \omega t + i \sin \omega t$, then $x_1$ and $x_2$ - the solutions of the two real differential equations

$$m\ddot{x} + r\dot{x} + kx = \cos \omega t \quad \text{and} \quad m\ddot{x} + r\dot{x} + kx = \sin \omega t,$$

combine to form the solution $x = x_1 + ix_2$ of the complex differential equation. Conversely, if we first solve the differential equation in complex form, the real part of the solution yields the function $x_1$ and the imaginary part the function $x_2$.

**11.3.2 Solution of the Non-homogeneous Equation:** We will solve the equation $m\ddot{x} + r\dot{x} + kx = ce^{i\omega t}$, by a naturally suggested device. We assume that $c$ is real and (for the time being) that $r \neq 0$. We now guess that there will exist a motion which has the same rhythm as the periodic external force, and attempt accordingly to find a solution of the differential equation in the form

$$x = \sigma e^{i\omega t},$$

where we need only determine the factor $\sigma$, which is independent of the time. If we substitute this expression and its derivatives $\dot{x} = i\omega\sigma e^{i\omega t}$, $\ddot{x} = -\omega^2\sigma e^{i\omega t}$ into the differential equation and remove the common factor $e^{i\omega t}$, we obtain the equation

$$-m\omega^2\sigma + ir\omega\sigma + k\sigma = c \text{ or } \sigma = \frac{c}{-m\omega^2 + ir\omega + k}.$$

Conversely, we see that for this value of $\sigma$, the expression $\sigma e^{i\omega t}$ is actually a solution of the differential equation. However, in order to express the meaning of this result clearly, we must perform a few transformations.

We start by writing the complex factor $\sigma$ in the form

$$\sigma = c\,\frac{k - m\omega^2 - ir\omega}{(k - m\omega^2)^2 + r^2\omega^2} = ca e^{-i\omega\delta},$$

where the positive **distortion factor** *a* and the **phase displacement** are expressed in terms of the given quantities *m*, *r* *k* by the equations

$$a^2 = \frac{1}{(k - m\omega^2)^2 + r^2\omega^2}, \quad \sin\omega\delta = r\omega a, \quad \cos\omega\delta = (k - m\omega^2)a.$$

In this notation, the solution assumes the form

$$x = ca e^{i\omega(t-\delta)},$$

and the meaning of the result is: There corresponds to the force $c \cos\omega t$, the effect $ca\cos\omega(t - \delta)$ and to the force $c \sin \omega t$ the effect $ca\sin\omega(t - \delta)$.

Hence we see that the effect is a function of the same type as the force, i.e., an undamped oscillation. This oscillation differs from the oscillation representing the force in that the amplitude is increased in the ratio $a : 1$ and the phase is altered by the angle $\omega\delta$. Of course, it is easy to obtain the same result without using complex notation, but at the cost of somewhat longer calculations.

According to the remark at the beginning of this section, we have solved completely the problem by finding this one solution; in fact, by superposition of any free oscillation, we can obtain the most general forced oscillation.

Collecting our results, we can make the statement:

The complete integral of the differential equation

$$m\ddot{x} + r\dot{x} + kx = ce^{i\omega t}$$

(where $x \neq 0$) is $x = cae^{i\omega(t-\delta)} + u$, where $u$ is the complete integral of the homogeneous equation $m\ddot{x} + r\dot{x} + kx = 0$ and the quantities $a$ and $\delta$ are defined by the equations

$$a^2 = \frac{1}{(k - m\omega^2)^2 + r^2\omega^2}, \quad \sin\omega\delta = r\omega a, \quad \cos\omega\delta = (k - m\omega^2)a.$$

The constants in this general solution allow us to make the solution suit an arbitrary initial state, i.e., for arbitrarily assigned values of $x_0$ and $\dot{x}_0$ the constants can be chosen in such a way that $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$.

**11.3.3 The Resonance Curve:** In order to acquire a grasp of the above solution and of its significance in applications, we shall study the distortion factor $a$ as a function of the exciting frequency $\omega$, i.e., the function

$$\phi(\omega) = \frac{1}{\sqrt{(k - m\omega^2)^2 + r^2\omega^2}}.$$

The reason for this detailed study is that for given constants $k$, $m$, $r$ or, as we say, for a given **oscillatory system**, we can think of the system as being acted upon by periodic exciting forces of very different circular frequencies, and it is important to consider the solution of the differential equation for these widely different exciting forces. In order to $\omega_0 = \sqrt{k/m}$. This number $\omega_0$ is the circular frequency which the system would have for free oscillations, if the friction $r$ were zero or, more briefly, the **natural frequency of the undamped system**. Owing to the friction $r$, the actual frequency of the free system is not equal to $\omega_0$, but is instead

$$\nu = \sqrt{\left(\frac{k}{m} - \frac{r^2}{4m^2}\right)},$$

where we assume that $4km - r^2 > 0$. (If this is not the case, the free system has no frequency; it is **aperiodic**.)

The function $\phi(\omega)$ tends asymptotically to the value 0 as the exciting frequency tends to infinity, and, in fact, it vanishes to the order $1/\omega^2$. Moreover, $\phi(0)=1/k$; in other words, an exciting force of frequency zero and magnitude 1, i.e., a constant force of magnitude 1 causes the displacement of the oscillatory system to rise to $1/k$. In the region of positive values of $\omega$, the derivative $\phi'(\omega)$ cannot vanish except when the derivative of the expression $(k - m\omega^2)^2 + r^2\omega^2$ vanishes, i.e., for a value $\omega = \omega_1 > 0$, for which the equation

$$-4m\omega(k - m\omega^2) + 2r^2\omega = 0$$

holds. In order that such a value may exist, we must obviously have in this case $2km - r^2 > 0$ ,

$$\omega_1 = \sqrt{\left(\frac{k}{m} - \frac{r^2}{2m^2}\right)} = \sqrt{\left({\omega_0}^2 - \frac{r^2}{2m^2}\right)}.$$

Since the function $\phi(\omega)$ is positive everywhere, increases monotonically for small values of $\omega$ and vanishes at infinity, this value must yield a maximum. We call the circular frequency $\omega_1$ the **resonance frequency** of the system.

By substituting this expression for $\omega_1$, we find that the value of the maximum is

$$\phi(\omega_1) = \frac{1}{r\sqrt{\left(\frac{k}{m} - \frac{r^2}{4m^2}\right)}}.$$

As $r \to 0$, this value increases beyond all bounds. For $r = 0$, i.e., for an undamped oscillatory system, the function $\phi(\omega)$ has an infinite discontinuity at the value $\omega = \omega_1$. This is a limiting case to which we shall give special consideration below.

The graph of the function $\phi(\omega)$ is called the **resonance curve** of the system. The fact that, for $\omega=\omega_1$ (and consequently for small values of $r$ in the neighbourhood of the natural frequency), the distortion of the amplitude $a=\phi(\omega)$ is particularly large is the mathematical expression of the **phenomenon of resonance**, which for fixed values of $m$ and $k$ is more and more evident as $r$ becomes smaller and smaller.

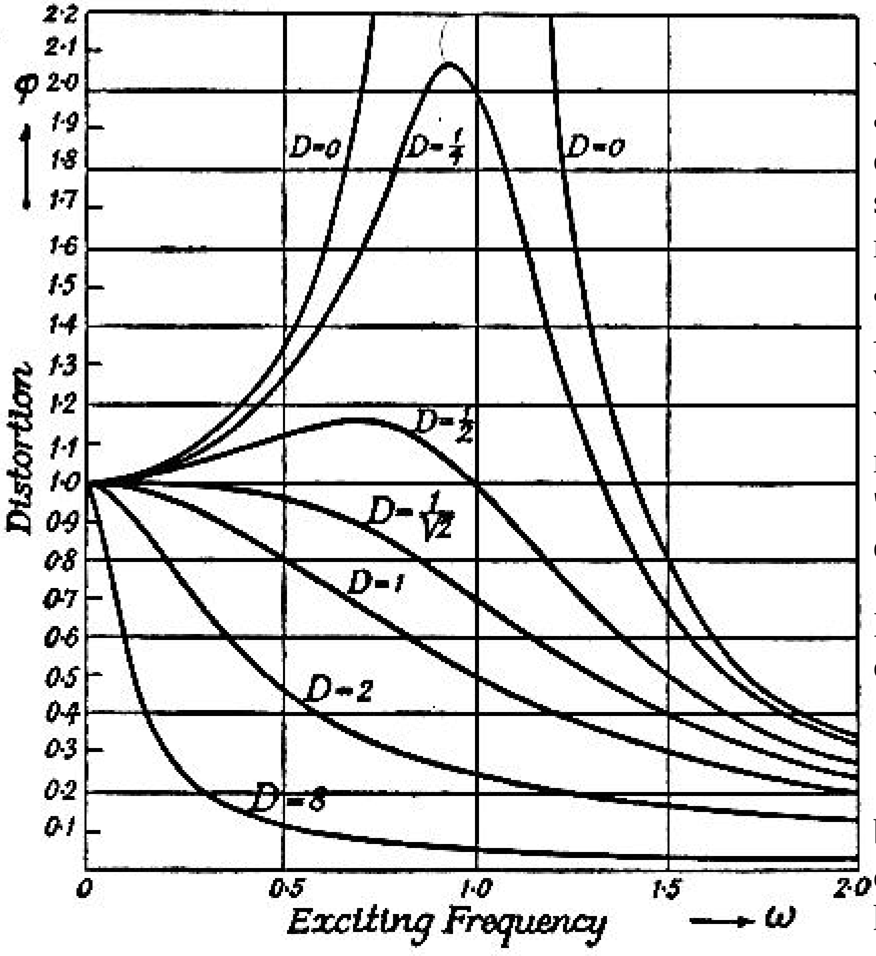



Fig. 3.—Resonance curves

We have sketched in Fig. 3 below a family of resonance curves, all of them corresponding to the values $m = 1$ and $k = 1$, and consequently to $\omega_0 = 1$, but with different values of $D = ½r$. We see that, for small values of $D$, there occurs well marked resonance near $\omega = 1$; in the limiting case $D$=0, there would be an infinite discontinuity of $\phi(\omega)$ at $\omega = 1$, instead of a maximum. As $D$ increases, the maxima move towards the left andm, for the value $D = 1/\sqrt{2}$, we have $\omega_1$=0. In this last case, the point, where the tangent is horizontal, has moved to the origin and the maximum has disappeared. If $D > 1/\sqrt{2}$, there is no zero of $\phi$ '($\omega$); the resonance curve has no longer a maximum and there does not occur any longer any resonance.

In general, the resonance phenomenon ceases as soon as the condition

$$2km - r^2 \leq 0$$

becomes true. In the case of the equality sign, the resonance curve reaches its largest height $\phi(0)=1/k$ at $\omega_1 = 0$; its tangent is horizontal there and after an initial course, which is almost

horizontal, it drops towards zero.

**11.3.4 Further Discussion of the Oscillation:**. However, we cannot be content with the above discussion. In order that we may really understand the phenomenon of forced motion, an additional point must be emphasized. The particular integral $cae^{i\omega(t-\delta)}$ is to be regarded as a limiting state which the complete integral

$$x(t) = cae^{i\omega(t-\delta)} + c_1u_1 + c_2u_2$$

approaches more and more closely as the time advances, since the free oscillation $c_1u_1 + c_2u_2$ superposed on the particular integral fades away with passing time. This fading away will take place slowly, if $r$ is small, rapidly if $r$ is large.

For example, let at the beginning of the motion, i.e., at time $t = 0$, the system be at rest, so that

$$x(0) = 0 \text{ and } \dot{x}(0) = 0.$$

We can determine from this condition the constants $c_1$ and $c_2$, and we see at once that not both are zero. Even when the exciting frequency is approximately or exactly equal to $\omega_1$, so that resonance occurs, the relatively large amplitude $a = \phi(\omega_1)$ will at first not appear. On the contrary, it will be masked by the function $c_1u_1 + c_2u_2$ and will first make its appearance when this function fades away, i.e., it will appear more slowly with a smaller $r$.

For the **undamped system**, i.e., for $r = 0$, our solution fails when the exciting frequency is equal to the natural circular frequency $\omega_0 = \sqrt{k/m}$, because then $\phi(\omega_0)$ is infinite. We therefore cannot obtain a solution of the equation $m\ddot{x} + kx = e^{i\omega t}$ in the form $\sigma e^{i\omega t}$. However, we can at once obtain a solution of the equation in the form $\sigma te^{i\omega t}$. If we substitute this expression into the differential equation, remembering that

$$\dot{x} = \sigma e^{i\omega t}(1 + i\omega t), \quad \ddot{x} = \sigma e^{i\omega t}(2i\omega - t\omega^2),$$

we have

$$\sigma(2im\omega - m\omega^2 t + kt) = 1,$$

and, since $m\omega^2 = k$,

$$\sigma = \frac{1}{2im\omega}.$$

Thus, when resonance occurs in an undamped system, we have the solution

$$x = \frac{t}{2im\omega} e^{i\omega t} = \frac{t}{2i\sqrt{km}} e^{i\omega t}.$$

In real notation, when $f(t) = \cos\omega t$, we have $x = \frac{1}{2}\frac{t}{\sqrt{km}}\sin\omega t$, and when $f(t)=\sin \omega t$, we find

$$x = -\frac{1}{2}\frac{t}{\sqrt{km}}\cos\omega t.$$

Thus, we see that we have found a function which may be referred to as an oscillation, but the amplitude of which increases proportionally with the time. The superposed free oscillation does not fade away, since it is not damped; but it retains its original amplitude and becomes unimportant in comparison with the increasing amplitude of the special forced oscillation. The fact that in this case the solution oscillates backwards and forwards between positive and negative bounds, which continually increase as time goes on, represents the real meaning of the infinite discontinuity of the resonance function in the case of an undamped system.

**11.3.5. Remarks on the Construction of Recording Instruments:**. For a great variety of applications in physics and engineering, the preceding discussion is of the utmost importance. With many instruments such as galvanometers, seismographs, oscillatory electrical circuits in radio receivers and microphone diaphragms, the problem is to record an oscillatory displacement, due to an external periodic force. In such cases, the quantity *x* satisfies our differential equation at least in first approximation.

If *T* is the period of oscillation of the external periodic force, we can expand the force in a Fourier series of the form

$$f(t) = \sum_{l=-\infty}^{\infty} \gamma_l e^{il(2\pi/T)t},$$

or, better still, we can think of it as represented with sufficient accuracy by a trigonometric sum $\sum_{l=-N}^{N} \gamma_l e^{il(2\pi/T)t}$ consisting of a finite number of terms. By the principle of superposition, the solution $x(t)$ of the differential equation, apart from the superposed free oscillation, will be represented by an infinite series (the convergence of which will not be discussed here) of the form

$$x(t) = \sum_{l=-\infty}^{\infty} \sigma_l e^{il(2\pi/T)t},$$

or approximately by a finite expression of the form

$$x(t) = \sum_{l=-N}^{N} \sigma_l e^{il(2\pi/T)t}.$$

By virtue of our previous results,

$$\sigma_l = \gamma_l \alpha_l e^{-i\delta_l(2\pi l/T)} \quad \text{and}$$

$$\alpha_l^2 = \frac{1}{\left(k - ml^2 \frac{4\pi^2}{T^2}\right)^2 + r^2 l^2 \frac{4\pi^2}{T^2}}, \quad \tan \frac{2\pi l}{T} \delta_l = \frac{2\pi l r}{T\left(k - m\frac{4\pi^2 l^2}{T^2}\right)}.$$

We can then describe the action of an arbitrary periodic external force in the following ways: If we analyze the exciting force into purely periodic components - the individual terms of the Fourier series -, then each component is subject to its own distortion of amplitude and phase displacement, and the separate effects are then superposed by addition. If we are only interested in the distortion of amplitude (the phase displacement is only of secondary importance in applications and, moreover, can be discussed in the same way as the distortion of amplitude, for example, it is imperceptible to the human ear), a study of the resonance curve yields complete information about the way in which the motions of the recording apparatus reproduce the external exciting force. For very large values of $l$ or $\omega\,(=2\pi lt/T)$, the effect of the exciting frequency on the displacements $x$ will be hardly perceptible. On the other hand, all exciting frequencies in the neighbourhood of $\omega_1$, the (circular) resonance frequency, will markedly affect the quantity $x$.

In the construction of physical measuring and recording apparatus, the constants $m$, $r$ and $k$ are at our disposal, at least within wide limits. These should be chosen so that the shape of the resonance curve is as well adapted as possible to the special requirements of the measurement in question. Here two considerations predominate. In the first place, it is desirable that the apparatus should be as sensitive as possible, i.e., for all frequencies $\omega$ in question, the value of $\alpha$ should be as large as possible. For small values of $\omega$, as we have seen, $\alpha$ is approximately proportional to l/$k$, so that the number 1/$k$ is a measure of the sensitivity of the instrument for small exciting frequencies. The sensitivity can therefore be increased by increasing l/$t$, i.e., by weakening the restoring force.

The other important point is the necessity for relative freedom from distortion. Let us assume that the representation $f(t) = \sum_{l=-N}^{N} \gamma_l e^{il(2\pi/T)t}$ is an adequate approximation to the exciting force. We then say that the apparatus records the exciting force $f(t)$ with relative freedom from distortion, if, for all circular frequencies $\omega \leq N$

$2\pi/T$, the distortion factor has approximately the same value. This condition is indispensable, if we wish to derive conclusions about the exciting process directly from the behaviour of the apparatus, for example, if a gramophone or wireless set is to reproduce both high and low musical notes with an approximately correct ratio of intensity. The requirement that the reproduction should be relatively **distortionless** can never be satisfied exactly, since no portion of the resonance curve is exactly horizontal. However, we can attempt to choose the constants $m$, $k$ and $r$ of the apparatus in such a way that no marked resonance occurs, and also in such a way that the curve has a horizontal tangent at the beginning, so that $\varphi(\omega) = \alpha$ remains approximately constant for small values of $\omega$. As we have learnt above, we can do this by setting

$$2km - r^2 = 0.$$

Given a constant $m$ and a constant $k$, we can satisfy this requirement by adjusting the friction $r$ properly, e.g., by inserting a properly chosen resistance in the electrical circuit. The resonance curve then shows that, from the frequency 0 to circular frequencies near the natural circular frequency $\omega_0$ of the undamped system, the instrument is nearly distortionless and that above this frequency the damping is considerable. Hence we obtain relative freedom from distortion in a given frequency interval by first choosing $m$ so small and $k$ so large that the natural circular frequency $\omega_0$ of the undamped system is larger than any of the exciting circular frequencies and then choosing a damping factor in accordance with the equation $2km - r^2$.

**Exercises 11.2:**

For the following equations find the solution, satisfying the initial conditions $x(0) = 0,\ \dot{x}(0) = 0.$ For Equations 1-4 state also the amplitude, the phase and the value of $\omega$ for which the amplitude is a maximum:

1. $\ddot{x} + 3\dot{x} + 2x = \cos\omega t.$
2. $\ddot{x} + \dot{x} + x = \cos\omega t.$
3. $\ddot{x} + \dot{x} + x = \sin\omega t.$
4. $2\ddot{x} + 2\dot{x} + x = \cos\omega t.$
5. $\ddot{x} + 4\dot{x} + 4x = \cos\omega t.$

Answers and Hints

## 11.4 Additional Remarks on Differential Equations

A more systematic study of differential equations is made Volume II, Chapter VI. We will present here only a few additions to the preceding special theory.

**11.4.1 Homogeneous Linear Differential Equations of Order *n* with Constant Coefficients:** More complicated vibration problems lead to linear differential equations for the unknown functions $x(t)$ of the independent variable of the form

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_n x = 0,$$

where $a_1, \cdots, a_n$ are constants and $n$ is a positive integer. We can solve this equation by a method similar to that for the case $n = 3$ (cf. 11.2). Let $x = e^{\lambda t}$. If we substitute this function and its derivatives into the differential equation and remove the common factor $e^{\lambda t}$, we obtain the equation of the $n$-th degree in $\lambda$:

$$f(\lambda) \equiv \lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0.$$

If $\lambda$ is a root of this equation, $e^{\lambda t}$ satisfies the differential equation. We shall now examine the various possibilities. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the roots of the equation $f(\lambda) = 0$, so that

$$f(\lambda) \equiv (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n).$$

First of all, let all the roots differ. If all the $\lambda_n$ are real, then we obtain $n$ linearly independent solutions $e^{\lambda_n t}$, just as before. The general solution is any linear combination

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}$$

of these solutions. The constants $c_n$ can be determined so that $x$ and its first $n$-1 derivatives take arbitrarily pre-assigned values at time $t = 0$; in order to do this, we must solve the system of $n$ linear equations:

$$\begin{aligned} c_1 + c_2 + \ldots + c_n &= x(0), \\ \lambda_1 c_1 + \lambda_2 c_2 + \ldots + \lambda_n c_n &= x'(0), \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \lambda_1^{n-1} c_1 + \lambda_2^{n-1} c_2 + \ldots + \lambda_n^{n-1} c_n &= x^{(n-1)}(0). \end{aligned}$$

This set of equations has always a solution, if the roots are not equal, because the determinant of the coefficients is not zero.

If two of the roots are equal, say $\lambda_1 = \lambda_2$, then not only $e^{\lambda_1 t}$ but also $te^{\lambda_1 t}$ is a solution. This can be verified as follows: since $f(\lambda) = 0$ has a double root $\lambda=\lambda_1=\lambda_2$, if follows by a well-known theorem of algebra that

$$f'(\lambda) \equiv n\lambda^{n-1} + (n-1)a_1\lambda^{n-2} + \ldots + a_{n-1} = 0.$$

Now, by Leibniz's rule for the derivative of a product,

$$\frac{d^k}{dt^k}(te^{\lambda t}) = t\frac{d^k}{dt^k}e^{\lambda t} + k\frac{dt}{dt}\frac{d^{k-1}}{dt^{k-1}}e^{\lambda t} = t\lambda^k e^{\lambda t} + k\lambda^{k-1}e^{\lambda t}.$$

Substituting in the differential equation, we find

$$\begin{aligned} te^{\lambda t}(\lambda^n + a_1\lambda^{n-1} + \ldots + a_n) + e^{\lambda t}(n\lambda^{n-1} + (n-1)a_1\lambda^{n-2} + \ldots + a_{n-1}) \\ = te^{\lambda t}f(\lambda) + e^{\lambda t}f'(\lambda) = 0, \end{aligned}$$

since $f(\lambda) = 0$ and, by the above remark regarding double roots, $f'(\lambda) = 0$.

In the same manner, if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are equal, we obtain the linearly independent solutions:

$$e^{\lambda_1 t}, \; te^{\lambda_1 t}, \; \ldots \, t^{\mu-1}e^{\lambda_1 t},$$

which may be combined to give a general solution depending on $c_1, c_2, \cdots, c_n$. These parameters again enable us to adapt the solution to $n$ pre-assigned conditions, so that, for $t = 0$, we can fix the value of $x(0)$ and its first $n - 1$ derivatives.

If the equation has complex roots, then, by a theorem of algebra, the roots occur in pairs, each one together with its conjugate root. Just as in the case $n$=2, we obtain solutions of the form

$$\cos\beta t \,.\, e^{\alpha t} \text{ and } \sin\beta t \,.\, e^{\alpha t}, \text{ where } \lambda_1 = \alpha + i\beta, \; \lambda_2 = \alpha - i\beta.$$

A few examples will serve to illustrate the above results.

*Example 1.* $$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0,$$

$$f(\lambda) = \lambda^3 + 2\lambda^2 - \lambda - 2 = 0.$$

The general solution is

$$x = c_1e^{-t} + c_2e^{t} + c_3e^{-2t}.$$

A particular solution for which $x = 2$, $x' = 0$ at $t = 0$ is $x = e^t + e^{-t}$.

*Example 2.* $$\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - \frac{dx}{dt} + x = 0.$$

The general solution is $x = c_1e^t + c_2te^t + c_3e^{-t}$.

*Example 3.* $$\frac{d^3x}{dt^3} - 2\frac{dx}{dt} + 4 = 0,$$

$$f(\lambda) = \lambda^3 - 2\lambda + 4 = (\lambda + 2)(\lambda - 1 + i)(\lambda - 1 - i).$$

The general solution is $x = c_1e^{-2t} + c_2e^t \cos t + c_3e^t \sin t$.

**11.4.2 Bernoulli Equation:** An equation of the type

$$\frac{dx}{dt} + A(t)x = B(t),$$

where $A$ and $B$ are only functions of t, is called a linear equation. In the case $B = 0$, if $x=\alpha(t)$, $x=\beta(t)$ are solutions, any linear combination of $\alpha$ and $\beta$ is also a solution. We shall now consider the slightly more general type of equation

$$\frac{dx}{dt} + A(t)x = B(t)x^n,$$

where $n$ is a positive integer; it is called **Bernoulli equation**. First, consider the simpler case where $B$ is zero, i.e., where

$$\frac{dx}{dt} + A(t)x = 0.$$

Rewriting the equation in the form $\frac{dx}{x} = -A(t)\,dt,$ we see that it can be integrated immediately:

$$\log x = -\int A(t)\,dt + c,$$
$$x = e^c e^{-\int A\,dt} = ve^{-\int A\,dt},$$

if we write $e^c = v$.

We now try to satisfy Bernolli' equation by a function of the form $x = ve^{-\int A\,dt},$ where we assume that $v$ is a variable, so that

$$\frac{dx}{dt} = \frac{dv}{dt}e^{-\int A\,dt} - vAe^{-\int A\,dt}.$$

On substitution, we find

$$\frac{dv}{dt}v^{-n} = Be^{-n\int A\,dt}e^{\int A\,dt},$$

which can be integrated at once, so that

$$x^{1-n} = (1-n)e^{(n-1)\int A\,dt}\left[\int Be^{(1-n)\int A\,dt}\,dt\right].$$

The above method is very important and may be applied in many cases. It is called **variation of parameters**; further details are given in Volume 2. Note that our solution is expressed in terms of integrals which, in general, cannot be expressed in terms of the elementary functions.

**Example:** Consider the equation

$$\frac{dx}{dt} - tx = t^3x^2.$$

Let

$$x = ve^{\int t\,dt} = ve^{\frac{1}{2}t^2};$$

then

$$\frac{dx}{dt} - tx = \frac{dv}{dt}e^{\frac{1}{2}t^2} + vte^{\frac{1}{2}t^2} - tve^{\frac{1}{2}t^2} = \frac{dv}{dt}e^{\frac{1}{2}t^2},$$

and the eqnation becomes

$$\frac{dv}{dt}e^{\frac{1}{2}t^2} = t^3v^2e^{t^2}, \quad \text{or} \quad \frac{dv}{v^2} = t^3e^{\frac{1}{2}t^2}dt.$$

By integration, we find

$$-\frac{1}{v} = (t^2 - 2)e^{\frac{1}{2}t^2} + c, \quad \text{or} \quad \frac{1}{x} = 2 - t^2 + ce^{-\frac{1}{2}t^2}.$$

This result could have been obtained by direct substitution in the formula given above, but actually carrying out the method is far more instructive.

**11.4.3 Other First Order Differential Equations Solvable by Simple Integration:** There are a few other types of first order differential equations which can be solved by integration (although in most cases the integration cannot be performed explicitly in terms of elementary functions).

The first method we consider is that of **separation of variables**. If the differential equation can be given the form

$$A(x)\,dx + B(y)\,dy = 0,$$

i.e., $y'B(y) + A(x) = 0$, the variables are said to be **separable**. Obviously, the solution is

$$\int A(x)\,dx + \int B(y)\,dy + c = 0.$$

**Example:** Consider the equation

$$yy' + xy^2 = x.$$

Here,

$$y\,dy + x(y^2 - 1)\,dx = 0, \quad \text{or} \quad \frac{y\,dy}{y^2 - 1} + x\,dx = 0,$$

whence

$$\tfrac{1}{2}\log(y^2 - 1) + \tfrac{1}{2}x^2 = c, \quad \text{or} \quad (y^2 - 1)e^{x^2} = k.$$

Another type of equation, which can be solved, is of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

where M and N are homogeneous functions of $x$ and $y$ of the same degree. In this case, the fraction $M/N$ is a function of $y/x$ only and we may write

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

If we set $y = xv$, this becomes

$$x\frac{dv}{dx} + v = f(v).$$

The variables *x*, *v* are now separable as shown:

$$\frac{dx}{x} = \frac{dv}{f(v) - v}.$$

Integrating, we have

$$\log x = \int \frac{dv}{f(v) - v} + c.$$

**Example:** Consider the equation

$$(2\sqrt{xy} - x)\,dy + y\,dx = 0.$$

Substituting $y = vx$, we have

$$(2v^{1/2} - 1)x\left(v + x\frac{dv}{dx}\right) + vx = 0,$$

$$v(2v^{1/2} - 1) + v + x(2v^{1/2} - 1)\frac{dv}{dx} = 0,$$

$$\frac{dx}{x} = -\frac{2v^{1/2} - 1}{2v^{3/2}}\,dv = -\frac{dv}{v} + \frac{dv}{2v^{3/2}}.$$

Now, integration yields

$$\log x = -\log v - v^{-\frac{1}{2}} + c \quad \text{or} \quad \log y + \sqrt{x/y} = c.$$

**11.4.4 Differential Equations of the Second Order:** There are a few types of non-linear differential equations the solutions of which can also be found by integration. One type has already been discussed implicitly in 5.4.4 in our study of the motion of a particle on a given curve. This type is as follows:

$$\frac{d^2x}{dt^2} = f(x).$$

We set $v = dx/dt$, so that

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

and our equation becomes

$$v\frac{dv}{dx} = f(x).$$

This equation may be regarded as one of the first order with $v$ as dependent and $x$ as independent variable. Separating the variables and integrating, we have

$$v\,dv = f(x)\,dx$$

$$v^2 = 2\int f(x)\,dx + c \quad \text{or} \quad v = \sqrt{2\int f(x)\,dx + c}.$$

Then

$$\frac{dx}{\sqrt{2\int f(x)\,dx + c}} = dt,$$

which can be solved by integration (although, in general, it is impossible to carry it out explicitly).

This device lets us solve equations of the types:

$$\phi\left(\frac{d^2x}{dt^2}, \frac{dx}{dt}\right) = 0,$$

$$\psi\left(\frac{d^2x}{dt^2}, \frac{dx}{dt}, t\right) = 0,$$

$$\theta\left(\frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right) = 0,$$

which, if we set $v = dx/dt$, reduce to

$$\phi\left(\frac{dv}{dt}, v\right) = 0,$$

$$\psi\left(\frac{dv}{dt}, v, t\right) = 0,$$

$$\theta\left(v\frac{dv}{dx}, v, x\right) = 0.$$

These are equations of the first order which may be solvable by the preceding methods. This solution, after $v$ has been replaced by $dx/dt$, will again be a differential equation of the first order, which must be solved for $x$. A few examples will make the process clear.

**Examples:**

1. $2a\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} = 1.$

Setting $dy/dx = p$, the equation becomes

$$2ap\frac{dp}{dx} = 1.$$

Integration by separation of variables yields

$$ap^2 = x + c_1 \quad \text{or} \quad \sqrt{a}\,\frac{dy}{dx} = \sqrt{x + c_1}$$

integration

$$\sqrt{a}(y + c_2) = \tfrac{2}{3}(x + c_1)^{3/2}$$

squaring

$$a(y + c_2)^2 = \tfrac{4}{9}(x + c_1)^3.$$

2. $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0.$

Setting *dy*/*dx* = *p* yields

$$(1 + x^2)\frac{dp}{dx} + xp = 0, \quad \text{or} \quad \frac{dp}{p} = -\frac{x\,dx}{1 + x^2},$$

integration

$$\log p = -\tfrac{1}{2}\log(1 + x^2) + c, \; p = c_1(1 + x^2)^{-1/2}, \; \text{or} \; \frac{dy}{dx} = \frac{c_1}{\sqrt{1 + x^2}},$$

whence

$$y = c_2 + c_1 \operatorname{arsinh} x.$$

3. $y\dfrac{d^2y}{dx^2} = 1 - \left(\dfrac{dy}{dx}\right)^2.$

Set *dy*/*dx* = *p*, then *dy/dx* = *pdp/dx*. Now

$$py\frac{dp}{dy} = 1 - p^2, \quad \text{or} \quad \frac{p\,dp}{1-p^2} = \frac{dy}{y}.$$

Integration yields

$$-\tfrac{1}{2}\log(1-p^2) = \log y + c, \text{ that is, } y = c_1(1-p^2)^{-1/2}, \text{ or } y^2(1-p^2) = c_1{}^2,$$

whence

$$\frac{dy}{dx} = p = \frac{\sqrt{y^2 - c_1{}^2}}{y}, \quad \text{or} \quad \frac{y\,dy}{\sqrt{y^2 - c_1{}^2}} = dx.$$

Integration yields

$$\sqrt{y^2 - c_1{}^2} = x + c_2, \text{ that is, } y^2 = x^2 + c_3x + c_4.$$

**Exercises 11.3:** Solve the differential equations

1. $(1 + y^2)dx - (y - \sqrt{1 + y})(1 + x)^{3/2} dy = 0.$

2. $(x^3 + y^3)dy = 3x^2y\,dx.$

3. $y(\log x - \log y)dy - x\,dx = 0.$

4. $xy' + y = y^2 \log x.$

5. $(1 + y^2)dx = (\arctan y - x)dy.$

6. $yy' + \frac{1}{2}y^2 = \sin x.$

7. $(x^3y^3 + x^2y^2 + xy + 1)y + (x^3y^3 - x^2y^2 - xy + 1)xy' = 0.$

8. $3y^2y' + y^3 = x - 1.$

9. $\sin x \cos y\,dx + \cos x \sin y\,dy = 0.$

10. $(1 + e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0.$

11. $\frac{d^3x}{dt^3} - 3\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - x = 0.$

12. $\frac{d^3x}{dt^3} - 6\frac{d^2x}{dt^2} + 9\frac{dx}{dt} = 0.$

13. $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0.$

14. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$

15. $\frac{d^5y}{dx^5} - 2\frac{d^4y}{dx^4} + y = 0.$

16. $a\frac{d^2y}{dx^2} = \frac{dy}{dx}.$

17. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0.$

18. $\frac{d^4y}{dx^4} = \frac{d^2y}{dx^2}.$

19. $(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0.$

20. $(1 - y)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right) = 0.$

21. $x\frac{d^2x}{dt^2} = 2\left(\frac{dx}{dt}\right)^2.$

22. $(1 - t^2)\frac{d^2s}{dt^2} - t\frac{ds}{dt} = 2.$

23. Find the motion of a particle moving along a straight line under the attraction of a force varying with the inverse square of the distance from the origin.

Answers and Hints

# Differential and Integral Calculus

**by R.Courant**

## Summary of Important Theorems and Formulae

### 1. Hyperbolic Functions 3.8

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}). \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}). \qquad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

$$\cosh^2 x - \sinh^2 x = 1. \qquad \cosh^2 x = \frac{1}{1 - \tanh^2 x}.$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y.$$

$$\cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1). \qquad \sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1).$$

$$\operatorname{ar\,sinh} x = \log\{x + \sqrt{(x^2 + 1)}\}.$$

$$\operatorname{ar\,cosh} x = \log\{x \pm \sqrt{(x^2 - 1)}\} \ (x \geqq 1).$$

$$\operatorname{ar\,tanh} x = \tfrac{1}{2} \log \frac{1 + x}{1 - x} \ (|x| < 1).$$

$$\operatorname{ar\,coth} x = \tfrac{1}{2} \log \frac{x + 1}{x - 1} \ (|x| > 1).$$

### 2. Convergence of Sequences and Series

#### 2.1 Infinite sequences 1.6:

Cauchy's convergence test 1.6.2. A sequence of numbers $a_n$ is convergent if, and only if, there exists for every positive constant $\varepsilon$ a number $N$ such that

$$|a_n - a_m| < \epsilon \text{ when } n > N,\ m > N.$$

Operating with limits 1.6.4. If $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist, then

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n;$$

$$\lim_{n\to\infty}(a_n \,.\, b_n) = \lim_{n\to\infty} a_n \,.\, \lim_{n\to\infty} b_n;$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}, \text{ provided } \lim_{n\to\infty} b_n \neq 0.$$

## 2.2 Infinite Series 8:

Cauchy's convergence test 8.1.1 The series $\Sigma a_n$ converges if, and only if, for there exists for every positive quantity $\varepsilon$ a number $N$ such that

$$|a_n + a_{n+1} + \ldots + a_m| < \epsilon \text{ when } m > n > N.$$

**Note:** All the following criteria are sufficient, but not necessary.

Principal of Comparison of Series 8.2 $\Sigma a_n$ converges if there exist numbers $b_n$ such that $b_n \geq |a_n|$ for all values of $n$ and $\Sigma b_n$ converges.

Ratio test and and root test 8.2.2 $\Sigma a_n$ converges if there is a number $N$, and also a number $q < 1$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < q \quad \text{or} \quad \sqrt[n]{|a_n|} < q$$

for all values $n > N$; in particular, if there is a number $k < 1$ such that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = k \quad \text{or} \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = k.$$

Leibnitz's Test 8.2 $\Sigma a_n$ converges if the terms have alternating signs and $|a_n|$ tends monotonically to zero.

## 3. Differentiation

### 3.1 General rule (Fundamental Ideas 2.3:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x).$$
$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x).$$
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2},\quad g(x) \neq 0$$

3.1 A3.3.2

Chain Rule

$$\text{If } f(x) = g\{\phi(x)\},$$
$$\frac{df}{dx} = \frac{dg}{d\phi}\,\frac{d\phi}{dx},$$
$$\frac{d^2f}{dx^2} = \frac{d^2g}{d\phi^2}\left(\frac{d\phi}{dx}\right)^2 + \frac{dg}{d\phi}\,\frac{d^2\phi}{dx^2}, \text{ and so on.}$$

3.4.1 A3.3.3

$$\text{If } u = f(\xi, \eta, \zeta, \ldots), \text{ where } \xi = \xi(x, y),\ \eta = \eta(x, y), \ldots,$$
$$\begin{aligned} u_x &= f_\xi \xi_x + f_\eta \eta_x + f_\zeta \zeta_x + \ldots, \\ u_{xx} &= f_{\xi\xi}\xi_x^{\,2} + f_{\eta\eta}\eta_x^{\,2} + f_{\zeta\zeta}\zeta_x^{\,2} + \ldots \\ &\quad + 2f_{\xi\eta}\xi_x\eta_x + 2f_{\xi\zeta}\xi_x\zeta_x + \ldots \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + f_\xi \xi_{xx} + f_\eta \eta_{xx} + f_\zeta \zeta_{xx} + \ldots, \end{aligned}$$

with corresponding formulae for $u_{xy}$ and $u_{yy}$ 10.4.2

Implicit Functions

If $F(x, y) = 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y},$$

$$\frac{d^2y}{dx^2} = -\frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{F_y^3}.$$

10.5.1

Functions expressed in terms of a parameter

If $x = x(t)$, $y = y(t)$, $\dfrac{dy}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt}$ 5.1.3

Inverse functions

$$\frac{dy}{dx} = 1\Big/\frac{dx}{dy}$$ 3.3

If $\xi = \phi(x, y)$, $\eta = \psi(x, y)$,

$$\frac{\partial x}{\partial \xi} = \frac{\psi_y}{D}, \quad \frac{\partial x}{\partial \eta} = -\frac{\phi_y}{D}, \quad \frac{\partial y}{\partial \xi} = -\frac{\psi_x}{D}, \quad \frac{\partial y}{\partial \eta} = \frac{\phi_x}{D},$$

where

$$D = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} = \phi_x\psi_y - \phi_y\psi_x$$

(functional determinant or Jacobian)

10.4.4

## 3.2 Special Formulae 2.3.3 3.1.3 3.3.3 3.6 3.8.3

$$(x^n)' = nx^{n-1}.$$

$$(\sin x)' = \cos x. \qquad (\text{arc} \sin x)' = \frac{1}{\sqrt{(1-x^2)}}.$$

$$(\cos x)' = -\sin x. \qquad (\text{arc} \cos x)' = -\frac{1}{\sqrt{(1-x^2)}}.$$

$$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x. \qquad (\text{arc} \tan x)' = \frac{1}{1+x^2}.$$

$$(\cot x)' = -\frac{1}{\sin^2 x} = -\text{cosec}^2 x. \qquad (\text{arc} \cot x)' = -\frac{1}{1+x^2}.$$

$$(\sinh x)' = \cosh x. \qquad (\text{ar} \sinh x)' = \frac{1}{\sqrt{(1+x^2)}}.$$

$$(\cosh x)' = \sinh x. \qquad (\text{ar} \cosh x)' = \pm \frac{1}{\sqrt{(x^2-1)}} \quad (x > 1).$$

$$(\tanh x)' = \frac{1}{\cosh^2 x} = \text{sech}^2 x. \qquad (\text{ar} \tanh x)' = \frac{1}{1-x^2} \quad (|x| < 1).$$

$$(\coth x)' = -\frac{1}{\sinh^2 x} = -\text{cosech}^2 x. \qquad (\text{ar} \coth x)' = \frac{1}{1-x^2} \quad (|x| > 1).$$

$$(\log_a x)' = \frac{1}{x} \log_a e; \qquad (a^x)' = a^x \log_e a;$$

in particular, $\qquad$ in particular,

$$(\log x)' = \frac{1}{x}. \qquad (e^x)' = e^x.$$

$$(u^v)' = u^v (vu'/u + v' \log u).$$

## 4. Integration

## 4.1 General Rules (Fundamental Ideas)

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$ 2.1.2

$$\int_a^b \{f(x) + g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx.$$

2.1.3 3.2.1

Estimation of Integrals

$$\text{If } f(x) \geqq g(x),\ b \geqq a, \quad \int_a^b f(x)\,dx \geqq \int_a^b g(x)\,dx$$ 2.7

Integration by Parts

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$ 4.4

Method of Substitution

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\{\phi(u)\}\,\phi'(u)\,du,$$ 4.2

Link between Differentiation and Integration

$$\frac{d}{dx}\int_a^x f(u)\,du = f(x).$$ 2.4.2

Improper Integrals 4.8

If $f(x)$ is continuous except at the point $x = b$, where it becomes infinite, $\int_a^b f(x)\,dx$ is (absolutely) convergent, if in the neighbourhood of $x = b$

$$|f(x)| \leq \frac{M}{(b-x)^{\nu}}, \text{ where } \nu < 1.$$ 4.8.2

where $\nu > 1$, for values of $x$

} *A.* 4.8.3 $\int_a^{\infty} f(x)\,dx$ converges (absolutely) if $|f(x)| \leq \frac{M}{x^{\nu}}$.

## 4.2 Special Formulae 2.2 2.7.2 3.2.2 3.3.4 3.6 4.1 4.2.1 4.2.3 4.4.2

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

$$\int \log x\,dx = x\log x - x.$$

$$\int \frac{dx}{x} = \log|x|.$$

$$\int \frac{1}{x}\log x\,dx = \tfrac{1}{2}(\log x)^2.$$

$$\int a^x\,dx = \frac{a^x}{\log a}.$$

$$\int \frac{1}{x\log x}\,dx = \log|\log x|.$$

$$\int x^a \log x\,dx = \frac{x^{a+1}}{a+1}\left(\log x - \frac{1}{a+1}\right); \quad a \neq -1.$$

$$\int \sin x\,dx = -\cos x. \qquad \int \sinh x\,dx = \cosh x.$$

$$\int \cos x\,dx = \sin x. \qquad \int \cosh x\,dx = \sinh x.$$

$$\int \tan x\,dx = -\log|\cos x|. \qquad \int \tanh x\,dx = \log\cosh x.$$

$$\int \cot x\,dx = \log|\sin x|. \qquad \int \coth x\,dx = \log|\sinh x|.$$

$$\int \operatorname{arc}\sin x\,dx = x \operatorname{arc}\sin x + \sqrt{(1-x^2)}.$$

$$\int \operatorname{arc}\cos x\,dx = x \operatorname{arc}\cos x - \sqrt{(1-x^2)}.$$

$$\int \operatorname{arc}\tan x\,dx = x \operatorname{arc}\tan x - \tfrac{1}{2}\log(1+x^2).$$

$$\int \operatorname{arc}\cot x\,dx = x \operatorname{arc}\cot x + \tfrac{1}{2}\log(1+x^2).$$

$$\int \operatorname{ar}\sinh x\,dx = x \operatorname{ar}\sinh x - \sqrt{(1+x^2)}.$$

$$\int \operatorname{ar}\cosh x\,dx = x \operatorname{ar}\cosh x - \sqrt{(x^2-1)}.$$

$$\int \operatorname{ar}\tanh x\,dx = x \operatorname{ar}\tanh x + \tfrac{1}{2}\log(1-x^2).$$

$$\int \operatorname{ar}\coth x\,dx = x \operatorname{ar}\coth x + \tfrac{1}{2}\log(x^2-1).$$

$$\int \frac{dx}{\sin x} = \log\left|\tan\frac{x}{2}\right|. \qquad \int \frac{dx}{\sinh x} = \log\left|\tanh\frac{x}{2}\right|.$$

$$\int \frac{dx}{\cos x} = \log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| \qquad \int \frac{dx}{\cosh x} = 2\operatorname{arc}\tan\left(\tanh\frac{x}{2}\right).$$

$$= 2\operatorname{ar}\tanh\left(\tan\frac{x}{2}\right).$$

$$\int \frac{dx}{\sin x\cos x} = \log|\tan x|. \qquad \int \frac{dx}{\sinh x\cosh x} = \log|\tanh x|.$$

$$\int \frac{dx}{\sin^2 x} = -\cot x. \qquad \int \frac{dx}{\sinh^2 x} = -\coth x.$$

$$\int \frac{dx}{\cos^2 x} = \tan x. \qquad \int \frac{dx}{\cosh^2 x} = \tanh x.$$

$$\left.\begin{aligned}\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} &= \frac{1}{ab} \operatorname{arc\,tan}\left(\frac{a}{b} \tan x\right)\\ \int \frac{dx}{a^2 \sin^2 x - b^2 \cos^2 x} &= -\frac{1}{ab} \operatorname{ar\,tanh}\left(\frac{a}{b} \tan x\right)\end{aligned}\right\} a,\ b \neq 0.$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \operatorname{arc\,tan} \frac{x}{a}.$$

$$\int \frac{dx}{x^2 - a^2} = \begin{cases} -\dfrac{1}{a} \operatorname{ar\,tanh} \dfrac{x}{a} = \dfrac{1}{2a} \log \dfrac{a - x}{a + x}, & \text{if } |x| < a.\\[2ex] -\dfrac{1}{a} \operatorname{ar\,coth} \dfrac{x}{a} = \dfrac{1}{2a} \log \dfrac{x - a}{x + a}, & \text{if } |x| > a,\ a > 0.\end{cases}$$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \begin{cases} +\operatorname{arc\,sin} \dfrac{x}{a}.\\[2ex] -\operatorname{arc\,cos} \dfrac{x}{a}.\end{cases} \qquad \int \frac{dx}{x\sqrt{(x^2 - a^2)}} = \begin{cases} -\dfrac{1}{a} \operatorname{arc\,sin} \dfrac{a}{x}.\\[2ex] +\dfrac{1}{a} \operatorname{arc\,cos} \dfrac{a}{x}.\end{cases}$$

$$\int \frac{x\,dx}{\sqrt{(a^2 + x^2)}} = \sqrt{(a^2 + x^2)}.$$

$$\int \frac{x\,dx}{\sqrt{(a^2 - x^2)}} = -\sqrt{(a^2 - x^2)}.$$

$$\int \frac{dx}{\sqrt{(a^2 + x^2)}} = \operatorname{ar\,sinh} \frac{x}{a} = \log\left\{\pm x + \sqrt{(x^2 + a^2)}\right\}.$$

$$\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \operatorname{ar\,cosh} \frac{x}{a} = \log\left\{x \pm \sqrt{(x^2 - a^2)}\right\}.$$

$$\int \frac{dx}{x\sqrt{(x^2 + a^2)}} = -\frac{1}{a} \operatorname{ar\,sinh} \frac{a}{x} = -\frac{1}{a} \log \frac{\pm a + \sqrt{(a^2 + x^2)}}{x}.$$

$$\int \frac{dx}{x\sqrt{(a^2 - x^2)}} = -\frac{1}{a} \operatorname{ar\,cosh} \frac{a}{x} = -\frac{1}{a} \log \frac{a \pm \sqrt{(a^2 - x^2)}}{x}.$$

$$\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1} \sin x\,dx.$$

$$\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1} \cos x\,dx.$$

$$\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\,dx.$$

$$\int (\log x)^n\,dx = x(\log x)^n - n\int (\log x)^{n-1}\,dx.$$

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx.$$

$$\int x^a (\log x)^n\,dx = \frac{x^{a+1}(\log x)^n}{a+1} - \frac{n}{a+1}\int x^a (\log x)^{n-1}\,dx \quad (a \neq -1).$$

$$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2(n-1)}\int \frac{dx}{(1+x^2)^{n-1}}.$$

$$\int \sqrt{(a^2 - x^2)}\,dx = -\frac{1}{2}a^2 \quad \operatorname{arcos}\frac{x}{a} + \frac{1}{2}x\sqrt{(a^2 - x^2)}.$$

$$\int \sqrt{(x^2 - a^2)}\,dx = -\frac{1}{2}a^2 \ \operatorname{arcosh}\frac{x}{a} + \frac{1}{2}x\sqrt{(x^2 - a^2)}.$$

$$\int \sqrt{(x^2 + a^2)}\,dx = \frac{1}{2}a^2 \ \operatorname{arsinh}\frac{x}{a} + \frac{1}{2}x\sqrt{(x^2 + a^2)}.$$

$$\int \frac{dx}{x^2 + 2bx + c} = -\frac{1}{\sqrt{(b^2 - c)}} \ \operatorname{artanh}\frac{x + b}{\sqrt{(b^2 - c)}}$$

$$= -\frac{1}{2\sqrt{(b^2 - c)}} \ \log\left|\frac{\sqrt{(b^2 - c)} - x - b}{\sqrt{(b^2 - c)} + x + b}\right|,$$

if $c < b^2$, i.e. $x^2 + 2bx + c = 0$ has real roots.

$$\int \frac{dx}{x^2 + 2bx + c} = \frac{1}{\sqrt{(c - b^2)}} \ \operatorname{ar\,tan}\frac{x + b}{\sqrt{(c - b^2)}},$$

if $c > b^2$, i.e. $x^2 + 2bx + c = 0$ has imaginary roots.

$$\int e^{ax}\sin bx\,dx = \frac{1}{a^2 + b^2}e^{ax}(a\sin bx - b\cos bx).$$

$$\int e^{ax}\cos bx\,dx = \frac{1}{a^2 + b^2}e^{ax}(a\cos bx + b\sin bx).$$

$$\int \sin^n x\cos x\,dx = \frac{\sin^{n+1}x}{n + 1}.$$

Recurrence Relations 4.4.3

$$\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx.$$

$$\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx.$$

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx.$$

$$\int (\log x)^n \, dx = x (\log x)^n - n \int (\log x)^{n-1} \, dx.$$

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.$$

$$\int x^a (\log x)^n \, dx = \frac{x^{a+1} (\log x)^n}{a+1} - \frac{n}{a+1} \int x^a (\log x)^{n-1} \, dx \quad (a \neq -1).$$

$$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2(n-1)} \int \frac{dx}{(1+x^2)^{n-1}}.$$

## 4.3 Integration of Special Functions

4.3.1 Rational functions These are reduced to the following three fundamental types by resolution into partial fractions 4.5.1

$$\int \frac{dx}{(x-a)^n} = -\frac{1}{n-1} \, \frac{1}{(x-a)^{n-1}};$$

$$\int \frac{dx}{(x^2+2bx+c)^n} = \frac{1}{(c-b^2)^{n-\frac{1}{2}}} \int \frac{du}{(1+u^2)^n},$$

where $\qquad c - b^2 > 0, \quad u = (x+b)/\sqrt{(c-b^2)},$

the integral on the right hand side being evaluated by the last recurrence relation above;

$$\int \frac{x\,dx}{(x^2+2bx+c)^n} = -\frac{1}{2(n-1)}\,\frac{1}{(x^2+2bx+c)^{n-1}} - b\int \frac{dx}{(x^2+2bx+c)^n},$$

where the integral on the right hand side is of the preceding type.

In the sequel, *R* denotes a rational function.

### **4.3.2** $\int R(\sin x, \cos x)\,dx$ 4.6.3

Substitution: $t = \tan x/2$, so that

$$\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \frac{dx}{dt} = \frac{2}{1+t^2}.$$

However, if *R* is an even function or only involves tan *x*, the following substitution is more convenient:

$$u = \tan x,\ \sin^2 x = \frac{u^2}{1+u^2},\ \cos^2 x = \frac{1}{1+u^2},\ \frac{dx}{du} = \frac{1}{1+u^2}.$$

### **4.3.3** $\int R(\cosh x, \sinh x)\,dx$ 4.6.3

Substitution $t = \tanh x/2$, so that

$$\sinh x = \frac{2t}{1-t^2},\ \cosh x = \frac{1+t^2}{1-t^2},\ \frac{dx}{dt} = \frac{2}{1-t^2}.$$

**4.3.4** $\int R(e^{mx})\,dx.$

Substitution $t = e^{mx}$, d$x$/d$t$ = 1/$mt$.

**4.3.5** $\int R(x, \sqrt{(1-x^2)})\,dx$ **4.6.4**

Substitution:

$$t = \sqrt{\left(\frac{1-x}{1+x}\right)},\quad x = \frac{1-t^2}{1+t^2},\quad \sqrt{(1-x^2)} = \frac{2t}{1+t^2},\quad \frac{dx}{dt} = -\frac{4t}{(1+t^2)^2}.$$

**4.3.6** $\int R(x, \sqrt{(x^2-1)})\,dx$ **4.6.5**

Substitution:

$$t = \sqrt{\left(\frac{x-1}{x+1}\right)},\quad x = \frac{1+t^2}{1-t^2},\quad \sqrt{(x^2-1)} = \frac{2t}{1-t^2},\quad \frac{dx}{dt} = \frac{4t}{(1-t^2)^2}.$$

**4.3.7** $\int R(x, \sqrt{(1+x^2)})\,dx$ **4.6.6**

Substitution:

$$t = x + \sqrt{(x^2+1)},\ x = \frac{t^2-1}{2t},\ \sqrt{(x^2+1)} = \frac{1+t^2}{2t},\ \frac{dx}{dt} = \frac{t^2+1}{2t^2}.$$

**4.3.8** $\int R(x, \sqrt{(ax^2+2bx+c)})\,dx$ **4.6.7**

The substitution $\xi = \dfrac{ax+b}{\sqrt{|ac-b^2|}}$ reduces this integral to one of the preceding three types.

4.3.9 $$\int R(x, \sqrt{(ax+b)}, \sqrt{(cx+d)})\, dx$$ 4.6.8

Substitution: $\xi = \sqrt{(cx+d)}$ or $x = \dfrac{1}{c}(\xi^2 - d)$, $\dfrac{dx}{d\xi} = \dfrac{2\xi}{c}$.

4.3.10 $$\int R\left(x, \sqrt[n]{\left(\frac{ax+b}{cx+d}\right)}\right) dx.$$ 4.6.8

Substitution:

$$\xi = \sqrt[n]{\left(\frac{ax+b}{cx+d}\right)},\ x = -\frac{d\xi^n - b}{c\xi^n - a},\ \frac{dx}{d\xi} = \frac{ad-bc}{(c\xi^n - a)^2} n\xi^{n-1}.$$

## 5. Uniform Convergence and Interchange of Infinite operations

For the definition of uniform convergence go to 8.4.2.

A series, which is uniformly convergent in a closed interval and the terms of which are continuous functions, represents a continuous function in the interval 8.4.3.

If $|f(x)| \leq a_n$ and $\Sigma a_n$ converges, $\Sigma f_n(x)$ converges uniformly ( and absolutely). 8.4.2.

Interchange of summation and differentiation 8.4.5

Any convergent series of continuous functions may be differentiated term by term, provided the resulting series converges uniformly.

Interchange of summation and integration 8.4.4

Any uniformly convergent series of continuous functions may be integrated term by term. The resulting series also converges uniformly.

## 6. Special Links

Stirling's Formula

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}} = 1.$$

Wallis' Product 4.4.4 A7 9.4.7

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left( \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right).$$

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{(n!)^2\, 2^{2n}}{(2n)!\sqrt{n}}.$$

Infinite products A8.3

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$ 3.6.6

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1-p^{-s}}, \quad s > 1$$ A8.3

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$ 9.4.8

Definition of the Gamma function 4.8.4

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dx \qquad (x \geqq 1){:}\ \Gamma(x+1) = x\Gamma(x);$$

if $x$ is a positive integer $n$,

$$\Gamma(n) = (n-1)!$$

Order of magnitude of functions 3.9

$$\lim_{x \to \infty} \frac{e^{cx}}{x^a} = \infty, \text{ if } c > 0,$$ 3.9.2

$$\lim_{x \to \infty} \frac{\log x}{x^a} = 0, \text{ if } a > 0,$$ 3.9.2

$$\lim_{x \to 0} x^a \log x = 0, \text{ if } a > 0.$$ 3.9.5

## 7. Special Definite integrals

Orthogonality relations of the trigonometric functions 4.3

$$\int_{-\pi}^{+\pi} \sin mx \, \sin nx \, dx = \begin{cases} 0, & \text{if } m \neq n. \\ \pi, & \text{if } m = n, \quad n \neq 0. \end{cases}$$

$$\int_{-\pi}^{+\pi} \sin mx \, \cos nx \, dx = 0.$$

$$\int_{-\pi}^{+\pi} \cos mx \, \cos nx \, dx = \begin{cases} 0, & \text{if } m \neq n. \\ \pi, & \text{if } m = n, \quad n \neq 0. \end{cases}$$

$$\int_0^\infty e^{-x^2} \, dx = \frac{1}{2}\sqrt{\pi}$$ 10.2.5

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{1}{2}\pi$$ 4.8.5

## 8. Mean Value theorems

Mean value theorem of the differential calculus 2.3.8

$$\frac{f(x+h)-f(x)}{h} = f'(x+\theta h), \quad 0<\theta<1.$$

If $f(x) = f(x+h) = 0$, this yields Rolle's theorem 2.3.8: Between two zeros of the function lies always a zero of the derivative.

Generalized Mean Value theorem A2.2 A3.3.3

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)},$$

where $\xi$ is a value between $a$ and $b$.

Taylor's theorem 6.2

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x) + R_n,$$

with the remainder

$$R_n = \frac{1}{n!}\int_0^h (h-\tau)^n f^{(n+1)}(x+\tau)\,d\tau$$

$$= \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(x+\theta h)$$

$$= \frac{h^{n+1}}{n!}(1-\theta)^n f^{(n+1)}(x+\theta h) \qquad (0<\theta<1).$$

Mean value theorem of the integral calculus 2.7.1

$$\int_a^b f(x)\,dx = (b-a)\,f(\xi), \text{ where } a \leqq \xi \leqq b.$$

$$\int^b f(x)\,p(x)\,dx = f(\xi)\int_a^b p(x)\,dx, \text{ if } p(x) \geqq 0.$$

## 9. Expansion in Series: Taylor Series, Fourier Series

### 1. Power series

Definition 8.5

#### 9.1.1 Power series in general

Any power series

$$\sum_{n=0}^{\infty} a_n x^n$$

in one variable has a **radius of convergence** $\rho$ (which may be zero or infinite); the series converges when $|x| < \rho$; in fact, it converges **uniformly and absolutely** in every interval $|x| \leq \eta$, where $\eta < \rho$; when $|x| > \rho$; the series diverges 8.5.1

If the remainder in Taylor's theorem tends to zero as $n$ increases, we have the infinite power series 6.2.3

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots.$$

#### 9.1.2 Special Taylor series 6.1.1 6.3 8.6 A8.4

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \dots + (-1)^{n-1}\frac{x^n}{n} + \dots$$
for $-1 < x \leqq 1$.

$$\left.\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + \dots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + \dots
\end{aligned}\right\} \text{for all values of } x.$$

$$\tan x = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{(2\nu)!} x^{2\nu-1} \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2},$$

$$x \cot x = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{2^{2\nu} B_{2\nu}}{(2\nu)!} x^{2\nu} \quad \text{for } -\pi < x < \pi,$$

where the $B_n$ are **Bernoulli numbers** A8.4.

$$\left.\begin{aligned}\operatorname{arc}\sin x &= x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \dots \\ \operatorname{ar}\sinh x &= x - \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\frac{x^7}{7} + - \dots \\ \operatorname{arc}\tan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - + \dots\end{aligned}\right\} \begin{array}{c}\text{for}\\ -1 \leqq x \leqq 1.\end{array}$$

$$\operatorname{ar}\tanh x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \qquad \text{for } |x| < 1.$$

### 9.1.3 Binomial series

$$(1+x)^a$$
$$= 1 + ax + \frac{a(a-1)}{2!}x^2 + \ldots + \frac{a(a-1)(a-2)\ldots(a-n+1)}{n!}x^n + .$$

for $-1 < x < 1$,
if $a > -1$ for $x = 1$ also,
if $a \geqq 0$ for $x = -1$ also;

in particular,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + - \ldots,$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + - \ldots,$$

$$\sqrt{(1+x)} = 1 + \frac{1}{2}x - \frac{1}{2.4}x^2 + \frac{1.3}{2.4.6}x^3 - \frac{1.3.5}{2.4.6.8}x^4 + - \ldots,$$

$$\frac{1}{\sqrt{(1+x)}} = 1 - \frac{1}{2}x + \frac{1.3}{2.4}x^2 - \frac{1.3.5}{2.4.6}x^3 + \frac{1.3.5.7}{2.4.6.8}x^4 - + \ldots.$$

### 9.1.4 Elliptic integral

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2\phi)}}$$
$$= \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \ldots\right\}.$$

### 9.2 Fourier series

If the function $f(x)$ is sectionally smooth in the interval $-\pi < x < \pi$, i.e., if its first derivative is sectionally continuous, the Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu \cos \nu x + b_\nu \sin \nu x),$$

is absolutely convergent throughout the entire interval. If $f(x)$ has a finite number of jump discontinuities, while $f'(x)$ is elsewhere sectionally continuous, the series converges uniformly in every closed subinterval which contains no discontinuities of $f(x)$. At every point at which $f(x)$ is continuous, the series represents the value of the function $f(x)$, while at every point of discontinuity of $f(x)$ it represents the arithmetic mean of the right hand and left hand limits of $f(x)$ 9.5.

## 10. Maxima and Minima

The following rule holds only for maxima and minima in the **interior** of the region under consideration.

In order that $\xi$ may be an extreme value of the function $y = f(x)$, $f'(x)$ must vanish. When this condition is satisfied, there is a maximum or minimum, if the first non-vanishing derivative of $f(x)$ is of even order; if it is of odd order, there is neither a maximum nor a minimum. In the former case, there is a maximum or a minimum according to whether the sign of the first non-zero derivative is negative or positive 3.5.

## 11. Curves

In what follows, $\xi$, $\eta$ are current co-ordinates.

Equation of the curve:

$$(a)\ \ y = f(x), \quad (b)\ \ F(x, y) = 0, \quad (c)\ \ x = \phi(t), \quad y = \psi(t).$$

Equation of the tangent at the point $(x,y)$ 5.1.3

$$(a)\ \ \eta - y = (\xi - x)f'(x), \quad (b)\ \ (\xi - x)F_x + (\eta - y)F_y = 0,$$
$$(c)\ \ \{\xi - \phi(t)\}\psi'(t) - \{\eta - \psi(t)\}\phi'(t) = 0.$$

Equation of the normal at the point $(x,y)$ 5.1.3

$$(a)\ \ \xi - x + (\eta - y)f'(x) = 0,\ \ (b)\ \ (\xi - x)F_y - (\eta - y)F_x = 0,$$
$$(c)\ \ \{\xi - \phi(t)\}\phi'(t) + \{\eta - \psi(t)\}\psi'(t) = 0.$$

Curvature 5.2.6

$$(a)\ \ k = \frac{y''}{(1+y'^2)^{\frac{3}{2}}},\quad (b)\ \ k = -\frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{(F_x^2 + F_y^2)^{\frac{3}{2}}},$$
$$(c)\ \ k = \frac{\dot\phi\ddot\psi - \ddot\phi\dot\psi}{(\dot\phi^2 + \dot\psi^2)^{\frac{3}{2}}}.$$

Radius of curvature 5.2.6

$$\rho = \frac{1}{|k|}.$$

Evolute (locus of centre of curvature) 5.2.6 A5.1

$$(a)\ \ \xi = x - y'\frac{1+y'^2}{y''},\quad \eta = y + \frac{1+y'^2}{y''};$$

$$(b)\ \ \xi = x + F_x\frac{F_x^2 + F_y^2}{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2},$$

$$\eta = y + F_y\frac{F_x^2 + F_y^2}{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2};$$

$$(c)\ \ \xi = \phi - \dot\psi\frac{\dot\phi^2 + \dot\psi^2}{\dot\phi\ddot\psi - \ddot\phi\dot\psi},\quad \eta = \psi + \dot\phi\frac{\dot\phi^2 + \dot\psi^2}{\dot\phi\ddot\psi - \ddot\phi\dot\psi}.$$

Involute A5.1

$$\xi = x + (a - s)\dot{x}, \quad \eta = y + (a - s)\dot{y},$$

where *a* is an arbitrary constant and *s* is the length of arc measured from a given point.

Point of inflection 3.5.1 5.1.5

Necessary condition for a point of inflection is

$$(a)\ y'' = 0, \ (b)\ F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2 = 0,$$
$$(c)\ \dot{x}\ddot{y} - \ddot{x}\dot{y} = 0.$$

Angle between two curves 5.1.3

$$(b)\ \cos\omega = \frac{F_xG_x + F_yG_y}{\sqrt{(F_x^2 + F_y^2)}\sqrt{(G_x^2 + G_y^2)}},$$

$$(c)\ \cos\omega = \frac{\dot{x}\dot{x}_1 + \dot{y}\dot{y}_1}{\sqrt{(\dot{x}^2 + \dot{y}^2)}\sqrt{(\dot{x}_1^2 + \dot{y}_1^2)}}.$$

In particular, the curves are orthogonal, if

$$(b)\ F_xG_x + F_yG_y = 0, \quad (c)\ \dot{x}\dot{x}_1 + \dot{y}\dot{y}_1 = 0;$$

the curves touch, if

$$(b)\ F_xG_y - F_yG_x = 0, \quad (c)\ \dot{x}\dot{y}_1 - \dot{x}_1\dot{y} = 0.$$

Two curves $y = f(x)$, $y = g(x)$ have contact of order *n* at a point *x*, if 6.4

$$f(x) = g(x), \quad f'(x) = g'(x), \ \ldots, \quad f^{(n)}(x) = g^{(n)}(x),$$
$$f^{(n+1)}(x) \neq g^{(n+1)}(x).$$

## 12. Length of Arc, Area, Volume

Length of arc 5.2.5

Let a plane curve be given by the equations

$$(a)\ \ y=f(x),\qquad (b)\ \ F(x,y)=0,\qquad (c)\ \ x=\phi(t),\ y=\psi(t),$$
$$(d)\ \ \text{(polar co-ordinates)}\ r=r(\theta).$$

The length of arc is

$$(a)\ \ s=\int_{x_0}^{x_1}\sqrt{(1+y'^2)}\,dx,\qquad\qquad (c)\ \ s=\int_{t_0}^{t_1}\sqrt{(\dot{x}^2+\dot{y}^2)}\,dt,$$

$$(b)\ \ s=\int_{x_0}^{x_1}\frac{1}{F_y}\sqrt{(F_x{}^2+F_y{}^2)}\,dx,\qquad (d)\ \ s=\int_{\theta_0}^{\theta_1}\sqrt{(r^2+r'^2)}\,d\theta.$$

Area of plane surface

The area bounded by the curve

$$r=r(\theta)$$

and two radius vectors $\theta_0$, $\theta_1$, where $r$, $\theta$ are polar co-ordinates, is given by 5.2.4

$$\frac{1}{2}\int_{\theta_0}^{\theta_1} r^2\,d\theta\,.$$

The area, enclosed by the curve

$$y=f(x),$$

the two ordinates $x = x_0$, $x = x_1$ and the $x$-axis, is 2.1.2

$$\int_{x_0}^{x_1} y\,dx$$

## Volume

The volume lying over a region *R* and bounded above by the surface with the equation

$$z = f(x, y)$$

is given by 10.6

$$V = \int\int_R f(x, y)\, dx\, dy\ .$$

End

# Differential and Integral Calculus

**by R.Courant**

# Miscellaneous Exercises

# Chapter I

## Introduction

1. Prove that if $p$ and $q$ are integers the expansion of $p/q$ as a decimal either terminates or recurs from a certain point onward. Prove also that every terminating or recurring decimal represents a rational number.

2. Express 39 in the ternary scale (scale of 3).

3. How would the number one hundred and fifty-six be written if (*a*) the binary scale (scale of 2), (*b*) the scale of 4, were in common use?

4. Express the following numbers in the scale of 12: (*a*) 1076, (*b*) 10,000, (*c*) 20,736, (*d*) 1/6, (*e*) 1/64, (*f*) 1/5.

5. We can find $\sqrt{2}$ to one decimal place thus: $1^2 = 1 < 2$, $2^2 = 4 > 2$, therefore $1 < \sqrt{2} < 2$. Next, $1{\cdot}3^2 = 1{\cdot}69 < 2$, $1{\cdot}4^2 = 1{\cdot}96 < 2$, $1{\cdot}5^2 = 2{\cdot}25 > 2$, therefore $1{\cdot}4 < \sqrt{2} < 1{\cdot}5$.

(*a*) Continue this process one step further.

(*b*) Calculate $\sqrt{7}$ to two decimal places by the same method.

6. For what values of $x$ do the following inequalities hold?

(*a*) $x^3 + 3x + 1 \geqq 0$.

(*b*) $x^2 - x + 1 \geqq 0$.

(*c*) $\left| x + \frac{1}{x} \right| \geqq 6$.

(*d*) $3x - 2 \leqq x^3$.

7. Prove that the arithmetic mean $\frac{a+b}{2}$ of two positive quantities $a$, $b$ is not less than the geometric mean $\sqrt{ab}$, i.e. that

$$\frac{a+b}{2} \geqq \sqrt{ab}.$$

State when the equality sign holds.

8. The quantity $\xi$ defined by $\frac{1}{\xi} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$ is called the harmonic mean of the two positive quantities $a$, $b$. Prove that the geometric mean is not less than the harmonic mean, i.e. that $\sqrt{ab} \geqq \xi$.

When does the equality sign hold?

9.* Show that the following inequalities hold, if $a$, $b$, $c$ are positive:

(*a*) $a^2 + b^2 + c^2 \geqq ab + bc + ca$.

(*b*) $(a + b)(b + c)(c + a) \geqq 8abc$.

(*c*) $a^2b^2 + b^2c^2 + c^2a^2 \geqq abc\,(a + b + c)$.

10. The numbers $x_1$, $x_2$, $x_3$ and $a_{ik}$ ($i, k = 1, 2, 3$) are all positive. In addition, $a_{ik} \leq M$ and $x_1^2 + x_2^2 + x_3^2 \leq 1$. Prove that

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + \ldots + a_{33}x_3^2 \leq 3M.$$

11.* Prove that if the numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ satisfy the inequalities $a_1 \geq a_2 \geq \ldots \geq a_n$, $b_1 \geq b_2 \geq \ldots \geq b_n$, then

$$n\sum_1^n a_i b_i \geq \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right).$$

12. Prove the following properties of the binomial coefficients:

(a) $1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots \pm \binom{n}{n} = 0.$

(b) $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} = n\,2^{n-1}.$

(c) $1.2\binom{n}{2} + 2.3\binom{n}{3} + \ldots + (n-1)n\binom{n}{n} = n(n-1)2^{n-2}.$

(d) $1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}.$

(e) $\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$

13. By summing

$$\nu(\nu+1)(\nu+2)\ldots(\nu+k+1) - (\nu-1)\nu(\nu+1)\ldots(\nu+k)$$

from $\nu = 1$ to $\nu = n$, show that

$$\sum_{\nu=1}^n \nu(\nu+1)(\nu+2)\ldots(\nu+k) = \frac{n(n+1)\ldots(n+k+1)}{k+2}.$$

14. Evaluate $1^3 + 2^3 + \ldots + n^3$ by using the relation

$$\nu^3 = \nu(\nu+1)(\nu+2) - 3\nu(\nu+1) + \nu.$$

15. Evaluate

(a) $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots + \frac{1}{n(n+1)(n+2)};$

(b) $\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \ldots + \frac{1}{n(n+2)}.$

(c) $\frac{1}{1.2.4} + \frac{1}{2.3.5} + \ldots + \frac{1}{n(n+1)(n+3)}.$

16. Find a formula for the $n$-th term of the following arithmetic progressions:

(a) 1, 2, 4, 7, 11, 16, . . . .

(b) −7, −10, −9, 1, 25, 68, . . . .

17.* Show that the sum of the first $n$ terms of an arithmetic progression of order $k$ is

$$aS_k + bS_{k-1} + \dots + pS_1 + qn,$$

where $S_\nu$ represents the sum of the first $n$ $\nu$-th powers, and $a, b, \dots, p, q$ are independent of $n$. Evaluate the sums for the arithmetic progressions of Ex. 16.

18.* Prove the binomial theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + b^n$$

by mathematical induction. (See also A3.3.1)

19. Find

(a) $\lim\limits_{n\to\infty}\left(\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \dots + \frac{1}{n(n+1)}\right)$.

(b) $\lim\limits_{n\to\infty}\left(\frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \dots + \frac{1}{n(n+1)(n+2)}\right)$.

(c) $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt[4]{n}} + \frac{1}{\sqrt[4]{n+1}} + \dots + \frac{1}{\sqrt[4]{2n}}\right)$.

20. If $\sum\limits_{i=0}^{k} a_i = 0$, prove that $\lim\limits_{n\to\infty} \sum\limits_{i=0}^{k} a_i\sqrt{n+i} = 0$.

21. Prove that $\lim\limits_{n\to\infty} \dfrac{n^5}{2^n} = 0$.

22. Prove that $\lim\limits_{n\to\infty} \dfrac{(n+1)^5}{2^n} = 0$.

23. Prove that $\lim\limits_{n\to\infty} \sqrt[n]{n^2} = 0$.

24. Prove that $\lim\limits_{n\to\infty} \sqrt[(2n+1)]{(n^2+n)} = 0$.

25. Use Cauchy's convergence test to show that the following sequences converge:

(a) $a_n = \dfrac{1}{n}$.

(b) $a_n = \dfrac{n+1}{n}$.

(c)* $a_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dots + \dfrac{1}{n!}$.

(d)* $a_n = 1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \dots \pm \dfrac{1}{n!}$.

26.* Show that the limits of the sequences (c), (d) of the previous example are reciprocals of one another (so that the limit of the sequence (d) is $1/e$!).

27.* Prove that the limit of the sequence

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \ldots$$

(a) exists, (b) is equal to 2.

28.* Prove that the limit of the sequence

$$a_n = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n}$$

exists. Show that the limit is less than 1 but not less than $\frac{1}{2}$.

29. Prove that the limit of the sequence

$$a_n = \frac{1}{n+1} + \ldots + \frac{1}{2n}$$

exists, is equal to the limit of the previous example, and is greater than $\frac{1}{2}$ but not greater than 1.

30. Obtain the following bounds for the limit $L$ of the two previous examples: $\dfrac{37}{60} < L < \dfrac{57}{60}$.

31.* Let $a_1$, $b_1$ be any two positive numbers, and let $a_1 < b_1$. Let

$$a_2 = \frac{2a_1 b_1}{a_1 + b_1}, \quad b_2 = \sqrt{a_1 b_1},$$

and in general

$$a_n = \frac{2a_{n-1}\, b_{n-1}}{a_{n-1} + b_{n-1}}, \quad b_n = \sqrt{a_{n-1}\, b_{n-1}}.$$

Prove that the sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ converge and have the same limit.

32.* If $a_n > 0$, and $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$, then $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L$.

33. Use Ex. 32 to evaluate the limits of the following sequences:

$$(a)\ \sqrt[n]{n}, \quad (b)\ \sqrt[n]{(n^5 + n^4)}, \quad (c)\ \sqrt[n]{\left(\frac{n!}{n^n}\right)}.$$

34. Use Ex. 33(*c*) to show that

$$n! = n^n e^{-n} a_n,$$

where $a_n$ is a number whose $n$-th root tends to 1. A7

35. Prove that $\lim_{x \to 0} \frac{x+2}{x+1} = 2$. Find a $\delta$ such that for $|x| < \delta$ the difference between 2 and $\frac{x+2}{x+1}$ is, in absolute value, (*a*) less than $\frac{1}{10}$, (*b*) less than $\frac{1}{1000}$, (*c*) less than $\varepsilon$, $\varepsilon > 0$.

36. (*a*) Prove that $\lim\limits_{x\to 1}\dfrac{x+2}{x+1}=\dfrac{3}{2}$. Find a $\delta$ such that for $|1-x|<\delta$ the difference between $\dfrac{3}{2}$ and $\dfrac{x+2}{x+1}$ is, in absolute value, less than $\varepsilon$, $\varepsilon>0$.

Do the same for (*b*) $\lim\limits_{x\to 2}\sqrt{(1+x^3)}$; (*c*) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$.

37. Prove that (*a*) $\lim\limits_{x\to 0}\dfrac{\sqrt{(1+x)}-1}{x}=\dfrac{1}{2}$.

$$\text{(b)}\ \lim_{x\to\infty}\sqrt{x+\tfrac{1}{2}}\,(\sqrt{x+1}-\sqrt{x})=\tfrac{1}{2}.$$

38. Prove that $\lim\limits_{m\to\infty}(\cos\pi x)^{2m}$ exists for each value of $x$ and is equal to 1 or 0 according as $x$ is an integer or not.

39.* Prove that $\lim\limits_{n\to\infty}\,[\lim\limits_{m\to\infty}(\cos\pi n!\,x)^{2m}]$ exists for each value of $x$ and is equal to 1 or 0 according as $x$ is rational or irrational.

40. Determine which of the following functions are continuous. For those which are discontinuous, find the points of discontinuity.

$$\text{(a)}\ f(x)=\frac{x^5+5x^3+3x^2}{\sin x},\quad f(0)=0.$$

$$\text{(b)}\ f(x)=\frac{x^5+5x^3+3x}{\sin x},\quad f(0)=0.$$

$$\text{(c)}\ f(x)=\lim_{m\to\infty}(\cos\pi x)^{2m}.$$

$$\text{(d)}\ f(x)=\lim_{n\to\infty}\,[\lim_{m\to\infty}(\cos\pi n!\,x)^{2m}].$$

41. Let $f(x)$ be continuous for $0\leq x\leq 1$. Suppose further that $f(x)$ assumes rational values only, and that $f(x)=\frac{1}{2}$ when $x=\frac{1}{2}$. Prove that $f(x)=\frac{1}{2}$ everywhere.

42. Has the function

$$f(x)=2\sin 3x+10\cos 5x$$

any real zeros?

43.* If $f(x)$ satisfies the functional equation

$$f(x+y)=f(x)+f(y)$$

for all values of $x$ and $y$, find the values of $f(x)$ at the rational points and prove that, if $f(x)$ is continuous, $f(x)=cx$, where $c$ is a constant.

44.* Prove a converse of the theorem of uniform continuity; namely, that if $f(x)$ is uniformly continuous in the half-open interval $a<x\leq b$, then $f(x)$ tends to a unique limit as $x\to a$ (which may be taken as the value of $f(a)$).

45. Plot the following graphs, and express the equations in Cartesian co-ordinates:

(a) $r = a + b\cos\theta$ (Limaçon).

(b) $r = \dfrac{2}{2 - \cos\theta}$ (Ellipse).

(c) $r = \dfrac{2a\sin^2\theta}{\cos\theta}$ (Cissoid).

(d) $r = \dfrac{3a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$ (Folium of Descartes).

46.* Show that the equation of an ellipse with one focus at the origin is

$$r = \frac{k}{1 - e\cos(\theta - \theta_0)}.$$

47. Let $c$ be the complex number $x + iy$ represented by a point in a Cartesian co-ordinate system. Plot the curves

(a) $\left|\dfrac{c - i}{c + i}\right| = 2.$

(b)* $\left|\dfrac{c - \alpha}{c - \beta}\right| = k,$ $\alpha$, $\beta$ complex constants.

(c) $|c^2 - 1| = k.$

48. Let $c_1$, $c_2$ be two complex numbers. Prove that

(a) $|c_1 \pm c_2| \leqq |c_1| + |c_2|.$

(b) $|c_1 \pm c_2| \geqq |c_1| - |c_2|.$

49. Prove the equality

$$|c_1 + c_2|^2 + |c_1 - c_2|^2 = 2|c_1|^2 + 2|c_2|^2$$

and state its geometrical interpretation.

50. Prove that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ by mathematical induction.

Answers and Hints

Chapter II

51.* Prove directly that the derivative of the function

$$f(x) = x^2 \sin \frac{1}{x}, \ x \neq 0; \ f(0) = 0$$

exists at every point and is equal to

$$-\cos \frac{1}{x} + 2x \sin \frac{1}{x}, \ x \neq 0; \ 0 \text{ at } x = 0.$$

Show that although $f'(x)$ is not continuous at $x = 0$, nevertheless the mean value theorem still applies and the property of Ex. 57 below holds good. (A3.2)

52. Draw the graph of the function

$$f(x) = x \sin \frac{1}{x}, \quad x \neq 0; \quad f(0) = 0$$

and find its derivative for $x \neq 0$. Show that its derivative does not exist at $x = 0$, but that the difference quotient $\dfrac{f(x) - f(0)}{x}$ as $x \to 0$ has the upper and lower limits 1 and $-1$ respectively. (See A3.5)

53. Investigate the behaviour of the function

$$f(x) = x \sin \frac{1}{x} + x^2 \sin \frac{1}{x}, \quad x \neq 0; \quad f(0) = 0$$

with regard to differentiability.

54. Prove that the derivative of the function

$$f(x) = \frac{1}{x} \sin x, \quad x \neq 0; \quad f(0) = 1$$

exists at every point and is equal to

$$f'(x) = -\frac{1}{x^2} \sin x + \frac{1}{x} \cos x, \quad x \neq 0; \quad f'(0) = 0.$$

Show that $f'(x)$ is continuous, and find $f''(x)$.

55. If $f(x)$ is continuous and differentiable for $a \leq x \leq b$, show that if $f'(x) \leq 0$ for $a \leq x < \xi$ and $f'(x) \geq 0$ for $\xi < x \leq b$, the function is never less than $f(\xi)$.

56.* If the continuous function $f(x)$ has a derivative $f'(x)$ at each point $x$ in the neighbourhood of $x = \xi$, and if $f'(x)$ approaches a limit $L$ as $x \to \xi$, then $f'(\xi)$ exists and is equal to $L$.

57.* If $f(x)$ possesses a derivative $f'(x)$ (not necessarily continuous at each point $x$ of $a \leq x \leq b$, and if $f'(x)$ assumes the values $m$ and $M$) it also assumes every value $\mu$ between $m$ and $M$.

58. If $f''(x) \geq 0$ for all values of $x$ in $a \leq x \leq b$, the graph of $y = f(x)$ lies above the tangent line at any point $x = \xi$, $y = f(\xi)$ of the graph. (The curve is convex upwards.)

59. If $f''(x) \geq 0$ for all values of $x$ in $a \leq x \leq b$, the graph of $y = y\,f(x)$ in the interval $x_1 \leq x \leq x_2$ lies below the line segment joining the two points of the graph for which $x = x_1$, $x = x_2$.

60. If $f''(x) \geq 0$, then $f\left(\dfrac{x_1 + x_2}{2}\right) \leq \dfrac{f(x_1) + f(x_2)}{2}$.

61. Given $f(x) = \frac{1}{3}x^3 - x^2 + 1$, find a number $\delta$ such that for every $h$ less in absolute value than $\delta$ and every $x$ in the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$ the following inequality holds:

$$\left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{1}{100}.$$

62. Differentiate directly and write down the corresponding integration formulae: (*a*) $x^{1/2}$; (*b*) $\tan x$.

63. Evaluate

$$(a)\ \lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}\right).$$

$$(b)\ \lim_{n\to\infty} \frac{1}{n}\left(1+\sec^2\frac{\pi}{4n}+\sec^2\frac{2\pi}{4n}+\ldots+\sec^2\frac{n\pi}{4n}\right).$$

64. Prove that

$$(a)\ \int_{-1}^{1}(x^2-1)^2\,dx=\frac{16}{15};\quad (b)\ (-1)^n\int_{-1}^{1}(x^2-1)^n\,dx=\frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

65. Show that

$$\frac{1}{\nu+1}<\int_{\nu}^{\nu+1}\frac{dx}{x}<\frac{1}{\nu}$$

and

$$\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}<\int_{1}^{n}\frac{dx}{x}<1+\frac{1}{2}+\ldots+\frac{1}{n-1}.$$

Prove that the sequence $1+\frac{1}{2}+\ldots+\frac{1}{\nu}-\int_{1}^{\nu}\frac{dx}{x}$, $\nu=1,2,\ldots$, is a decreasing sequence and is bounded below.

66.* Let $f(x)$ be a function such that $f''(x)\geqq 0$ for all values of $x$, and let $u=u(t)$ be an arbitrary continuous function. Then

$$\frac{1}{a}\int_{0}^{a}f(u(t))\,dt\geqq f\left(\frac{1}{a}\int_{0}^{a}u(t)\,dt\right).$$

67.* If a particle traverses distance 1 in time 1, beginning and ending at rest, then at some point in the interval it must have been subjected to an acceleration $\geqq 4$.

# Answers and Hints

## Chapter III

### Differentiation and Integration of the Elementary Functions

**68.** Differentiate the following functions:

(a) $e^{\tan^2 x + \log \sin x}$.

(b) $(x + 2)^4(1 - x^3)^{1/3}(x^3 + 1)^{5/7}$.

(c) $\dfrac{x^3 \sin x - x^5 \cos x}{x^2 \tan x}$.

**69.** What conditions must the coefficients $\alpha$, $\beta$, $a$, $b$, $c$ satisfy in order that

$$\frac{\alpha x + \beta}{\sqrt{(ax^2 + 2bx + c)}}$$

shall everywhere have a finite derivative which is never zero?

70. Sketch the graph of the function

$$y = (x^2)^x, \quad y(0) = 1.$$

Show that the function is continuous at $x = 0$. Has the function maxima, minima, or points of inflection?

71. Among all triangles with given base and given perimeter, the isosceles triangle has the maximum area.

72. Among all triangles with given base and given vertical angle, the isosceles triangle has the maximum area.

73. Among all triangles with given base and given area, the isosceles triangle has the maximum vertical angle.

74.* Among all triangles with given area, the equilateral triangle has the least perimeter.

75.* Among all triangles with given perimeter, the equilateral triangle has the maximum area.

76.* Among all triangles inscribed in a circle, the equilateral triangle has the maximum area.

77. Prove the following inequalities:

(a) $e^x > \dfrac{1}{1+x}, \quad x > 0.$

(b) $e^x > 1 + \log(1+x), \quad x > 0.$

(c) $e^x > 1 + (1+x)\log(1+x), \quad x > 0.$

78.* Let $a$, $b$ be two positive numbers, $p$ and $q$ any non-zero numbers, $p < q$. Prove that

$$\frac{[\theta a^p + (1-\theta)b^p]^{1/p}}{[\theta a^q + (1-\theta)b^q]^{1/q}} \leq 1$$

for all values of $\theta$ in the interval $0 < \theta < 1$.

(This is Jensen's inequality, which states that the $p$-th power mean $[\theta a^p + (1-\theta)b^p]^{1/p}$ of two positive quantities $a$, $b$ is an increasing function of $p$.)

79. Show that the equality sign in the above inequality holds if, and only if, $a = b$.

80. Prove that $\lim\limits_{p\to 0} [\theta a^p + (1-\theta)b^p]^{1/p} = a^\theta b^{1-\theta}$.

81. Defining the zero-th power mean of $a$, $b$ as $a^\theta b^{1-\theta}$, show that Jensen's inequality applies to this case, and becomes $(a \neq b)$

$$a^\theta b^{1-\theta} \gtrless [\theta a^q + (1-\theta)b^q]^{1/q} \text{ according as } q \lessgtr 0.$$

For $q = 1$, $$a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b.$$

82. Prove the inequality

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b,$$

$a$, $b > 0$, $0 < \theta < 1$, without reference to Jensen's inequality, and show

that equality holds only if $a = b$. (This inequality states that the $\theta$, $1 - \theta$ geometric mean is less than the corresponding arithmetic mean.)

83. If $\varphi(x) \to \infty$ as $x \to \infty$, show that $\log \varphi(x)$ is of a lower order and $e^{\phi(x)}$ of a higher order of magnitude than $\varphi(x)$.

84. If the order of magnitude of the positive function $f(x)$ as $x \to \infty$ is higher, the same, or lower than that of $x^m$, prove that $\int_a^x f(\xi)\,d\xi$ has the corresponding order of magnitude relative to $x^{m+1}$.

85. Compare the order of magnitude as $x \to \infty$ of $\int_a^x f(\xi)\,d\xi$ relative to $f(x)$ for the following functions $f(x)$:

(a) $\dfrac{e^{\sqrt{x}}}{\sqrt{x}}$.

(b) $e^x$.

(c) $xe^{x^2}$.

(d) $\log x$.

86. Prove that if $f(x)$ is continuous and

$$f(x) = \int_0^x f(t)\,dt,$$

then $f(x)$ is identically zero.

87. Prove that $\sum_{i=1}^{n-1} ix^{i-1} = \dfrac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}$.

88. Show that $\dfrac{d^n(e^{x^2/2})}{dx^n} = u_n(x)e^{x^2/2}$,

where $u_n(x)$ is a polynomial of degree $n$. Establish the recurrence relation

$$u_{n+1} = xu_n + u_n'.$$

89.* By applying Leibnitz's rule to

$$\frac{d}{dx}(e^{x^2/2}) = xe^{x^2/2},$$

obtain the recurrence relation

$$u_{n+1} = xu_n + nu_{n-1}.$$

90.* By combining the recurrence relations of Ex. 88, 89, obtain the differential equation

$$u_n'' + xu_n' - nu_n = 0$$

satisfied by $u_n(x)$.

91. Find the polynomial solution

$$u_n(x) = x^n + a_1x^{n-1} + \ldots + a_n$$

of the differential equation $u_n'' + xu_n' - nu_n = 0$.

92.* If $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$, prove the relations

(a) $P_{n+1}' = \frac{x^2 - 1}{2(n+1)} P_n'' + \frac{(n+2)x}{x+1} P_n' + \frac{n+2}{2} P_n.$

(b) $P_{n+1}' = xP_n' + (n+1)P_n.$

(c) $\frac{d}{dx}((x^2 - 1)P_n') - n(n+1)P_n = 0.$

93. Find the polynomial solution

$$P_n = \frac{(2n)!}{2^n (n!)^2} x^n + a_1 x^{n-1} + \dots + a_n$$

the differential equation

$$\frac{d}{dx}((x^2 - 1)P_n') - n(n+1)P_n = 0.$$

94. Determine the polynomial $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$ by using e binomial theorem.

95.* Let $\lambda_{n,p}(x) = \binom{p}{n} x^n (1 - x)^{p-n}$, $n = 0, 1, 2, \dots, p$. Show at

$$1 = \sum_{n=0}^{p} \lambda_{n,p}(x).$$

$$x = \sum_{n=1}^{p} \frac{n}{p} \lambda_{n,p}(x).$$

$$\dots\dots\dots$$

$$x^k = \sum_{n=k}^{p} \frac{\binom{n}{k}}{\binom{p}{k}} \lambda_{n,p}(x).$$

$$\dots\dots\dots$$

$$x^p = \lambda_{p,p}(x).$$

# Answers and Hints

## Chapter IV

### Further development of the Differential Calculus

Perform the integrations in Ex. 96–101.

96. $\displaystyle\int \frac{1+\sqrt[3]{x}}{1+\sqrt[4]{x}}\,dx.$

97. $\displaystyle\int \frac{e^{2x}}{\sqrt[4]{(e^x+1)}}\,dx.$

98. $\displaystyle\int \frac{x\,dx}{\sqrt[3]{(1+x)}-\sqrt{(1+x)}}.$

99. $\displaystyle\int \frac{x^2-1}{x^4+x^2+1}\,dx.$

100. $\displaystyle\int \frac{dx}{x\sqrt{(x^{2n}-1)}}.$

101. $\displaystyle\int \frac{dx}{x(x+1)\ldots(x+n)}.$

Evaluate the integrals in Ex. 102–107.

102. $\displaystyle\int_0^{\pi/2} \cos^n x\,dx.$

103. $\displaystyle\int_0^{\pi/6} \cos^7 3\theta \sin^4 6\theta\,d\theta.$

104. $\int_0^1 \frac{x^{2n}\,dx}{\sqrt{(1-x^2)}}$.

105. $\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{(1-x^2)}}$.

106. $\int_0^1 x^2\sqrt{(1-x^2)}\,dx$.

107. $\int_0^1 x^2(1-x^2)^{3/2}\,dx$.

Obtain recurrence formulæ for the integrals of Ex. 108–112.

108. $\int x^a(\log x)^m\,dx$.

109. $\int x^n e^{ax}\sin bx\,dx$.

110. $\int x^n e^{ax}\cos bx\,dx$.

111. $\int e^{ax}\sinh bx\,dx$.

112. $\int e^{ax}\cosh bx\,dx$.

113. Integrate $\int \frac{dx}{\sqrt{(a^2-x^2)}}$ in three different ways and compare the results.

114.* Let $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$. Show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0, \text{ if } m \neq n.$$

115. Prove that $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{n+1}$.

116. Prove that $\int_{1}^{-1} x^m P_n(x)\,dx = 0$, if $m < n$.

117. Evaluate $\int_{-1}^{1} x^n P_n(x)\,dx$.

Test whether the improper integrals in Ex. 118–131 converge or diverge.

118. $\int_0^a \frac{dx}{\sqrt{(ax-x^2)}}$.

119. $\int_1^\infty \frac{dx}{x\sqrt{(x^2-1)}}$.

120. $\int_0^1 \left(\log\frac{1}{x}\right)^n dx$.

121. $\int_0^1 x^m\left(\log\frac{1}{x}\right)^n dx$.

122. $\int_0^\infty e^{-x}x^m(\log x)^n\,dx$.

123. $\int_0^\pi \log\sin x\,dx$.

124. $\int_0^\pi \frac{1}{x}\log\sin x\,dx$.

125. $\int_0^\pi x\log\sin x\,dx$.

126. $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

127. $\int_0^\infty x^{2n-1}e^{-x^2}\,dx$.

128. $\int_0^{\pi/2} \frac{x^m\,dx}{(\sin x)^n}$.

129. $\int_0^\infty \frac{dx}{1+x^4\sin^2 x}$.

130. $\int_0^\infty \frac{x\,dx}{1+x^2\sin^2 x}$.

131.* $\int_0^\infty \frac{x^a\,dx}{1+x^\beta\sin^2 x}$.

**132.*** If $\int_a^\infty \frac{f(x)}{x}\,dx$ converges for any positive value of $a$, and if $f(x)$ tends to a limit $L$ as $x \to 0$, show that $\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx$ converges and has the value $L \log \frac{\beta}{\alpha}$.

**133.** By reference to the previous example, show that

$$(a)\quad \int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\,dx = \log \frac{\beta}{\alpha}.$$

$$(b)\quad \int_0^\infty \frac{\cos \alpha x - \cos \beta x}{x}\,dx = \log \frac{\beta}{\alpha}.$$

**134.*** If $\int_a^b \frac{f(x)}{x}\,dx$ converges for any positive values of $a$ and $b$, and if $f(x)$ tends to a limit $M$ as $x \to \infty$ and a limit $L$ as $x \to 0$, show that

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx = (L - M) \log \frac{\beta}{\alpha}.$$

**135.** Obtain the following expressions for the gamma function:

$$\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2}\,dx,$$

$$\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx.$$

## Answers and Hints

### Chapter V

Applications

136. Plot the following curves and find their equations in non-parametric form:

(a) $x = \dfrac{5at^2}{1+t^5}, \quad y = \dfrac{5at^3}{1+t^5}.$

(b) $x = at + b\sin t, \quad y = a - b\cos t.$

137.* Show that the two families of ellipses and hyperbolas,

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} = 1, \quad \text{for } \lambda < b,$$

$$\frac{x^2}{a^2-\tau} + \frac{y^2}{b^2-\tau} = 1, \quad \text{for } a < \tau < b,$$

are confocal and intersect at right angles.

138. Find the pedal curves (see p. 267, Ex. 11) of the following:

(a) the ellipse $x = a\cos\theta$, $y = b\sin\theta$ with respect to the origin;
(b) the hyperbola $x = \cosh\theta$, $y = b\sinh\theta$ with respect to the origin;
(c) the parabola $y^2 = 4px$ with respect to the origin;
(d) the parabola $y^2 = 4px$ with respect to the focus.

139. Show that the tangent to an ellipse is equally inclined to the focal radii drawn to the point of contact.

140. Show that the tangent to a hyperbola is equally inclined to the focal radii drawn to the point of contact.

141. A constant length $l$ is measured off along the normal to a parabola. Find the curve described by the extremity of this segment.

142. Find the area bounded by the loop of the curve

$$x^5 + y^5 - 5ax^2y^2 = 0.$$

143. Find the area enclosed by the curve

$$a^2(x^2 + y^2)^2(b^2x^2 + a^2y^2) = (a^2 - b^2)^2b^2x^4.$$

144. Find the length of arc of the epicycloid

$$x = (a + b)\cos t - b\cos\frac{a+b}{b}t$$

$$y = (a + b)\sin t - b\sin\frac{a+b}{b}t$$

reckoned from the initial point $t = 0$.

145. Prove that the radius of curvature at a point of the polar curve $r = f(\theta)$ is

$$\frac{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 - r\dfrac{d^2r}{d\theta^2} + 2\left(\dfrac{dr}{d\theta}\right)^2}.$$

146.* If the curvature of a curve in the $xy$-plane is a monotonic function of the length of arc, prove that the curve is not closed and that it has no double points.

147. Find the moment of inertia of a rod of length $L$

(a) with respect to its centre;
(b) with respect to one end;
(c) with respect to a point on the line of the rod at a distance $d$ from the centre;
(d) with respect to any point at a distance $d$ from the centre.

148. Find the equation of the curves which everywhere intersect the straight lines through the origin at the same angle $\alpha$.

149. Find the equation of the curves whose normal is of constant length $k$. (The "length" of the normal is the length of the portion of the normal intercepted between the curve and the $x$-axis.)

150. Show that the only curves whose curvature is a fixed constant $k$ are circles of radius $1/k$.

151. Find the equation of the curves whose centre of curvature lies on the $x$-axis and whose radius of curvature is therefore equal to the length of the normal.

152. Find the equation of the curves whose radius of curvature is equal to the length of the normal but whose centre of curvature does not lie on the $x$-axis.

153.* Obtain the formula for the length of a curve in polar co-ordinates.

# Answers and Hints

## Chapter VI

### Taylor's Theorem and the Approximate Expressions of Functions by Polynomials

154. Deduce the integral formula for the remainder $R_n$ by applying integration by parts to

$$f(x+h)-f(x)=\int_0^h f'(x+\tau)\,d\tau.$$

155. Integrate the formula

$$R_n=\frac{1}{n!}\int_0^h (h-\tau)^n f^{(n+1)}(x+\tau)\,d\tau,$$

and so obtain

$$R_n=f(x+h)-f(x)-hf'(x)-\ldots-\frac{h^n}{n!}f^{(n)}(x).$$

156.* Suppose that in some way a series for the function $f(x)$ has been obtained, namely

$$f(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n+R_n(x),$$

where $a_0, a_1, \ldots, a_n$ are constants, $R_n(x)$ is $n$ times continuously differentiable, and $\frac{R_n(x)}{x^n}\to 0$ as $x\to 0$. Show that $a_k=\frac{f^k(0)}{k!}$ $(k=0,\ldots,n)$, i.e. that the series is a Taylor series.

157.* Find the first three non-vanishing terms of the Taylor series for $\sin^2 x$ in the neighbourhood of $x=0$ by multiplying the Taylor series for $\sin x$ by itself. Justify this procedure.

158.* Find the first three non-vanishing terms of the Taylor series for $\tan x$ in the neighbourhood of $x=0$, by using the relation $\tan x=\frac{\sin x}{\cos x}$, and justify the procedure.

159.* Find the first three non-vanishing terms of the Taylor series for $\sqrt{\cos x}$ in the neighbourhood of $x=0$, by applying the binomial theorem to the Taylor series for $\cos x$, and justify the procedure.

160. Find the first four non-vanishing terms of the Taylor series for the following functions in the neighbourhood of $x=0$:

(a) $x\cot x$. (c) $\sec x$. (e) $e^{e^x}$.

(b) $\frac{\sqrt{\sin x}}{\sqrt{x}}$. (d) $e^{\sin x}$. (f) $\log\sin x-\log x$.

161. Find the Taylor series for arc $\sin x$ in the neighbourhood of $x = 0$ by using

$$\text{arc} \sin x = \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

(Cf. p. 203, Ex. 5.)

162.* Find the Taylor series for (arc $\sin x)^2$. (Cf. p. 203, Ex. 5.)

163. Find the Taylor series for the following functions in the neighbourhood of $x = 0$:

$$(a)\ \sinh^{-1} x. \qquad (b)\ \int_0^x e^{-t^2}\,dt. \qquad (c)\ \int_0^x \frac{\sin t}{t}\,dt.$$

164.* Estimate the error involved in using the first $n$ terms in the series in Ex. 163.

165.* Two oppositely charged particles $+e$, $-e$ situated at a small distance $d$ apart form an electric dipole with moment $M = ed$. Show that the potential energy (a) at a point situated on the axis of the dipole at a distance $r$ from the centre of the dipole is $\frac{M}{r^2}(1 + \varepsilon)$, where $\varepsilon$ is approximately equal to $\frac{d^2}{4r^2}$;

(b) at a point situated on the perpendicular bisector of the dipole is 0;

(c) at a point with polar co-ordinates $r$, $\theta$ relative to the centre and axis of the dipole is $\frac{M \cos\theta}{r^2}(1 + \varepsilon)$, where $\varepsilon$ is approximately equal to $\frac{d^2}{8r^2}(5\cos^2\theta - 3)$.

(The potential energy of a single charge $q$ at a point at a distance $r$ from the charge is $q/r$; the potential energy of several charges is the sum of the potential energies of the separate charges.)

166.* Find the first three terms of the Taylor series for $\left(1 + \frac{1}{x}\right)^x$ in powers of $\frac{1}{x}$.

167. Evaluate the following limits:

$$(a)\ \lim_{x\to\infty} x\left[\left(1+\frac{1}{x}\right)^x - e\right].$$

$$(b)\ \lim_{x\to\infty} \frac{e}{2}x + x^2\left[\left(1+\frac{1}{x}\right)^x - e\right].$$

$$(c)^*\ \lim_{x\to\infty} x\left[\left(1+\frac{1}{x}\right)^x - e\log\left(1+\frac{1}{x}\right)^x\right].$$

$$(d)\ \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}. \qquad (e)\ \lim_{x\to\infty}\left(\frac{\sin x}{x}\right)^{1/x^2}.$$

168.* Show that the osculating circle at a point where the radius of curvature is a maximum or minimum does not cross the curve.

169. Find the maxima and minima of the following functions: (a) $|x|$, (b) $x \sin(1/x)$.

# Answers and Hints

## Chapter VII

## Numerical Methods

**170.** Show that the length of the ellipse $x = a\cos t$, $y = b\sin t$ is

$$s = 4a\int_0^{\pi/2}\sqrt{1 - e^2\cos^2 t}\,dt, \quad \text{where } e^2 = \frac{a^2 - b^2}{a^2}.$$

Calculate the length of the ellipse for which $e = \frac{1}{2}$ to four significant figures, by using Simpson's rule with six divisions.

**171.** Expand the integral of Ex. 170 as a series, and estimate the number of terms necessary for accuracy to four significant figures.

**172.** Evaluate $\displaystyle\int_0^1 \frac{\log(1+x)}{x}\,dx$, using Simpson's rule with $h = 0{\cdot}1$.

**173.** The hypotenuse of a right-angled triangle is measured accurately as 40, and one angle is measured as 30° with a possible error of $\frac{1}{2}$°. Find the possible error in the lengths of each of the sides and in the area of the triangle.

**174.*** By considering $\displaystyle\int_{1/2}^{n+1/2}\log(\alpha + x)\,dx$, $\alpha > 0$, show that

$$\alpha(\alpha+1)\ldots(\alpha+n) = a_n n!\, n^\alpha,$$

where $a_n$ is bounded below by a positive number. Show that $a_n$ is monotonically decreasing for sufficiently large values of $n$. (The limit of $a_n$ as $n \to \infty$ is $1/\Gamma(\alpha)$.)

**175.** Find an approximate expression for $\log \dfrac{n_1!\, n_2! \ldots n_l!}{n!}$, where $n_1 + n_2 + \ldots + n_l = n$.

**176.** Show that the coefficient of $x^n$ in the binomial expansion of $\dfrac{1}{\sqrt{1-x}}$ is asymptotically given by $\dfrac{1}{\sqrt{\pi n}}$.

# Answers and Hints

## Chapter VIII

177. Prove that if $\sum_{\nu=1}^{\infty} a_\nu^2$ converges, so does $\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu}$.

178. If $a_n$ is a monotonic increasing sequence with positive terms, when does the series $\frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \ldots$ converge?

179.* If the series $\sum_{\nu=1}^{\infty} a_\nu$ with decreasing positive terms converges, then $\lim_{n \to \infty} n a_n = 0$.

180. Show that the series $\sum_{\nu=1}^{\infty} \sin \frac{\pi}{\nu}$ diverges.

181.* Prove that if $\Sigma a_\nu$ converges and if $b_1, b_2, b_3, \ldots$ is a bounded monotonic sequence of numbers, then $\Sigma a_\nu b_\nu$ converges.

182.* Prove that if $\Sigma a_\nu$ oscillates between finite bounds and if $b_\nu$ is a monotonic sequence tending towards zero, then $\Sigma a_\nu b_\nu$ converges.

183. Discuss the convergence or divergence of the following series:

(a) $\Sigma \dfrac{(-1)^\nu}{\nu}$. (b) $\Sigma \dfrac{(-1)^\nu \cos(\theta/\nu)}{\nu}$. (c) $\Sigma \dfrac{\cos \nu\theta}{\nu}$.

(d) $\Sigma \dfrac{\sin \nu\theta}{\nu}$. (e) $\Sigma \dfrac{(-1)^\nu \cos \nu\theta}{\nu}$. (f) $\Sigma \dfrac{(-1)^\nu \sin \nu\theta}{\nu}$.

184. Find the sums of the following derangements of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$ for $\log 2$:

(a) $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + - - \ldots$.

(b) $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + + + \ldots$.

185. For what values of $\alpha$ do the following series converge?

(a) $1 - \dfrac{1}{2^\alpha} + \dfrac{1}{3} - \dfrac{1}{4^\alpha} + \dfrac{1}{5} - \dfrac{1}{6^\alpha} + - \ldots$.

(b) $1 + \dfrac{1}{3^\alpha} - \dfrac{1}{2^\alpha} + \dfrac{1}{5^\alpha} + \dfrac{1}{7^\alpha} - \dfrac{1}{4^\alpha} + + - \ldots$.

186. Find whether the following series converge or diverge:

(a) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + + - \ldots$.

(b) $1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + + - \ldots$.

187. Show that

(a) $\sum\limits_{\nu=1}^{\infty} \dfrac{\nu!}{(2\nu)!}$ converges.

(b) $\sum\limits_{\nu=2}^{\infty} \dfrac{\log(\nu+1) - \log\nu}{(\log\nu)^2}$ converges.

(c) $\sum\limits_{\nu=1}^{\infty} \dfrac{1 . 2 . 3 \ldots \nu}{(\alpha+1)(\alpha+2)\ldots(\alpha+\nu)}$ converges if $\alpha > 1$ and diverges if $\alpha \leqq 1$.

188.* By comparison with the series $\sum\limits_{\nu=1}^{\infty} \dfrac{1}{\nu^\alpha}$, prove the following test:

If $\dfrac{\log(1/|a_n|)}{\log n} > 1 + \varepsilon$ for every sufficiently large $n$, the series $\Sigma a_\nu$ converges absolutely; if $\dfrac{\log(1/|a_n|)}{\log n} < 1 - \varepsilon$ for every sufficiently large $n$, the series $\Sigma a_\nu$ does not converge absolutely.

189. Show that the series $\sum\limits_{\nu=1}^{\infty} \left(1 - \dfrac{1}{\sqrt{\nu}}\right)^\nu$ converges.

190. By comparison with the series $\Sigma \frac{1}{\nu(\log \nu)^\alpha}$, prove the following test:

The series $\Sigma \mid a_\nu \mid$ converges or diverges according as

$$\frac{\log(1/n \mid a_n \mid)}{\log \log n}$$

is greater than $1 + \varepsilon$ or less than $1 - \varepsilon$ for every sufficiently large $n$.

191. Derive the $n$-th root test from the test of Ex. 188.

192.* Prove the following comparison test: if the series $\Sigma b_\nu$ of positive terms converges, and

$$\left| \frac{a_{n+1}}{a_n} \right| < \frac{b_{n+1}}{b_n}$$

from a certain term onwards, the series $\Sigma a_\nu$ is absolutely convergent; if $\Sigma b_\nu$ diverges and

$$\left| \frac{a_{n+1}}{a_n} \right| > \frac{b_{n+1}}{b_n}$$

from a certain term onwards, the series $\Sigma a_\nu$ is not absolutely convergent.

193. Obtain the ratio test by comparison with the geometric series.

194.* By comparison with $\sum\limits_{\nu=1}^{\infty} \frac{1}{\nu^\alpha}$, prove Raabe's test:

The series $\Sigma \mid a_\nu \mid$ converges or diverges according as

$$n\left(\frac{\mid a_n \mid}{\mid a_{n+1} \mid} - 1\right)$$

is greater than $1 + \varepsilon$ or less than $1 - \varepsilon$ for every sufficiently large $n$.

195. By comparison with $\Sigma \frac{1}{\nu(\log \nu)^\alpha}$, prove the following test:

The series $\Sigma \mid a_\nu \mid$ converges or diverges according as

$$n \log n\left(\frac{\mid a_n \mid}{\mid a_{n+1} \mid} - 1 - \frac{1}{n}\right)$$

is greater than $1 + \varepsilon$ or less than $1 - \varepsilon$ for every sufficiently large $n$.

196. Prove Gauss's test:

If
$$\frac{\mid a_n \mid}{\mid a_{n+1} \mid} = 1 + \frac{\mu}{n} + \frac{R_n}{n^{1+\varepsilon}},$$

where $\mid R_n \mid$ is bounded, then $\Sigma \mid a_\nu \mid$ converges if $\mu > 1$, diverges if $\mu \leqq 1$.

197. Test the following series for convergence or divergence:

(a) $\frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \dots$ .

(b) $1 + \frac{\alpha \, . \, \beta}{1 \, . \, \gamma} + \frac{\alpha(\alpha+1) \, . \, \beta(\beta+1)}{1 \, . \, 2 \, . \, \gamma(\gamma+1)} + \frac{\alpha(\alpha+1)(\alpha+2) \, . \, \beta(\beta+1)(\beta+2)}{1 \, . \, 2 \, . \, 3 \, . \, \gamma(\gamma+1)(\gamma+2)} + \dots$ .

198. (a) Show that the series $\sum_{\nu=1}^{\infty} \frac{1}{\nu^x}$ converges uniformly for $x \geqq 1 + \varepsilon$.

(b) Show that the derived series $-\sum \frac{\log \nu}{\nu^x}$ converges uniformly for $x \geqq 1 + \varepsilon$.

199.* Show that the series $\sum \frac{\cos \nu x}{\nu^a}$, $\alpha > 0$, converges uniformly for $\varepsilon \leqq x \leqq 2\pi - \varepsilon$.

200. The series

$$\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots$$

converges uniformly for $\varepsilon \leqq x \leqq N$.

201. Find the regions in which the following series are convergent:

(a) $\sum x^{\nu!}$.

(b) $\sum \frac{(\nu!)^2 x^\nu}{(2\nu)!}$.

(c) $\sum \frac{1}{\nu^x}$.

(d) $\sum \frac{(-1)^\nu}{\nu^x}$.

(e) $\sum \frac{a^\nu}{\nu^x}$, $a < 1$.

(f) $\sum \frac{a^\nu}{\nu^x}$, $a > 1$.

(g) $\sum \frac{\log \nu}{\nu^x}$.

(h) $\sum \frac{x^\nu}{1 - x^\nu}$.

202.* Prove that if the series $\sum \frac{a^\nu}{\nu^x}$ converges for $x = x_0$, it converges for any $x > x_0$; if it diverges for $x = x_0$, it diverges for any $x < x_0$. Thus, there is an "abscissa of convergence" such that for any greater value of $x$ the series converges, and for any smaller value of $x$ the series diverges.

203. If $\sum \frac{a_\nu}{\nu^x}$ converges for $x = x_0$, the derived series $-\sum \frac{a_\nu \log \nu}{\nu^x}$ converges for any $x > x_0$.

204. If $a_\nu > 0$ and $\sum a_\nu$ converges, then

$$\lim_{x \to 1-0} \sum a_\nu x^\nu = \sum a_\nu.$$

205. If $a_\nu > 0$ and $\sum a_\nu$ diverges,

$$\lim_{x \to 1-0} \sum a_\nu x^\nu = \infty.$$

206.* Prove Abel's theorem:

If $\sum a_\nu X^\nu$ converges, then $\sum a_\nu x^\nu$ converges uniformly for $0 \leqq x \leqq X$.

207.* If $\sum a_\nu X^\nu$ converges, then $\lim_{x \to X-0} \sum a_\nu x^\nu = \sum a_\nu X^\nu$.

208. Find the rational functions represented by the following Taylor series:

(a) $x + x^2 - x^3 - x^4 + x^5 + x^6 - - + + \cdots$.

(b) $1 + 2x - 4x^3 - 5x^4 + 7x^6 + 8x^7 - - + + \cdots$.

209. Show that

$$(a)\ \frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots=1.$$

$$(b)\ \frac{1}{2}+\frac{1.3}{2.4.6}+\frac{1.3.5.7}{2.4.6.8.10}+\ldots=\frac{1}{2}\sqrt{2}.$$

210. Let $z=re^{i\theta}=r(\cos\theta+i\sin\theta)$. From the expansion $\frac{1}{1-z}=\sum_{\nu=0}^{\infty}z^{\nu}$, show that

$$\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}=\sum_{\nu=0}^{\infty}r^{\nu}\cos\nu\theta$$

and

$$\frac{r\sin\theta}{1-2r\cos\theta+r^2}=\sum_{\nu=0}^{\infty}r^{\nu}\sin\nu\theta.$$

Answers and Hints

Chapter IX

Fourier Series

**211.*** Using the expression for the cotangent in partial fractions, expand $\pi x \cot \pi x$ as a power series in $x$. By comparing this with the series given on p. 423, show that

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^{2m}} = (-1)^{m-1} \frac{(2\pi)^{2m}}{2\,.\,(2m)!} B_{2m}.$$

**212.** Show that

$$\sum_{\nu=1}^{\infty} \frac{1}{(2\nu-1)^{2m}} = \frac{(-1)^{m-1}(2^{2m}-1)\pi^{2m}}{2(2m)!} B_{2m}.$$

**213.** Show that

$$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu}}{\nu^{2m}} = \frac{(-1)^{m}(2^{2m}-2)\pi^{2m}}{2\,.\,(2m)!} B_{2m}.$$

**214.** Prove that

$$(a)\quad \int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6};$$

$$(b)\quad \int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}.$$

**215.** Using the infinite products for the sine and cosine, show that

$$(a)\quad \log\left(\frac{\sin x}{x}\right) = -\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}2^{2\nu-1}B_{2\nu}}{(2\nu)!\,\nu} x^{2\nu};$$

$$(b)\quad \log \cos x = -\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}2^{2\nu-1}(2^{2\nu}-1)B_{2\nu}}{(2\nu)!\,\nu} x^{2\nu}.$$

**216.** Using the infinite products for the sine and cosine, evaluate

$$(a)\quad \tfrac{2}{1}\cdot\tfrac{2}{3}\cdot\tfrac{6}{5}\cdot\tfrac{6}{7}\cdot\tfrac{10}{9}\cdot\tfrac{10}{11}\cdot\tfrac{14}{13}\cdots;$$

$$(b)\quad 2\,.\,\tfrac{2}{3}\cdot\tfrac{4}{3}\cdot\tfrac{8}{9}\cdot\tfrac{10}{9}\cdot\tfrac{14}{15}\cdot\tfrac{16}{15}.$$

**217.** Express the hyperbolic cotangent in terms of partial fractions.

Answers and Hints

# Chapter XI

## The Differential Equations for the Simplest Types of Vibrations

218. Find the curves whose tangent is of constant length $a$. (The "length" of the tangent is the length of the portion of the tangent intercepted between the curve and the $x$-axis.)

219. Find the curves which are orthogonal to the family $y = ce^{kx}$.

220. If $s$ denotes the length of arc of a chain measured from a point at which the tangent is horizontal, the form of the chain is determined by the differential equation

$$\frac{d}{dx}(\log s) = \frac{d}{dx}\left(\log \frac{dy}{dx}\right).$$

Show that the equation of the chain is $y = c \cosh \frac{x}{c} + a$.

221. Integrate the equation for the electric circuit

$$\mu \dot{I} + \rho I = E,$$

where $E = E_0 \sin \omega t$, and $\mu$, $\rho$, $E_0$, $\omega$ are constants.

222. A particle falls towards a point which attracts inversely as the cube of the distance and directly as the mass. Find the motion and time of descent if $v = 0$ and $x = a$ at $t = 0$.

223.* Integrate $y = -xp + x^4p^2$, where $p = \frac{dy}{dx}$.

224. Integrate $y = p + \log p$.

225.* Solve the difference equation

$$u_{n+2} + 2au_{n+1} + bu_n = 0,$$

where $a$, $b$ are constants, by putting $u_n = \lambda^n$. Show that the solution can be expressed in the form $u_n = \alpha r_1^n + \beta r_2^n$, where $r_1$, $r_2$ are the roots (supposed distinct) of the equation $\lambda^2 + 2a\lambda + b = 0$. Show that the form of the solution when $b = a^2$ is $u_n = \alpha(-a)^n + \beta n(-a)^n$.

Answers and Hints

# End

# Exercises. Answers and Hints

## Exercises 1.1

**1.** (*d*), (*e*). Show that $x$ satisfies an equation of the type

$$x^6 + a_1x^5 + \ldots + a_6 = 0,$$

where $a_1, \ldots, a_6$ are integers; prove that $x$ is then either irrational or an integer.

**2.** Use the irrationality of $\sin 60° = \sqrt{3}/2$.

**4.** Write $ax^2 + 2bx + c$ as $a\left(x + \frac{b}{a}\right)^2 + \frac{ac - b^2}{a}$.

**7.** If $a > 0$ and $b^2 - ac \leq 0$, it is possible to make $ax^2 + 2bx + c = 0$ for some value of $x$ if, and only if, $b^2 - ac = 0$; then use Example 6.

**8.** The cosine of the angle between two straight lines is $\leq 1$ in absolute value.

**9.** Use Schwarz's inequality.

**10.** Square both sides and then use Schwarz's inequality. The sum of the lengths of two sides of a triangle is not less than the third side.

## Exercises 1.2

**2.** (*a*), (*d*), (*e*), (*g*) odd; (*b*) even.

**3.** (*b*), (*c*), (*h*) monotonic; (*a*), (*d*), (*e*), (*l*), (*m*) even; (*d*) and (*e*) identical.

## Exercises 1.3

**2.** $(n + 1)(2n + 1)(2n + 3)/3$.

**3.** (*c*) Expand $(1 + 1)^n$ by the binomial theorem.

**4.** (*a*) $n(n + 1)(n + 2)/3$.

(*b*) Sum $\frac{1}{\nu + 1} - \frac{1}{\nu}$ from $\nu = 1$ to $\nu = n$. $n/(n + 1)$.

(*c*) Sum $\frac{1}{(\nu + 1)^2} - \frac{1}{\nu^2}$ from $\nu = 1$ to $\nu = n$. $n(n + 2)/(n + 1)^2$.

**5.** 3; 193.

**7.** $\frac{1}{6}(2n^3 + 3n^2 - 11n + 30)$.

## Exercises 1.4

**1.** (*a*) 1; (*b*) 333; (*c*) 333,333.
**2.** (*a*) 0; (*b*) $\infty$; (*c*) 6; (*d*) $a_0/b_0$; (*e*) 1/3.
**4.** 19.
**5.** (*a*) 6; (*b*) 10; (*c*) 14.
**6.** (*a*) 25; (*b*) 2500; (6) 250,000.
**9.** (*a*) 0; (*b*) no; (*c*) yes; (*e*) 30.
**15.** The greatest of $a_1, \ldots, a_k$.
**16.** 2.
**17.** Use the fact that $n/2^n \to 0$.

## Exercises 1.5

**1.** (*a*) For every number $M$, no matter how large, there exists an $n$ such that $|a_n| > M$.

(*b*) There exists a positive number $\varepsilon$ such that for every number $M$ there exist numbers $n$, $m$ greater than $M$ for which $|a_n - a_m| \geqq \varepsilon$.

**5.** Error is less than $\dfrac{1}{n(n!)}$; $e = 2{\cdot}71828\ldots$.

## Exercises 1.6

**1.** (*a*) 6; (*b*) 15; (*c*) $\frac{1}{2}$; (*d*) 3.
**3.** Limits (*a*) and (*b*) do not exist; limit (*c*) exists and is equal to 1.

## Exercises 1.7

**3.** (*a*) 1/60; 1/600; 1/6000.
(*b*) $1/10(1 + 2|\xi - 1|)$, &c.
(*c*) $1/120(1 + |\xi|)^3$, &c.
(*d*) 1/100; 1/10000; 1/1000000. (*e*) 1/10; 1/100; 1/1000.

**4.** (*a*) 1/600; $\varepsilon/6$. (*b*) 1/400; $\varepsilon/4$. (*c*) 1/77600; $\varepsilon/776$. (*d*) 1/10000; $\varepsilon^2$. (*e*) 1/100; $\varepsilon$.

**5.** (*a*), (*b*), (*c*), (*d*), (*g*) continuous;
(*e*) discontinuous at $x = 2, 4$;
(*f*) ,, ,, $x = 3$;
(*h*), (*k*), (*m*) ,, ,, $x = (n + \frac{1}{2})\pi$;
(*i*), (*j*) ,, ,, $x = n\pi$;
(*l*) ,, ,, $x = n\pi$, $n \neq 0$.

## Exercises 1.8

**1.** (a) Upper bound $= \frac{2\sqrt{3}}{5}$,* lower $= 0$, upper limit $= 0$, lower $= 0$.
(b) „ $= \frac{1}{2}$,* „ $= -1$,* „ $= 0$,* „ $= 0$.*
(c) „ $= \frac{9}{10}$,* „ $= -\frac{2}{3}$,* „ $= \frac{1}{2}$, „ $= \frac{1}{2}$.
(d) „ $= \frac{19}{10}$,* „ $= -\frac{1}{2}$,* „ $= \frac{3}{2}$, „ $= \frac{1}{2}$.
(e) „ $= 2$,* „ $= 0$, „ $= 1$, „ $= 0$.

The quantities marked * belong to the sequence

**2.** Divide the interval into a finite number of sub-intervals by points $a = x_0, x_1, x_2, \ldots, x_n = b$ so close that $|f(x) - f(\bar{x})| < \varepsilon$ if $x$ and $\bar{x}$ lie in the same sub-interval. Join adjacent points $x = x_i$, $y = f(x_i)$ by straight lines.

**3.** The expression $-\frac{k}{2}|x - x_i| + \frac{k}{2}|x - x_{i-1}|$ has the slope zero outside the interval $(x_{i-1}, x_i)$. Add suitable terms of this kind.

$$\frac{3}{2} + x - \frac{3}{2}|x - 2| + |x - 3| - \frac{1}{2}|x - 5|.$$

**4.** (a) $\varepsilon/6$; (b) $\varepsilon/na^{n-1}$, $n > 0$; (c) $\varepsilon^2/2$.

## Exercises 1.9

**2.** (a) $r = a$; (b) $r = 2a\cos(\varphi - \varphi_0)$; (c) $r = a/\cos(\varphi - \varphi_0)$.

**3.** $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$;
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$;
$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$, $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.

**4.** (a) $-6i$; $\theta = \pi$, $r = 3$; $\theta = \pi/2$, $r = 2$; $\theta = 3\pi/2$, $r = 6$.

(b) $1 + \sqrt{3} + i(1 - \sqrt{3})$; $\theta = \pi/4$, $r = 4\sqrt{2}$; $\theta = \pi/3$, $r = \frac{1}{2}$; $\theta = 7\pi/12$, $r = 2\sqrt{2}$.

(c) $2$; $\theta = \pi/4$, $r = \sqrt{2}$; $\theta = 7\pi/4$, $r = \sqrt{2}$; $\theta = 2\pi$, $r = 2$.

(d) $2 - 2i\sqrt{3}$; $\theta = 5\pi/6$, $r = 2$; $\theta = 5\pi/3$, $r = 4$.

(e) $\pm 1$; $\theta = 0$, $r = 1$; $\theta = 0$, $r = \pm 1$.

(f) $\pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$; $\theta = \pi/2$, $r = 1$; $\theta = \pi/4$, $r = \pm 1$.

(g) $\{\sqrt{\sqrt{2}+1} + i\sqrt{\sqrt{2}-1}\}/\sqrt{2}$; $\theta = \pi/4$, $r = \sqrt{2}$; $\theta = \pi/8$, $r = \sqrt[4]{2}$.

(h) $-\sqrt[3]{18}\,(\sqrt{3} + i)/2$; $\theta = 7\pi/4$, $r = 3\sqrt{2}$; $\theta = 7\pi/6$, $r = (3\sqrt{2})^{2/3} = \sqrt[3]{18}$.

(k) $1$, $(-1 \pm i\sqrt{3})/2$; $\theta = 0$, $r = 1$; $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, $r = 1$.

(l) $\sqrt[4]{2}\{\sqrt{\sqrt{2}+1} + i\sqrt{\sqrt{2}-1}\}$; $\theta = \pi/2$, $r = 16$; $\theta = \pi/8$, $r = \pm 2$.

**5.** Note that $\varepsilon^\nu$ satisfies the equation $x^n - 1 = 0$; then factorize $-1$.

Exercises 2.1

**1.** Use formulæ of § 2 and the basic rules: $70/3$.

**2.** Required area may be regarded as the difference between the area under the line and the area under the parabola, taken between the points of intersection of curve and line: $10\sqrt{5}/3$.

**3.** $\sqrt{5}/6$.

**4.** $\frac{1}{6}(a^2 + 4b)^{3/2}$.

**5.** (a) $\{(1 + b)^{1+\alpha} - (1 + a)^{1+\alpha}\}/(1 + \alpha)$; (b) $-(\cos\alpha b - \cos\alpha a)/\alpha$; (c) $(\sin\alpha b - \sin\alpha a)/\alpha$.

**7.** $(b^4 - a^4)/4$.

**8.** $1/(n + 1)$.

Exercises 2.2

**1.** For every number $a$ there exists an $\varepsilon$ such that for every positive number $\delta$ there exists an $x$ for which

$$|x - \xi| \leqq \delta \quad \text{and} \quad \left| a - \frac{f(x) - f(\xi)}{x - \xi} \right| \geqq \varepsilon.$$

**2.** (a) $-1/(x+1)^2$; (b) $-2x/(x^2+2)^2$; (c) $-4x/(2x^2+1)^2$;
(d) $-\cos x/\sin^2 x$; (e) $3\cos 3x$; (f) $-a\sin ax$; (g) $2\sin x\cos x$;
(h) $-2\cos x\sin x$.

**3.** (a) $\xi$ has any value; (b) $\xi = (x_1 + x_2)/2$; (c) $\xi = \sqrt{\left(\dfrac{x_2^2 + x_2x_1 + x_1^2}{3}\right)}$.

$$(e)\ \ \xi = \left(\frac{x_2^{2/3} + x_2^{1/3}x_1^{1/3} + x_1^{2/3}}{3}\right)^{3/2}.$$

Exercises 2.3

**2.** (a) $\frac{1}{2}$; (b) $\frac{1}{2}$.

Exercises 2.4

**1.** $\pi/4 = 0.785$.

Exercises 2.5

**1.** (b) $\xi = \dfrac{a+b}{2}$; (c) $\xi = \sqrt[n]{\left(\dfrac{a^n + a^{n-1}b + \ldots + b^n}{n+1}\right)}$; (d) $\xi = \sqrt{ab}$.

**3.** (a) $I_n = a^{1+1/n}/(1 + 1/n)$, $\lim\limits_{n\to\infty} I_n = a$;

(b) $I_n = a^{n+1}/(n+1)$, $\lim\limits_{n\to\infty} I_n = 0$ for $-1 \leqq a \leqq 1$, $\infty$ for $a > 1$.

**4.** $|F(x) - f(x)| \leqq \dfrac{1}{2\delta}\displaystyle\int_{-\delta}^{\delta} |f(x+t) - f(x)|\, dt$. Use the uniform continuity of $f(x)$ in $a \leqq x \leqq b$. Also, we may write

$$F(x) = \frac{1}{2\delta}\left\{\int_{x-\delta}^{c} f(t)\,dt + \int_{c}^{x+\delta} f(t)\,dt\right\},$$

where $c$ is a fixed number.

**5.** Express the integrals as limits of sums, using equal subdivisions of $a \leqq x \leqq b$, and applying Schwarz's inequality (p. 12) to these sums. Another method is to integrate $\{f(x) + tg(x)\}^2 \geqq 0$ and use **6.** in Exercises 1.1.

Exercise 2.6:

**1.** Let $\varphi(x) = f'(x)$ where $f'(x) \geqq 0$, $\varphi(x) = 0$ elsewhere. Let $\psi(x) = f'(x) - \varphi(x)$; then $\psi(x) \leqq 0$. Consider $\int_a^x \varphi(x)\,dx$, $\int_a^x \psi(x)\,dx$.

Exercises 3.1:

**1.** $f'(1) = 1$, $f''(1) = 8$, $f'''(1) = 36$, $f^{iv}(1) = 96$, $f^{v}(1) = 120$, $f^{vi}(1) = 0$, $f^{vii}(1) = 0, \ldots$ .

**2.** 0.

**3.** (*a*) $a$; (*b*) $175\,cx^6$; (*c*) $2(b + cx)$; (*d*) $\dfrac{ad - bc}{(cx + d)^2}$;

(*e*) $\dfrac{2x^2(a\beta - \alpha b) + 2x(a\gamma - \alpha c) + 2(b\gamma - \beta c)}{(\alpha x^2 + 2\beta x + \gamma)^2}$;

(*f*) $\dfrac{4x(1 + x^4)}{(1 - x^2)^2(1 + x^2)^2}$; (*g*) 0.

**4.** (*a*) $F(x) = a_n x^n + (a_{n-1} + na_n)x^{n-1} + \{a_{n-2} + (n-1)(a_{n-1} + na_n)\}x^{n-2} + \ldots$ .

(*b*) $F(x) = \dfrac{a_n}{c_0}x^n + \left(a_{n-1} - na_n\dfrac{c_1}{c_0}\right)x^{n-1} + \left\{\dfrac{a_{n-2}}{c_0} - (n-1)a_{n-1}\dfrac{c_1}{c_0^2} - n(n-1)a_n\dfrac{(c_0c_2 - c_1^2)}{c_0^3}\right\}x^{n-2} + \ldots$

**5.** (*a*) $2\cos 2x$; (*b*) $-1/(1 + \sin 2x)$; (*c*) $\tan x + x/\cos^2 x$;

(*d*) $-2/(1 - \sin 2x)$; (*e*) $-\dfrac{\sin x}{x^2} + \dfrac{\cos x}{x}$.

**6.** $\sec^3 x + \sec x \tan^2 x$.

**7.** $24\sec^5 x - 20\sec^3 x + \sec x$.

**8.** $\cos x\,(\operatorname{cosec}^2 x - 6\operatorname{cosec}^4 x)$.

**9.** $24\sec^5 x - 20\sec^3 x + \sec x - \cos x$.

**10.** $\infty$.

**11.** $ax^2/2 + bx$.

**12.** $ax^3/3 + bx^2 + cx$.

**13.** $x^9 + x^7 + x^5 + x^3 + x$.

**14.** $-(1/x + 1/2x^2 + 1/3x^3)$.

**15.** $x^3/3 - 1/x$.

**16.** $a\sin x - b\cot x$.

**17.** $3x^2/2 - 7\cos x - 5/2x^2 - 9\tan x$.

**18.** $\sec x$.

Exercises 3.2:

1. 4.

3. $\sqrt[n]{x}\{1-(n-1)x\}/nx(1+x)^2$.

4. $\cos^2 x/2\sqrt{x} - 2\sqrt{x}\sin x\cos x$.

5. $1/\sqrt{x}(1-\sqrt{x})^2$.

6. $\dfrac{(1-\tan x)+3x(1+\tan^2 x)}{3x^{2/3}(1-\tan x)^2}$.

7. $(\text{arc}\cos x - \text{arc}\sin x)/\sqrt{(1-x^2)}$.

8. $2/(1+x^2)(1-\text{arc}\tan x)^2$.

9. $\dfrac{1}{\sqrt{1-x^2}\,\text{arc}\tan x} - \dfrac{\text{arc}\sin x}{(1+x^2)(\text{arc}\tan x)^2}$.

10. $-\dfrac{5}{1+x^2}+\dfrac{1}{\sqrt{1-x^2}(\text{arc}\cos x)^2}$.

11. 0·785.

Exercises 3.3:

**1.** $3(x+1)^2$. **2.** $6(3x+5)$. **3.** $15x^{14}(3x^6-6x^3-1)(x^6-3x^3-$
**4.** $-1/(1+x)^2$. **5.** $2x/(1-x^2)^2$. **6.** $an(ax+b)^{n-1}$.

**7.** $-\dfrac{1}{\sqrt{x^2-1}(x+\sqrt{x^2-1})}$.

**8.** $\dfrac{x^2(am-bl)+2x(an-cl)+(bn-cm)}{2\sqrt{\{(ax^2+bx+c)(lx^2+mx+n)^3\}}}$.

**9.** $-\frac{5}{3}(1-x)^{2/3}$. **10.** $\sin 2x$. **11.** $2x\cos(x^2)$.

**12.** $\sin x\cos x/\sqrt{(1+\sin^2 x)}$. **13.** $2\left(x\sin\dfrac{1}{x^2}-\dfrac{1}{x}\cos\dfrac{1}{x^2}\right)$.

**14.** $\dfrac{2}{(1-x)^2\cos^2\left(\dfrac{1+x}{1-x}\right)}$. **15.** $(2x+3)\cos(x^2+3x+2)$.

**16.** $3x^2/\sqrt{\{1-(3+x^3)^2\}}$. **17.** $-1$. **18.** $1$.

**19.** $\dfrac{\sqrt{2}}{x}(x^{\sqrt{2}}+x^{-\sqrt{2}})$. **20.** $\sqrt[3]{5}\cos(x+7)\{\sin(x+7)\}^{\sqrt[3]{5}-1}$.

**21.** $-\dfrac{a\alpha\sin x}{\sqrt{\{1-(a\cos x+b)^2\}}}\cdot\{\arcsin(a\cos x+b)\}^{\alpha-1}$.

Exercises 3.4

1. (a) Max. for $x = -\sqrt{2}$, min. for $x = \sqrt{2}$, infl. for $x = 0$.

(b) Max. for $x = \frac{2}{5}$, min. for $x = 0$, infl. for $x = -\frac{1}{10}$.

(c) Max. for $x = 1$, min. for $x = -1$, infl. for $x = 0, \pm\sqrt{3}$.

(d) Max. for $x = \sqrt[4]{3}$, min. for $x = -\sqrt[4]{3}$,

infl. for $x = 0, \pm\sqrt[4]{6 \pm \sqrt{33}}$.

(e) Max. for $x = (n + \frac{1}{2})\pi$, min. for $x = n\pi$, infl. for $x = \dfrac{2n+1}{4}\pi$.

2. Max. for $x = -\sqrt{-p}$; min. for $x = \sqrt{-p}$; infl. for $x = 0$. No maxima or minima when $p \geqq 0$. Roots are all real, or two complex and one real, according as $q^2 + 4p^3 \leqq 0$ or $> 0$.

3. The point (0, 1).

4. Equation of line is $(y - y_0)/(x - x_0) = -\sqrt[3]{(y_0/x_0)}$.

5. $\sqrt{189}$ ft.

6. The point dividing the line $ab$ in the ratio $\sqrt[3]{a} : \sqrt[3]{b}$.

7. The square.

8. The rectangle with corners $x = \pm a/\sqrt{2}$, $y = \pm b/\sqrt{2}$.

9. The right-angled triangle, i.e. $c^2 = a^2 + b^2$.

10. The side of rectangle opposite to $g$ must be at the distance $\frac{1}{4}\{\sqrt{(8r^2 + h^2)} + h\}$ from the centre.

11. The cylinder whose height is equal to the diameter of its base.

13. If $\varphi$ is the angle of the prism and $n$ its index of refraction, the angle of incidence must be $\arcsin\left(n \sin \dfrac{\varphi}{2}\right)$.

14. $x = (\sum_i a_i)/n$.

15. Find the minimum of $x^p - px$.

16. Find the minima of $x - \sin x$ and $\sin x - \dfrac{2}{\pi}x$ in the interval $0 \leqq x \leqq \dfrac{\pi}{2}$. Or, show that $\dfrac{\sin x}{x}$ is monotonic in that interval.

18. $\left(\dfrac{a_1 + a_2 + \dots + a_{n-1}}{n-1}\right)^{\frac{n-1}{n}} \dfrac{1}{\sqrt[n]{(a_1 a_2 \dots a_{n-1})}}$.

Exercises 3.5:

**1.** $0{\cdot}693$. **2.** $\log x$. **3.** $1/x \log x$. **4.** $1/\sqrt{(1 + x^2)}$.

**5.** $$\frac{1 - 2x\sqrt{1 + \log x}\cos x}{2x\sqrt{1 + \log x}(\sqrt{1 + \log x} - \sin x)}.$$

**6.** $x/(x^2 + 1) - 1/3(2 + x)$.

**7.** $$\frac{\sqrt[3]{7x^2 + 1}}{\sqrt{x - 2}\sqrt{x^4 + 1}}\left(\frac{14x}{3(7x^2 + 1)} - \frac{1}{4(x - 2)} - \frac{2x^3}{(x^4 + 1)}\right).$$

**11.** $x = 1/\lambda$, provided $\lambda \neq 0$; if $\lambda = 0$, no maximum exists.

**12.** $(\log a)^2 \cdot a^{(a^x)} \cdot a^x$.

**13.** $$a^{\sin x (\log x)^3} \cdot \log a \left\{\cos x (\log x)^2 + \frac{2 \sin x \log x}{x}\right\}.$$

Exercises 3.6:

**1.** (*a*) Keep $x$ fixed and differentiate with respect to $y$; then put $y$ equal to zero.

(*b*) Evaluate $f(x)$ first for rational $x$, and then by continuity for irrational $x$.

**2.** Differentiate with respect to $y$, and then put $y$ equal to 1.

**3.** 2315 years.

**4.** (*a*) $y = \beta + ce^{\alpha x}$; (*b*) $y = -\dfrac{\beta}{\alpha} + ce^{\alpha x}$, $\alpha \neq 0$; $y = \beta x + c$, $\alpha = 0$.

(*c*) $y = \beta x e^{\alpha x} + ce^{\alpha x}$; (*d*) $y = \dfrac{\beta}{\gamma - \alpha} e^{\gamma x} + ce^{\alpha x}$, $\gamma \neq \alpha$.

Exercises 3.7:

**1.** $\sinh a - \sinh b = 2\cosh\left(\dfrac{a+b}{2}\right)\sinh\left(\dfrac{a-b}{2}\right).$

$\cosh a + \cosh b = 2\cosh\left(\dfrac{a+b}{2}\right)\cosh\left(\dfrac{a-b}{2}\right).$

$\cosh a - \cosh b = 2\sinh\left(\dfrac{a+b}{2}\right)\sinh\left(\dfrac{a-b}{2}\right).$

**2.** $\tanh(a \pm b) = \dfrac{\tanh a \pm \tanh b}{1 \pm \tanh a \tanh b}.$

$\coth(a \pm b) = \dfrac{1 \pm \coth a \coth b}{\coth a \pm \coth b}.$

$\sinh \tfrac{1}{2}a = \sqrt{\left(\dfrac{\cosh a - 1}{2}\right)};\quad \cosh \tfrac{1}{2}a = \sqrt{\left(\dfrac{\cosh a + 1}{2}\right)}.$

**3.** (*a*) $\sinh x + \cosh x$; (*b*) $-4\,\dfrac{e^{\tanh x + \coth x}}{\cosh 4x - 1}$;

(*c*) $(1 + \sinh 2x)\coth(x + \cosh^2 x)$; (*d*) $1/\sqrt{(x^2 - 1)} + 1/\sqrt{(x^2 + 1)}$;

(*e*) $\alpha \sinh x/\sqrt{(\alpha^2 \cosh^2 x + 1)}$; (*f*) $2/(1 - x^2)$.

**4.** $\sinh b - \sinh a.$

Exercises 3.8:

**1.** (*a*) Higher than $x^N$; (*b*) lower than $x^{\varepsilon}$; (*c*) same as 1; (*d*) higher than $x^N$; (*e*), (*f*) higher than $x^{\frac{1}{2}-\varepsilon}$, lower than $x^{\frac{1}{2}+\varepsilon}$; (*g*) same as $x$; (*h*) higher than $x^N$; (*j*) lower than $x^{\varepsilon}$.

**2.** Higher than $e^{ax}$, $(\log x)^a$, same as $e^{x^\beta}$; (*b*) lower than $e^{ax}$, $e^{x^a}$; (*c*) bounded; (*d*) same as $e^x$, lower than $e^{x^a}$, higher than $(\log x)^\alpha$; (*e*), (*f*), (*g*) lower than $e^{\alpha x}$, $e^{x^a}$, higher than $(\log x)^a$; (*h*) higher than $e^{x^{1-\varepsilon}}$, lower than $e^{x^{1+\varepsilon}}$, higher than $e^{ax}$, $(\log x)^a$; (*j*) same as $\log x$, lower than $e^{ax}$, $e^{x^\alpha}$.

**3.** (*a*) Same as $x^\beta$; (*b*) lower than $\left(\frac{1}{x}\right)^{\varepsilon}$; (*c*) same as $x$; (*d*) same as $x$; (*e*) same as $x^{5/2}$; (*f*) same as $x^{3/2}$; (*g*) higher than $x^N$; (*h*) higher than $x^{1-\varepsilon}$, lower than $x$; (*j*) lower than $\left(\frac{1}{x}\right)^{\varepsilon}$.

**4.** Yes; 0.

**5.** 0, 1.

**6.** $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = f'(0) = 0.$

**8, 9.** Use result of Ex. 7.

**Exercises 3.9:**

1. $f''[g\{h(x)\}]g'^2\{h(x)\}h'^2(x)+f'[g\{h(x)\}]g''\{h(x)\}h'^2(x)$
$+f'[g\{h(x)\}]g'\{h(x)\}h''(x)$.

2. (a) $x^{\sin x}\left(\dfrac{\sin x}{x}+\log x \,.\cos x\right)$.

(b) $(\cos x)^{\tan x}\left(-\tan^2 x+\dfrac{\log\cos x}{\cos^2 x}\right)$.

(c) $\dfrac{u'(x)}{u(x)\log v(x)}-\dfrac{v'(x)\log u(x)}{v(x)\{\log v(x)\}^2}$.

4. (a) $e^{\alpha x}[\alpha^n x^3+3n\alpha^{n-1}x^2+3n(n-1)\alpha^{n-2}x+n(n-1)(n-2)\alpha^{n-3}]$;

(b) $\dfrac{2(-1)^n(n-1)!}{x^n}\left(\sum_{\nu=1}^{n-1}\dfrac{1}{\nu}-\log x\right)$;

(c) $\dfrac{(-1)^m}{2}\{\cos x-3^{2m}\cos 3x\}$, for $n=2m$;

$\dfrac{(-1)^m}{2}\{3^{2m+1}\sin 3x-\sin x\}$, for $n=2m+1$.

(d) $\dfrac{(-1)^l}{2}[(m+k)^{2l}\sin(m+k)x-(m-k)^{2l}\sin(m-k)x]$, for $n=2l$;

$\dfrac{(-1)^l}{2}[(m+k)^{2l+1}\cos(m+k)x-(m-k)^{2l+1}\cos(m-k)x]$, for $n=2l+1$.

(e) $$e^x\left[\left(\sum_{l=0}^{2l\leq n}(-1)^l\binom{n}{2l}2^{2l}\right)\cos 2x+\left(\sum_{l=0}^{2l+1\leq n}(-1)^{l+1}\binom{n}{2l+1}2^{2l+1}\right)\sin 2x\right]=5^{n/2}e^x\cos(2x+n\alpha),$$

where $\tan\alpha=2$ (expanding $(1+2i)^n$ by the binomial theorem, and grouping real and imaginary terms).

(f) $e^x.\sum_{\nu=0}^{6}\binom{6}{\nu}\binom{n}{\nu}(1+x)^{6-\nu}$.

5. Let $y=\arcsin x$. Then

$$\frac{d^n y}{dx^n}=\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{\sqrt{(1-x^2)}}\right)=\frac{d^{n-2}}{dx^{n-2}}\left(\frac{x}{(1-x^2)^{3/2}}\right).$$

Apply Leibnitz's rule to this last expression:

$$\left.\frac{d^n y}{dx^n}\right|_{x=0}=(n-2)\frac{d^{n-3}}{dx^{n-3}}\left(\frac{1}{(1-x^2)^{3/2}}\right)_{x=0}$$
$$=3.(n-2)\frac{d^{n-4}}{dx^{n-4}}\left(\frac{x}{(1-x^2)^{5/2}}\right),$$

and continue the process:

$$\left.\frac{d^n y}{dx^n}\right|_{x=0}=1.3.5\ldots(2\nu-1).(n-2)(n-4)\ldots(n-2\nu+2)\frac{d^{n-2\nu}}{dx^{n-2\nu}}\left(\frac{x}{(1-x^2)^{(2\nu+1)/2}}\right).$$

If $n=2l$, $\left.\dfrac{d^n y}{dx^n}\right|_{x=0}=0$; if $n=2l+1$, $\left.\dfrac{d^n y}{dx^n}\right|_{x=0}=1^2.3^2.5^2.\ldots.(2l-1)^2$.

$$\left.\frac{d^{2l}}{dx^{2l}}(\arcsin x)^2\right|_{x=0}=\sum_{k=0}^{l-1}\binom{2l}{2k+1}1^2.3^2.\ldots.(2k-1)^2.1^2.3^2.\ldots.(2l-2k-3)^2.$$

$$\left.\frac{d^{2l+1}}{dx^{2l+1}}(\arcsin x)^2\right|_{x=0}=0.$$

6. Differentiate $(1+x)^n$ twice and put $x=1$.

Exercises 4.1:

1. $\frac{1}{2}e^{x^2}$. 2. $-\frac{1}{4}e^{-x^4}$. 3. $\frac{2}{9}(1+x^3)^{3/2}$. 4. $\frac{1}{2}(\log x)^2$.

5. $-\dfrac{1}{n-1}\left(\dfrac{1}{\log x}\right)^{n-1}$.

6. Hint: write denominator in the form $(3x-1)^2+1$: arc tan $(3x-1)$.

7. $\log\left\{\dfrac{x-1}{2}+\sqrt{1+\left(\dfrac{x-1}{2}\right)^2}\right\}$.

8. Hint: $6x/(2+3x) = 2-4/(2+3x)$: $2x-\frac{4}{3}\log|2+3x|$.

9. $\arcsin x - \sqrt{(1-x^2)}$. 10. $\log\left\{\dfrac{x+1}{2}+\sqrt{1+\left(\dfrac{x+1}{2}\right)^2}\right\}$.

11. $\arcsin\dfrac{x+1}{2}$. 12. $\frac{1}{2}\log(x^2-x+1)+\dfrac{1}{\sqrt{3}}\arctan\dfrac{2x-1}{\sqrt{3}}$.

13. $2\operatorname{ar\,cosh}\left(\dfrac{x-2}{\sqrt{3}}\right)+\sqrt{(x^2-4x+1)}$.

14. $-\frac{1}{3}\sqrt{(2+2x-3x^2)}+\dfrac{4}{3\sqrt{3}}\arcsin\dfrac{3x-1}{\sqrt{7}}$.

15. $\dfrac{2}{\sqrt{3}}\arctan\dfrac{2x+1}{\sqrt{3}}$. 16. $\dfrac{2}{\sqrt{3}}\arctan\dfrac{2x-1}{\sqrt{3}}$.

17. $\dfrac{1}{\sqrt{(b-a^2)}}\arctan\dfrac{x+a}{\sqrt{(b-a^2)}}$, if $b-a^2>0$; $-\dfrac{1}{x+a}$, if $b-a^2=0$;

$-\dfrac{1}{\sqrt{(a^2-b)}}\operatorname{ar\,tanh}\dfrac{x+a}{\sqrt{(a^2-b)}}$, if $b-a^2<0$.

18. $-x^4/4-x^3/3-x^2/2-x-\log|x-1|$.

19. Hint: $\sin^3 x\cos^4 x = \sin x\cos^4 x(1-\cos^2 x) = \sin x\cos^4 x - \sin x\cos^6 x$:

$-\dfrac{\cos^5 x}{5}+\dfrac{\cos^7 x}{7}$.

20. $\dfrac{\sin^3 x}{3}-2\dfrac{\sin^5 x}{5}+\dfrac{\sin^7 x}{7}$. 21. $\frac{1}{9}(1-x^2)^{9/2}-\frac{1}{7}(1-x^2)^{7/2}$.

22. $\frac{1}{2}\arcsin x-\frac{1}{2}x\sqrt{(1-x^2)}$. 23. $\pi^2/32$.

24. $\dfrac{1+(-1)^n}{n+1}$. 25. 2. 26. $\dfrac{1}{2(1+a^2)}-\dfrac{1}{2(1+b^2)}$.

27. $\frac{1}{3}(a^3-b^3)+\frac{1}{2}(a^2-b^2)+(a-b)+\log\dfrac{a-1}{b-1}$.

28. $\frac{1}{2}\left(1-\cos\dfrac{\pi^2}{2}\right)$. 29. Cf. Ex. 8, p. 88: $1/(n+1)$.

* Here and elsewhere the constants of integration are omitted.

## Exercises 4.2:

**1.** Take $f = x$, $g' = \cos x/\sin^2 x$: $-x/\sin x + \log\tan x/2$.

**2.** Take $f = x^4/4$, $g' = 4x^3/(1-x^4)^2$: $x^4/4(1-x^4) + \frac{1}{4}\log|1-x^4|$.

**3.** $(x^2-2)\sin x + 2x\cos x$.

**4.** $-\frac{1}{2}(x^2+1)e^{-x^2}$. **5.** $4\pi(-1)^n/n^2$. **6.** $0$.

**7.** $\frac{1}{2}(x^2\sin x^2 + \cos x^2)$. **8.** $\frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + \frac{3}{8}x$.

**9.** $\frac{1}{192}\sin 6x + \frac{3}{64}\sin 4x + \frac{15}{64}\sin 2x + \frac{5}{16}x$.

**10.** Put $x = \cos\theta$: $x\sqrt{1-x^2}\left(-\frac{1}{16} - \frac{1}{24}x^2 + \frac{1}{6}x^4\right) + \frac{1}{16}\arcsin x$.

**11.** $e^x(x^2-2x+2)$. **12.** $-\dfrac{1}{(n-1)x^{n-1}}\log x - \dfrac{1}{(n-1)^2x^{n-1}}$.

**13.** $\dfrac{x^{m+1}}{m+1}\log x - \dfrac{x^{m+1}}{(m+1)^2}$. **14.** $\frac{1}{3}x^3\{(\log x)^2 - \frac{2}{3}\log x + \frac{2}{9}\}$.

**16.** Put $x^2 = t$, then use Ex. 15.

**17.** Integrate by parts repeatedly.

**19.** Use mathematical induction: assume that the $n$-th iterated integral $f_n(x)$ is given by $\dfrac{1}{(n-1)!}\displaystyle\int_0^x f(u)(x-u)^{n-1}\,du$, and expand the integrand by the binomial theorem. Then $f_{n+1}(x) = \displaystyle\int_0^x f_n(t)\,dt$; integrate each term by parts.

## Exercises 4.3:

1. $\log\sqrt{\left|\frac{x}{2-3x}\right|}$.  2. $\log\left|1-\frac{1}{x}\right|$.

3. $\log\left|\frac{x}{x+1}\right|^3+\frac{3}{x+1}+\frac{3}{2(x+1)^2}$.

4. $\frac{x}{3}-\frac{1}{8}\log|x+1|+\frac{49}{72}\log|3x-5|$.

5. $-\frac{1}{2(x-1)}+\log\sqrt[4]{\frac{1+x^2}{(1-x)^2}}$.  6. $\frac{-1}{2(x-1)}-\log\sqrt[4]{\frac{1+x^2}{(1-x)^2}}$.

7. $\log\frac{1}{\sqrt[3]{|x-1|}}+\frac{1}{6}\log(x^2+x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$.

8. $\log\sqrt[3]{|x+1|}-\frac{1}{6}\log|x^2-x+1|+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$.

9. $\log\frac{1}{\sqrt[5]{(x-2)^2}}+\log\sqrt[5]{1+x^2}+\frac{2}{5}\arctan x$.

10. $\frac{2}{3}\log|x+2|+\frac{5}{6}\log|x-1|-\frac{3}{2}\log|x+1|$.

11. $-\frac{x^3}{3}+\log\sqrt[4]{\left|\frac{x+1}{x-1}\right|}-\frac{1}{2}\arctan x$.

12. $\frac{1}{3}\arctan x+\frac{\sqrt{3}}{12}\log\frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1}+\frac{1}{6}\arctan(2x+\sqrt{3})$
$$+\frac{1}{6}\arctan(2x-\sqrt{3}).$$

13. $\frac{1}{6}\log\frac{x-1}{x+1}+\frac{\sqrt{2}}{3}\arctan\frac{x}{\sqrt{2}}$.  14. $-\frac{3x^2+2}{2x(x^2+1)}-\frac{3}{2}\arctan x$.

**Exercises 4.4:**

1. $-\dfrac{2}{1+\tan\dfrac{x}{2}}$.  2. $\tan\dfrac{x}{2}$.  3. $\dfrac{2}{\sqrt{3}}\,\text{arc tan}\left(\dfrac{2\tan\dfrac{x}{2}+1}{\sqrt{3}}\right)$.

4. $\dfrac{1}{8}\left(\tan^2\dfrac{x}{2}-\cot^2\dfrac{x}{2}\right)+\dfrac{1}{2}\log\left|\tan\dfrac{x}{2}\right|$.  5. $\log\left|\dfrac{\tan\dfrac{x}{2}+1}{\tan\dfrac{x}{2}-1}\right|$.

6. $\dfrac{1}{\sqrt{2}}\,\text{arc tan}\,\dfrac{1}{2}\sqrt{2}$.  7. $\dfrac{1}{\sqrt{2}}\,\text{arc tan}\,\dfrac{\tan x}{\sqrt{2}}$.

8. $\dfrac{1}{2\sqrt{3}}\,\text{arc tan}\,\dfrac{2\tan x}{\sqrt{3}}$.  9. $\dfrac{1}{2\cos^2 x}+\log\cos x$.

10. $\dfrac{1}{\sqrt{2}}\log\left|\dfrac{\tan\dfrac{x}{2}-1+\sqrt{2}}{\tan\dfrac{x}{2}-1-\sqrt{2}}\right|$.

11. $\dfrac{1}{4}\log\dfrac{\cos^2 x-\cos x+1}{(\cos^2 x+\cos x+1)^3}+\dfrac{1}{2\sqrt{3}}\,\text{arc tan}\,\dfrac{2\cos x-1}{\sqrt{3}}$
$-\dfrac{1}{2\sqrt{3}}\,\text{arc tan}\,\dfrac{2\cos x+1}{\sqrt{3}}$.

12. $\dfrac{1}{2}x\sqrt{x^2-4}-2\,\text{ar cosh}\,\dfrac{x}{2}$.

13. $\dfrac{1}{2}x\sqrt{4+9x^2}+\dfrac{2}{3}\,\text{ar sinh}\,\dfrac{3}{2}x$.

14. $2\,\text{arc tan}\sqrt{\dfrac{x-3}{x-1}}$.

15. $\frac{1}{3}\sqrt{(x^2+4x)^3}-(x+2)\sqrt{(x^2+4x)}+4\,\text{ar cosh}\,\dfrac{x+2}{2}$.

16. $\sqrt{x}-\sqrt{(1-x)}+\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{(\sqrt{x}-\sqrt{\frac{1}{2}})(\sqrt{1-x}+\sqrt{\frac{1}{2}})}{(\sqrt{x}+\sqrt{\frac{1}{2}})(\sqrt{1-x}-\sqrt{\frac{1}{2}})}\right|$.

17. $\log\left|x\,.\,\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right|+\sqrt{(1-x^2)}$.

18. $\frac{1}{2}\,\text{ar cosh}(2x-2a+1)+\sqrt{(x-a)^2+(x-a)}-2\sqrt{x-a}$.

19. $\dfrac{2}{[illegible]}\,.\,(\sqrt{(x-a)^3}-\sqrt{(x-b)^3})$.

Exercises 4.5:

**1.** Div. **2.** Conv. **3.** Conv. **4.** Conv. **5.** Div.
**6.** Conv. **7.** Conv. **8.** Div. **9.** Conv. **10.** Conv.
**11.** Conv. **14.** (*a*) For $0 < s < 1$. (*b*) For $0 < s < 2$.
**15.** Yes.

Chapter IV, Mixed Erercises:

**1.** Put $\arcsin x = t$: $\frac{1}{2}e^{\arcsin x}(x + \sqrt{1 - x^2})$.

**2.** $\frac{1}{9}\cos^9 x - \frac{1}{7}\cos^7 x$.

**3.** $x\{(\log x)^2 - 2\log x + 2\}$. **4.** $\frac{1}{4}\log \dfrac{2 - \cos x}{2 + \cos x}$.

**5.** Put $\sqrt{(1 - e^{-2x})} = t$: $x - \sqrt{(1 - e^{-2x})} + \log\{1 + \sqrt{(1 - e^{-2x})}\}$.

**6.** 0. **7.** 0. **10.** 0.

**12.** Consider the function $1/x$ for the interval $1 \leqq x \leqq 2$. Subdivide the interval into $n$ equal parts and form the lower sum as in Chap. II, § 1 (p. 76 *et seq.*). This turns out to be $\alpha_n$. Now let $n \to \infty$. The result is $\log 2$.

**13.** Compare with $1/\sqrt{(1 - x^2)}$ at $x = 0, 1/n, 2/n, \ldots, (n-1)/n$: $\pi/2$.

**14.** Evaluate

$$\lim_{n\to\infty} \log \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{1}{n}\left[\log 1 + \log\left(1 - \frac{1}{n}\right) \ldots + \ldots + \log\left(1 - \frac{n-1}{n}\right)\right],$$

using the definition of the definite integral.

**15.** $1/(1 + \alpha)$.

Exercises 5.1:

**1.** $(x^2 + y^2)^3 = a^2(x^2 - y^2)^2$.

**2.** Take $c$ as rotating with constant velocity, and measure the time so that at time $t = 0$ the point $P$ is in contact with the circle $C$: $x = (R + r)\cos\theta - r\cos\{(R + r)\theta/r\}$, $y = (R + r)\sin\theta - r\sin\{(R + r)\theta/r\}$.

**3.** $x = 2R\cos\theta(1 - \cos\theta) + R$, $y = 2R\sin\theta(1 - \cos\theta)$.

**4.** $x = (R - r)\cos\theta + r\cos\{(R - r)\theta/r\}$, $y = (R - r)\sin\theta - r\sin\{(R - r)\theta/r\}$.

**6.** Take rectangular co-ordinates so that the origin is at the centre of $C$ and the point $P$ lies on the $x$-axis at time $t = 0$: $x^{2/3} + y^{2/3} = R^{2/3}$.

**7.** $x = 3at/(1 + t^3)$, $y = 3at^2/(1 + t^3)$.

**10.** $\alpha = \arctan\left(\dfrac{r(f' - g')}{r^2 + f'g'}\right)$.

**11.** $x = \dfrac{f'(y_0 g' + x_0 f') - g'(gf' - fg')}{f'^2 + g'^2}$, $y = \dfrac{g'(y_0 g' + x_0 f') + f'(gf' - fg')}{f'^2 + g'^2}$.

**12.** (*a*) $C$ itself; (*b*) the cardioid of the circle with diameter $PM$, having its vertex at $P$.

Exercise 5.2:

1. $\frac{3}{5}(b^{5/3} - a^{5/3})$. 2. $3a^2/4$. 3. $\frac{1}{6}a^2(\theta_2^3 - \theta_1^3)$. 4. $6\pi R^2$.

5. $6\pi r^2$. 6. $\pi(1 + \frac{1}{2}x_0^2)$. 7. $\frac{1}{2}\pi(a^2 + b^2 + x_0^2)$.

8. $x = R + s(1 - s/2R + s^2/32R^2)(1 - s/16R)$,
$y = R(s/R - s^2/16R^2)^{3/2}(1 - s/8R)$, for $0 \leqq s \leqq 16R$

9. $x = 2a \operatorname{arc} \cos(1 - s/4a) - (1 - s/4a)\sqrt{\{s(1 - s/8a)/2a\}}$,
$y = s - s^2/8a$, for $0 \leqq s \leqq 8a$.

10. $s = \sqrt{(4/9 + x)^3} - 8/27$. 11. $6R$.

12. (a) $\frac{1}{2}a\{\operatorname{ar} \sinh \theta + \theta\sqrt{(1 + \theta^2)}\}$.

(b) $\dfrac{\sqrt{(1 + m^2)}}{m}(e^{m\theta} - e^{m\theta_0})$.

(c) $8R(1 - \cos\frac{1}{2}\theta)$. (d) $a\{\frac{1}{3}(\theta^3 - \theta_0^3) + \theta - \theta_0\}$.

13. (a) $\frac{1}{2}(1 + 4x^2)^{3/2}$: min. $\frac{1}{2}$ at $x = 0$.

(b) $(a^2 \sin^2\varphi + b^2 \cos^2\varphi)/ab$: if $a > b$, min. $b/a$ at $\varphi = 0$, $\pi$, max. $a/b$ at $\varphi = \pi/2$, $3\pi/2$.

14. $\rho = 1/\sqrt{t}$.

17. Vol. $\pi r^2(h_2 - h_1) - \frac{1}{3}\pi(h_2^3 - h_1^3)$. Surface $2\pi(h_2 - h_1)r$.

18. If $\rho$ is the radius of circle and $r$ the distance of its centre from the line, the volume is $2\pi^2 r\rho^2$, the surface area $4\pi^2 r\rho$.

19. $\pi(x_1 - x_0) + \dfrac{\pi}{2}(\sinh 2x_1 - \sinh 2x_0)$.

20. $k = \pi s$.

21. $y = -\operatorname{ar} \cosh \dfrac{1}{x} + \sqrt{(1 - x^2)} + \text{const.}$; $s = \log\left(\dfrac{x}{x_0}\right)$;

$x = e^s$, $y = -\operatorname{ar} \cosh e^{-s} + \sqrt{(1 - e^{2s})} + \text{const.}$

22. Let $ds$, $ds'$ be the lengths of arc, $l$, $l'$ the total lengths, $A$, $A'$ the areas, and $k$, $k'$ the curvatures of the curve and the parallel curve respectively. Then

$$ds' = (1 + pk)ds; \qquad k' = k/(1 + pk);$$
$$A' = A + lp + \pi p^2; \quad l' = l + 2\pi p.$$

23. (a) $\xi = r(\sin\varphi_2 - \sin\varphi_1)/(\varphi_2 - \varphi_1)$,
$\eta = -r(\cos\varphi_2 - \cos\varphi_1)/(\varphi_2 - \varphi_1)$,

where $\varphi_1$, $\varphi_2$ are the $\theta$-co-ordinates of the extremities of the arc.

(b) $\xi = (x_2 \sinh x_2 - x_1 \sinh x_1 - \cosh x_2 + \cosh x_1)/(\sinh x_2 - \sinh x_1)$

$\eta = \{2(x_2 - x_1) + \sinh 2x_2 - \sinh 2x_1\}/4(\sinh x_2 - \sinh x_1)$,

where $(x_1, y_1)$, $(x_2, y_2)$ are the extremities of the arc.

24. $(\alpha^2 + \beta^2)(b - a) + \frac{2}{3}(\beta^3 - \alpha^3)$.

25. (a) $\sinh x_2 - \sinh x_1 + \frac{1}{3}(\sinh^3 x_2 - \sinh^3 x_1)$,

(b) $(x_2^2 + 2)\sinh x_2 - (x_1^2 + 2)\sinh x_1 - 2x_2 \cosh x_2 + 2x_1 \cosh x_1$,

if $0 \leq x_1 \leq x_2$.

## Exercises 5.3:

1. $$\frac{dx}{dt} = -\frac{r \sin(2t/r)}{2\sqrt{(l^2 - r^2 \sin^2(t/r))}} - \sin\frac{t}{r};$$

$$\frac{d^2x}{dt^2} = -\frac{l^2 \cos(2t/r) + r^2 \sin^4(t/r)}{\sqrt{(l^2 - r^2 \sin^2(t/r))^3}} - \frac{1}{r}\cos\frac{t}{r}.$$

2. Horizontal.

3. $u = v_0/(1 + ksv_0)$, $t = s/v_0 + \frac{1}{2}ks^2$.

4. (a) $x = 4 \arctan e^t - \pi$; $x = \pi$.

5. (a) $$t = \frac{1}{\sqrt{(2\mu M)}}\left(y_0\sqrt{(y_0 - y)} - y_0^{3/2} \arctan\sqrt{\{y/(y_0 - y)\}} + \tfrac{1}{2}\pi y_0\right).$$

(b) $\sqrt{\{2\mu M(1/R - 1/y_0)\}}$; (c) $\sqrt{\dfrac{2\mu M}{R}}$.

6. $$\theta = at,\quad r = \frac{k}{1 - e\cos at},\quad \text{where } a = \frac{(1-e)^2}{k^2}\sqrt{ck};$$

$$\text{period} = \frac{2\pi}{a} = \frac{2\pi}{(1-e)^2 c^{1/2}} \cdot k^{3/2}.$$

## Exercise 6.1:

1. 0·28. 2. 0·182. 3. Impossible; series not valid.

## Exercise 6.2:

2. $$\frac{1}{1-x}:\quad \theta = \frac{1 - (1-x)^{1/(n+2)}}{x}.$$

$$\frac{1}{1+x}:\quad \theta = \frac{(1+x)^{1/(n+2)} - 1}{x}.$$

Exercise 6.3:

1. $1 + \frac{1}{2}x - \frac{1}{4(1+\theta x)^{3/2}}$; $-\frac{1}{4} < R < -\frac{\sqrt{2}}{16}$.

2. 1·5; error just over 6%.  3. $1 + \frac{1}{3}x$; $|x| < 0{\cdot}3$,

4. $1 + \frac{1}{3}x - \frac{1}{9}x^2$; $\frac{5}{81} \times 10^{-6}$.

5. (a) $1 + \frac{x}{n}$; $\frac{1}{2n}\left(\frac{1}{n} - 1\right) \times 10^{-4}$.

(b) $1 + \frac{x}{n} + \frac{1}{2n}\left(\frac{1}{n} - 1\right)x^2$; $\frac{1}{6n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right) \times 10^{-6}$.

6. 0·0100.  7. (a) 0·9999; (b) 5·0133; (c) 9·8489.  8. 0·515.

9. $x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \frac{x^8}{8!}(-128\cos(2\theta x))$.

10. $1 - \frac{3x^2}{2} + \frac{7x^4}{8} + \frac{3}{4}\frac{x^6}{6!}(243\cos(3\theta x) + \cos(\theta x))$.

11. $-\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6$

$$- 16\frac{x^8}{8!}(17 + 248\tan^2(\theta x) + 756\tan^4(\theta x) + 840\tan^6(\theta x) + 315\tan^8(\theta x)).$$

12. $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$

$$+ 16\frac{x^7}{7!}(17 + 248\tan^2(\theta x) + 756\tan^4(\theta x) + 840\tan^6(\theta x) + 315\tan^8(\theta x)).$$

13. $\frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6$

$$+ 16\frac{x^8}{8!}(17 + 248\tan^2(\theta x) + 756\tan^4(\theta x) + 840\tan^6(\theta x) + 315\tan^8(\theta x)).$$

14. $1 - x^2 + \frac{1}{2}x^4 - \frac{x^6}{3!}e^{-\theta^2x^2}$.

15. $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$

$$+ \frac{x^6}{6!}(720\sec^7(\theta x) - 840\sec^5(\theta x) + 182\sec^3(\theta x) - \sec(\theta x)).$$

16. $-\frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \ldots$.

17. $\frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \ldots$.

18. $x - \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{x^4}{4!}\frac{1}{(1+\theta x)^5}(-50 + 24\log\overline{1+\theta x})$.

19. $1 + x + \frac{1}{2}x^2 - \frac{3}{24}x^4 + \frac{x^5}{5!}e^{\sin\theta x}(\cos^5(\theta x) - 10\cos^3(\theta x) + \cos(\theta x)$

$$-10\sin(\theta x)\cos^3(\theta x) + 15\sin(\theta x)\cos(\theta x) + 6\sin^2(\theta x)\cos(\theta x)).$$

20. $x + \frac{1}{3}x^3$; $0 < x < \pi/4$.

21. (a) $y = x^2 + x^4 + 2x^6 + \ldots$; (b) $y = 1 - x^2 - x^4 - 2x^6 - \ldots$;
(c) $y = x^3 + x^9 + \ldots$.

Exercises 6.4:

**1.** 2. **2.** 4. **3.** $a = 8/3$, $b = 16/3$, $c = -5/3$, $d = -5/3$.

**4.** Third order and also zero order at (0, 0); zero order at $(\frac{1}{2}, \frac{1}{2})$.

**5.** Third order at (0, 0).

**7.** Take $P$ as origin and the tangent to the curve at $P$ as $x$-axis. Let the co-ordinates of $Q$ be $(x, y)$. Then the centre of the circle in question lies on the $y$-axis at the point $\eta = \dfrac{y}{2} + \dfrac{x^2}{2y}$; use Ex. 6.

**8.** Take axes as in Ex. 7; let the slope of the curve at $Q$ be $y'$. Then the two normals intersect on the $y$-axis at the point $\eta = y + \dfrac{x}{y'}$. Now write $y = \dfrac{y''(0)}{2!}x^2 + \ldots$, and let $x \to 0$.

**9.** At a point $P$ where $\rho = \dfrac{(1+y'^2)^{3/2}}{y''}$ is a maximum or minimum, we necessarily have $y''' = \dfrac{3y'y''}{(1+y'^2)}$. Take axes as in Ex. 7; then $y'''(0) = 0$, so that the equation of the curve in the neighbourhood of $x = 0$ is $y = \dfrac{1}{2\rho}x^2 + ax^4 + \ldots$. The equation of the osculating circle is $y = \dfrac{1}{2\rho}x^2 + bx^4 + \ldots$, and the contact is at least of order 3.

**10.** Minimum at $x = 0$.

Exercise 6.5:

**1.** $na^{n-1}$. **2.** 1/6. **3.** 1/30. **4.** 2. **5.** 1.

**6.** Write expression as $\cot x/\cot 5x$: 1/5.

**7.** 1/2. **8.** 1/3. **9.** Take logarithms: 1.

**10.** $e$. **11.** 2. **12.** $-2$.

Exercises 7.1:

**1.** (*a*) 3·14; (*b*) 3·1416. **2.** ·89.

**3.** 0·93.

Exercises 7.2:

**1.** Error $< -\cdot 03$ metre, $< 0\cdot 007\%$. **2.** $0\cdot 693$. **3.** $1\cdot 609438$.

**4.** $3\cdot 14159$.

Exercises 7.3:

**1.** $1\cdot 0755$. **2.** $4\cdot 4934$. **3.** $1\cdot 475$.

**4.** $0$, $1\cdot 90$, $-1\cdot 90$. **5.** $1\cdot 045$.

**6.** Write equation in form $x = 1 + 0\cdot 3x^3 - 0\cdot 1x^4$; $1\cdot 519$.

**7.** $-1\cdot 2361$, $3\cdot 2361$, $5\cdot 0000$.

Exercises 8.1:

**1.** Use the fact that $\dfrac{1}{\nu(\nu+1)} = \dfrac{1}{\nu} - \dfrac{1}{\nu+1}$.

**2.** Split up $1/x(x+1)(x+2)$ into partial fractions: in the result substitute $x = 1$, $x = 2, \ldots, x = \nu$ in turn and add.

**4.** Convergent for $\alpha > 0$.

**5.** Put $\Sigma a_\nu = A$. For every positive $\varepsilon$, $|s_n - A| < \varepsilon$ if $n$ is greater than a certain $m$. Write

$$\frac{s_1 + \ldots + s_N}{N} = \frac{s_1 + \ldots + s_m}{N} + \frac{N-m}{N}\,\frac{s_{m+1} + \ldots + s_N}{N-m}$$

and let $N \to \infty$.

**6.** Yes. **7.** No.

Exercises 8.2:

**1.** Convergent.

**2.** Prove first that $n!/n^n \leq 2/n^2$ when $n > 2$: convergent.

**3.** Divergent. **4.** Cf. Chap. III, § 9, p. 189: divergent.

**5.** Note that $(\log n)^{\log n} = n^{\log(\log n)}$ and $\log(\log n) > 2$ when $n$ is large: convergent.

**6.** Convergent. **7.** $1/(n+1)^2$.

**8.**
$$\text{Error} = \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots\right)$$
$$< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right)$$
$$< \frac{1}{(n+1)!}\,\frac{1}{1 - \dfrac{1}{n+1}} < \frac{1}{n\,.\,n!}.$$

**9.**
$$\text{Error} = \frac{1}{(n+1)^{n+1}} + \frac{1}{(n+2)^{n+2}} + \dots$$
$$< \frac{1}{(n+1)^{n+1}} + \frac{1}{(n+1)^{n+2}} + \frac{1}{(n+1)^{n+3}} + \dots < \frac{1}{n(n+1)^n}.$$

**10.** $\text{Error} = \dfrac{n+1}{2^{n+1}} + \dfrac{n+2}{2^{n+2}} + \dots.$ Now for $n > 1$,
$$n + 2 < \tfrac{3}{2}(n+1), \quad n + 3 < \tfrac{3}{2}(n+2) < (\tfrac{3}{2})^2(n+1), \dots;$$
hence
$$\text{Error} < \frac{n+1}{2^{n+1}}\left(1 + \tfrac{3}{4} + (\tfrac{3}{4})^2 + \dots\right) < \frac{n+1}{2^{n-1}}.$$

**12.** Convergent. **13.** Compare with $\displaystyle\int \frac{dx}{x(\log x)^\alpha}$.

**14.** Compare with $\displaystyle\int \frac{dx}{x \log x\,(\log\log x)^\alpha}$.

**16.** Use Schwarz's inequality.

**17.**
$$1 + \tfrac{1}{2} - \tfrac{2}{3} + \dots - \frac{2}{3n+3} = \sum_{\nu=1}^{3n+3} \frac{1}{\nu} - 3\sum_{\nu=1}^{n+1} \frac{1}{3\nu} = \sum_{\nu=n+2}^{3n} \frac{1}{\nu};$$
then use the formula on p. 381,
$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{n} = \log n + C + \varepsilon_n,$$
where $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

**18.** Take the sum from $\nu = 1$ to $\nu = mn$:
$$\sum_{\nu=1}^{mn} \frac{\alpha_\nu{}^n}{\nu} = \sum_{\nu \neq kn} \frac{1}{\nu} - \sum_{\nu = kn} \frac{n-1}{\nu} = \sum_{\nu=1}^{mn} \frac{1}{\nu} - \sum_{k=1}^{m} \frac{n}{kn} = \sum_{\nu=m+1}^{mn} \frac{1}{\nu}.$$

Exercises 8.3:

**3.** (a) $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases}$

(b) $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \quad (\alpha > 0).$

Convergence is non-uniform, and $\lim_{n\to\infty} \int_{-1}^{1} f_n(x)\,dx = \int_{-1}^{1} \lim_{n\to\infty} f_n(x)\,dx.$

**4.** $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ \frac{1}{2} & \text{if } |x| = 1 \\ 1 & \text{if } |x| > 1. \end{cases}$

**9.** Consider $\lim_{n\to\infty} \sqrt[2n]{(1 - x^{2n})}$ for $-1 < x < +1$ and $\lim_{n\to\infty} \sqrt[2n]{(1 - y^{2n})}$ for $-1 < y < +1$.

**10.** Let $\varepsilon > 0$. Divide up the interval by points $x_0 = a, x_1, \ldots, x_m = b$ into sub-intervals of length less than $\varepsilon/3M$. At each point $x_i$ we can choose $n_i$ so large that $|f_n(x_i) - f_m(x_i)| < \varepsilon/3$ when $n$ and $m > n_i$. Let $N$ be the greatest of $n_0, n_1, \ldots, n_m$. Then prove by the mean value theorem that in each sub-interval the inequality $|f_n(x) - f_m(x)| < \varepsilon$ holds when $n$ and $m > N$.

Exercises 8.4:

Note on Ex. 1–20: in most of these problems the ratio test is effective, but for Ex. 12–15 the root test is preferable.

1. $|x| < 1$. 2. $|x| < 1$. 3. $|x| < 1$.

4. $|x| < 1$. 5. $|x| < 1$. 6. $-\infty < x < +\infty$.

7. $|x| < 1$. 8. $|x| < 1$. 9. $|x| < 1$.

10. $|x| < 1$. 11. $|x| < 1$. 12. $|x| < 1/a$.

13. $|x| < 1$. 14. $|x| < 1$. 15. $-\infty < x < +\infty$.

16. $|x| < 4$. 17. $|x| < 1$. 18. $|x| < 1$ or $a$, whichever is the greater.

19. $|x| < 1$.

20. Note that $1/n^{1+1/n}$ lies between $n^{-1}$ and $n^{-2}$: $|x| < 1$.

21. $\sum\limits_{\nu=0}^{\infty} \dfrac{(\log a)^\nu}{\nu!} x^\nu$.

22. $-\dfrac{1}{2} - \dfrac{x}{3} - \dfrac{x^2}{4} - \ldots - \dfrac{x^n}{n+2} - \ldots = -\dfrac{1}{x^2} \sum\limits_{\nu=2}^{\infty} \dfrac{x^\nu}{\nu}$.

23. Write $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$: $\sum\limits_{\nu=1}^{\infty} \dfrac{(-1)^{\nu-1} 2^{2\nu-1}}{(2\nu)!} x^{2\nu}$.

24. $1 + \sum\limits_{\nu=1}^{\infty} \dfrac{(-1)^\nu 2^{2\nu-1}}{(2\nu)!} x^{2\nu}$.

25. $\sum\limits_{\nu=3}^{\infty} \dfrac{(-1)^{\nu-1}(2x)^{2\nu}}{32(2\nu)!}(15 + 3^{2\nu} - 6 \,.\, 2^{2\nu})$.

26. $x^3 + \dfrac{1}{2}\dfrac{x^9}{3} + \dfrac{1\,.\,3}{2\,.\,4}\dfrac{x^{15}}{5} + \ldots = x^3 + \sum\limits_{\nu=2}^{\infty} \dfrac{(x^3)^{2\nu-1}}{2\nu-1} \cdot \dfrac{1\,.\,3\ldots(2\nu-3)}{2\,.\,4\ldots(2\nu-2)}$.

27. 1·4142.

28. (a) $1 - \dfrac{1}{3\,.\,3!} + \dfrac{1}{5\,.\,5!} - \dfrac{1}{7\,.\,7!} + - \ldots$.

(b) $\dfrac{1}{2} + \dfrac{1}{320} + \dfrac{1}{3\,.\,2^{12}} + \ldots$.

(c) $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dfrac{1}{5^2} - + \ldots$.

(d) Put $x = 1/t$: $\dfrac{1}{10} - \dfrac{2^5 - 1}{10^5} + \dfrac{2^9 - 1}{24\,.\,10^9} - + \ldots$.

29. (a) $x + x^2 + \dfrac{x^3}{3}$. (b) $x^2 - x^3 + \dfrac{11x^4}{12}$.

(c) $x + \dfrac{x^2}{2} + \dfrac{13x^3}{24} + \dfrac{19x^4}{48}$. (d) $x^2 - \dfrac{x^4}{3}$.

31. $|x| < \rho$. 32. $f(x) = 4e^x - x - 1$.

# Exercises. Answers and Hints 2

Exercises 8.5:

**1.** Break off the series at the $n$-th term; then

$$\frac{1}{2}x + \frac{1}{2.4}x^2 + \frac{1.3}{2.4.6}x^3 + \ldots + \frac{1.3\ldots(2n-3)}{2.4\ldots2n}x^n < 1 - \sqrt{(1-x)} \leqq 1.$$

Put $x = 1$: the partial sums all $\leqq 1$.

**2.** Use Ex. 1. Show that the greatest error occurs when $x = 1$ and that it can be made less than $\varepsilon$.

**3.** Write $|t| = \sqrt{t^2} = \sqrt{\{1-(1-t^2)\}}$: then put $x = 1 - t^2$ in Ex. 2.

**4.** The substitution $x = a + (b-a)t$ transforms the function $f(x)$ into a function $\varphi(t)$, $0 \leqq t \leqq 1$. Approximate to $\varphi(t)$ by a polygonal function $\psi(t)$ to within $\varepsilon/2$ (cf. Ex. 2, p. 70). Represent $\psi(t)$ as a sum of the form $a + bt + \Sigma c_i\,|t - t_0|$. Approximate to this by a polynomial (cf. Ex. 3) and replace $t$ by its expression in terms of $x$.

**7.** If there were only a finite number of primes, the identity would be valid for any positive $s$, in particular for $s = 1$. (Multiplication of absolutely convergent series.)

**8.** First prove by induction that

$$(1-x)\prod_{\nu=0}^{n-1}(1+x^{2^\nu}) = 1 - x^{2^n}.$$

Exercises 9.1:

**3.** $\sum_{\nu=1}^{n} \sin \nu\alpha =$ imaginary part of $\sum_{\nu=0}^{n} e^{i\nu\alpha}$: $\sin\left(\frac{n+1}{2}\right)\alpha \sin\frac{n}{2}\alpha / \sin\frac{1}{2}\alpha$.

**4.** Use the formula $\sigma_n(\alpha) = \frac{1}{2}(1-e^{i\alpha})^{-1}(e^{-in\alpha} - e^{(n+1)i\alpha})$ on p. 436.

**5.** Evaluate $\frac{1}{\pi}\int_{-\pi}^{\pi}\sigma_k(\alpha)d\alpha$, and then use expression for $s_m(\alpha)$ in terms of $\sigma_k(\alpha)$.

Exercises 9.2:

1. (a) $\dfrac{e^{a\pi}-e^{-a\pi}}{\pi}\left\{\dfrac{1}{2a}+\sum_{\nu=1}^{\infty}\dfrac{(-1)^{\nu}}{a^2+\nu^2}(a\cos\nu x-\nu\sin\nu x)\right\}$.

(b) $\dfrac{8}{15}\pi^4-48\sum_{\nu=1}^{\infty}\dfrac{(-1)^{\nu}}{\nu^4}\cos\nu x$.

(c) $\dfrac{\sin a\pi}{a}\sum_{\nu=1}^{\infty}(-1)^{\nu}\nu\left[\dfrac{1}{\nu^2-(a+1)^2}+\dfrac{1}{\nu^2-(a-1)^2}-\dfrac{2}{\nu^2-a^2}\right]\sin\nu x$,

if $a$ is not an integer; $\frac{1}{2}\sin(a-1)x+\sin ax+\frac{1}{2}\sin(a+1)x$, if $a$ is an integer.

(d) $\dfrac{b-a}{2\pi}+\dfrac{1}{\pi}\sum_{\nu=1}^{\infty}\left(\dfrac{\sin\nu b-\sin\nu a}{\nu}\cos\nu x-\dfrac{\cos\nu b-\cos\nu a}{\nu}\sin\nu x\right)$.

2. Apply the transformation $x=-\pi+2\pi t$ to § 4. No. 2 (p. 440).

3. $B_2(t)=t^2-t+\frac{1}{6}$; $B_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{2}t$; $B_4(t)=t^4-2t^3+t^2-\frac{1}{30}$.

4. $B_1(t)$ has already been given in Ex. 2. By (b) of the definition in Ex. 3, the other expansions are obtained by successive integration. The constants of integration must be proved to be zero.

5. In the results for $B_2(t)$ and $B_4(t)$ in Ex. 3 and 4 put $t=0$.

6. In the results for $B_3(t)$ in Ex. 3 and 4 put $t=\frac{1}{4}$.

8. $\cos\pi x=\prod_{\nu=1}^{\infty}\left(1-\dfrac{x^2}{(\nu+\frac{1}{2})^2}\right)$.

Exercises 10.2:

3. (a) Discontinuous on the line $x=0$; (b) discontinuous for $x=y=0$; (c) discontinuous on the line $x=-y$; (d) discontinuous for $y=-x^2$.

Exercises 10.3:

**1.** (a) $\dfrac{\partial f}{\partial x} = \dfrac{2x}{3\sqrt[3]{(x^2+y^2)^2}}, \quad \dfrac{\partial f}{\partial y} = \dfrac{2y}{3\sqrt[3]{(x^2+y^2)^2}}.$

(b) $\dfrac{\partial f}{\partial x} = 2x\cos(x^2-y), \quad \dfrac{\partial f}{\partial y} = -\cos(x^2-y).$

(c) $\dfrac{\partial f}{\partial x} = e^{x-y}, \quad \dfrac{\partial f}{\partial y} = -e^{x-y}.$

(d) $\dfrac{\partial f}{\partial x} = -\dfrac{1}{2\sqrt{(1+x+y^2+z^2)^3}}, \quad \dfrac{\partial f}{\partial y} = -\dfrac{y}{\sqrt{(1+x+y^2+z^2)^3}},$

$\dfrac{\partial f}{\partial z} = \dfrac{-z}{\sqrt{(1+x+y^2+z^2)^3}}.$

(e) $\dfrac{\partial f}{\partial x} = yz\cos(xz), \quad \dfrac{\partial f}{\partial y} = \sin(xz), \quad \dfrac{\partial f}{\partial z} = xy\cos(xz).$

(f) $\dfrac{\partial f}{\partial x} = \dfrac{x}{1+x^2+y^2}, \quad \dfrac{\partial f}{\partial y} = \dfrac{y}{1+x^2+y^2}.$

**2.** (a) $\dfrac{\partial f}{\partial x} = y, \quad \dfrac{\partial f}{\partial y} = x, \quad \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial y^2} = 0, \quad \dfrac{\partial^2 f}{\partial x\,\partial y} = 1.$

(b) $\dfrac{\partial f}{\partial x} = \dfrac{1}{x}, \quad \dfrac{\partial f}{\partial y} = \dfrac{1}{y}, \quad \dfrac{\partial^2 f}{\partial x^2} = -\dfrac{1}{x^2}, \quad \dfrac{\partial^2 f}{\partial x\,\partial y} = 0, \quad \dfrac{\partial^2 f}{\partial y^2} = -\dfrac{1}{y^2}.$

(c) $\dfrac{\partial f}{\partial x} = \dfrac{1+y^2}{(1-xy)^2}, \quad \dfrac{\partial f}{\partial y} = \dfrac{1+x^2}{(1-xy)^2}, \quad \dfrac{\partial^2 f}{\partial x^2} = \dfrac{2(1+y^2)y}{(1-xy)^3},$

$\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{2(x+y)}{(1-xy)^3}, \quad \dfrac{\partial^2 f}{\partial y^2} = \dfrac{2(1+x^2)x}{(1-xy)^3}.$

(d) $\dfrac{\partial f}{\partial x} = yx^{y-1}, \quad \dfrac{\partial f}{\partial y} = x^y\log x, \quad \dfrac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2},$

$\dfrac{\partial^2 f}{\partial x\,\partial y} = x^{y-1}(1+y\log x), \quad \dfrac{\partial^2 f}{\partial y^2} = x^y(\log x)^2.$

(e) $\dfrac{\partial f}{\partial x} = yx^{y-1}e^{(x^y)}, \quad \dfrac{\partial f}{\partial y} = x^y\log[xe^{(x^y)}],$

$\dfrac{\partial^2 f}{\partial x^2} = yx^{y-2}e^{(x^y)}(yx^y+y-1),$

$\dfrac{\partial^2 f}{\partial x\,\partial y} = x^{y-1}e^{(x^y)}(1+y\log x+yx^y\log x),$

$\dfrac{\partial^2 f}{\partial y^2} = x^y(\log x)^2e^{(x^y)}(1+x^y).$

**3.** Differentiate $\varphi(x^2+y^2) = \psi(x)\psi(y)$ partially with respect to $x$ and with respect to $y$. Eliminate $\varphi'(x^2+y^2)$, put $y = 1$, and solve the resulting differential equation: $f(x, y) = ae^{b(x^2+y^2)}$.

Exercises 10.3:

**1.** (a) $\dfrac{\partial f}{\partial x} = -\dfrac{x + y\cos z}{\sqrt{(x^2 + y^2 + 2xy\cos z)^3}}$, $\dfrac{\partial f}{\partial y} = -\dfrac{y + x\cos z}{\sqrt{(x^2 + y^2 + 2xy\cos z)^3}}$,

$$\frac{\partial f}{\partial z} = \frac{xy\sin z}{\sqrt{(x^2 + y^2 + 2xy\cos z)^3}};$$

(b) $\dfrac{\partial f}{\partial x} = \dfrac{1}{\sqrt{(z^2 + 2zy^2 + y^4 - x^2)}}$, $\dfrac{\partial f}{\partial y} = -\dfrac{2xy}{(z + y^2)\sqrt{(z^2 + 2zy^2 + y^4 - x^2)}}$,

$$\frac{\partial f}{\partial z} = -\frac{x}{(z + y^2)\sqrt{(z^2 + 2zy^2 + y^4 - x^2)}};$$

(c) $\dfrac{\partial f}{\partial x} = 2x\left(1 + \dfrac{y}{1 + x^2 + y^2 + z^2}\right)$,

$$\frac{\partial f}{\partial y} = \log(1 + x^2 + y^2 + z^2) + \frac{2y^2}{1 + x^2 + y^2 + z^2},$$

$$\frac{\partial f}{\partial z} = \frac{2yz}{1 + x^2 + y^2 + z^2};$$

(d) $\dfrac{\partial f}{\partial x} = \dfrac{1}{2(1 + x + yz)\sqrt{(x + yz)}}$, $\dfrac{\partial f}{\partial y} = \dfrac{z}{2(1 + x + yz)\sqrt{(x + yz)}}$,

$$\frac{\partial f}{\partial z} = \frac{y}{2(1 + x + yz)\sqrt{(x + yz)}}.$$

**2.** (a) $\dfrac{\partial f}{\partial x} = x^{(x^x)}x^x\left(\log x + (\log x)^2 + \dfrac{1}{x}\right)$; (b) $\dfrac{\partial f}{\partial x} = \dfrac{1}{x^{3+1/x^3}}(2\log x - 1)$.

**5.** $z_x = 3$, $z_y = 1$; $z_r = z_x\cos\theta + z_y\sin\theta$, $z_\theta = -z_x r\sin\theta + z_y r\cos\theta$.

**7.** (a) $ad - bc$; (b) $1/r$; (c) $4xy$.

Exercises 10.4:

1. (a) $a^2b^2(a^2 - b^2)/8$; (b) $-4$; (c) $\log 2$; (d) $e^{ab}/b - 1/b - a$; (e) $\pi/16$; (f) $4/3$.

2. $2\pi$.

3. Use the fact that the figure is symmetrical; $\frac{1}{16}$ of the volume lies above the triangle with vertices (0, 0), (1, 0), (1, 1) and below the surface $x^2 + z^2 = 1$; 16/3.

4. $\frac{1}{3}\pi(r - h)^2(2r + h)$.

5.

| | Area | Centre of Gravity | Moment about x-axis | Moment about y-axis | Moment of Inertia about x-axis | Moment of Inertia about y-axis |
|---|---|---|---|---|---|---|
| (a) | $\frac{1}{2}r^2$ | $(0, 4r/3\pi)$ | $\frac{2}{3}r^3$ | 0 | $\pi r^4/8$ | $\pi r^4/8$ |
| (b) | $ab$ | $(\frac{1}{2}a, \frac{1}{2}b)$ | $\frac{1}{2}ab^2$ | $\frac{1}{2}a^2b$ | $\frac{1}{3}ab^3$ | $\frac{1}{3}a^3b$ |
| (c) | $4ab$ | $(0, 0)$ | 0 | 0 | $4ab^3/3$ | $4a^3b/3$ |
| (d) | $\pi ab$ | $(0, 0)$ | 0 | 0 | $\pi ab^3/4$ | $\pi a^3b/4$ |
| (e) | $\frac{1}{2}ab$ | $(\frac{1}{3}a, \frac{1}{3}b)$ | $\frac{1}{6}ab^2$ | $\frac{1}{6}a^2b$ | $ab^3/12$ | $a^3b/12$ |

6.

| | Volume | Centre of Gravity | Moment of Inertia about x-axis | Moment of Inertia about y-axis | Moment of Inertia about z-axis |
|---|---|---|---|---|---|
| (a) | $abc$ | $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$ | $\frac{1}{3}abc(b^2 + c^2)$ | $\frac{1}{3}abc(c^2 + a^2)$ | $\frac{1}{3}abc(a^2 + b^2)$ |
| (b) | $\frac{2}{3}\pi a^3$ | $(0, 0, 3a/8)$ | $4\pi a^5/15$ | $4\pi a^5/15$ | $4\pi a^5/15$ |
| (c) | $\frac{1}{6}abc$ | $(\frac{1}{4}a, \frac{1}{4}b, \frac{1}{4}c)$ | $abc(b^2 + c^2)/60$ | $abc(c^2 + a^2)/60$ | $abc(a^2 + b^2)/60$ |

Exercises 10.5:

1. (a) $a^2b^2(a^2 - b^2)/8$; (b) $-4$; (c) $\log 2$; (d) $e^{ab}/b - 1/b - a$; (e) $\pi/16$; (f) $4/3$.

2. $2\pi$.

3. Use the fact that the figure is symmetrical; $\frac{1}{16}$ of the volume lies above the triangle with vertices (0, 0), (1, 0), (1, 1) and below the surface $x^2 + z^2 = 1$; 16/3.

4. $\frac{1}{3}\pi(r - h)^2(2r + h)$.

| | Area | Centre of Gravity | Moment about $x$-axis | Moment about $y$-axis | Moment of Inertia about $x$-axis | Moment of Inertia about $y$-axis |
|---|---|---|---|---|---|---|
| 5. (a) | $\frac{1}{2}r^2$ | $(0, 4r/3\pi)$ | $\frac{2}{3}r^3$ | 0 | $\pi r^4/8$ | $\pi r^4/8$ |
| (b) | $ab$ | $(\frac{1}{2}a, \frac{1}{2}b)$ | $\frac{1}{2}ab^2$ | $\frac{1}{2}a^2b$ | $\frac{1}{3}ab^3$ | $\frac{1}{3}a^3b$ |
| (c) | $4ab$ | (0, 0) | 0 | 0 | $4ab^3/3$ | $4a^3b/3$ |
| (d) | $\pi ab$ | (0, 0) | 0 | 0 | $\pi ab^3/4$ | $\pi a^3b/4$ |
| (e) | $\frac{1}{2}ab$ | $(\frac{1}{3}a, \frac{1}{3}b)$ | $\frac{1}{6}ab^2$ | $\frac{1}{6}a^2b$ | $ab^3/12$ | $a^3b/12$ |

| | Volume | Centre of Gravity | Moment of Inertia about $x$-axis | Moment of Inertia about $y$-axis | Moment of Inertia about $z$-axis |
|---|---|---|---|---|---|
| 6. (a) | $abc$ | $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$ | $\frac{1}{3}abc(b^2 + c^2)$ | $\frac{1}{3}abc(c^2 + a^2)$ | $\frac{1}{3}abc(a^2 + b^2)$ |
| (b) | $\frac{2}{3}\pi a^3$ | $(0, 0, 3a/8)$ | $4\pi a^5/15$ | $4\pi a^5/15$ | $4\pi a^5/15$ |
| (c) | $\frac{1}{6}abc$ | $(\frac{1}{4}a, \frac{1}{4}b, \frac{1}{4}c)$ | $abc(b^2 + c^2)/60$ | $abc(c^2 + a^2)/60$ | $abc(a^2 + b^2)/60$ |

Exercises 11.1

1. $c_1e^t + c_2e^{2t}$; $e^{2t} - e^t$.

2. $c_1e^{-t} + c_2e^{-2t}$; $e^{-t} - e^{-2t}$.

3. $c_1e^{\frac{1}{2}t} + c_2e^{-t}$; $\frac{2}{3}(e^{\frac{1}{2}t} - e^{-t})$.

4. $c_1e^{-2t} + c_2te^{-2t}$; $te^{-2t}$.

5. $c_1e^{-\frac{1}{2}t} + c_2te^{-\frac{1}{2}t}$; $te^{-\frac{1}{2}t}$.

6. $e^{-\frac{1}{2}t}\left(c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t\right) = ae^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} (t - \delta)$;
$\frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t$; $\nu = \sqrt{3}/2$, $T = 4\pi/\sqrt{3}$, $a = 2/\sqrt{3}$, $\delta = \pi/\sqrt{3}$.

7. $\sqrt{2}e^{-\frac{1}{2}t} \cos \frac{1}{2}(t + \frac{1}{2}\pi)$; $a = \sqrt{2}$, $\delta = -\pi/2$, $\nu = \frac{1}{2}$.

Exercises 11.2:

1. $$-\frac{e^{-t}}{1+\omega^2}+\frac{2e^{-2t}}{4+\omega^2}+\frac{(2-\omega^2)\cos\omega t+3\omega\sin\omega t}{(1+\omega^2)(4+\omega^2)};$$

$$\alpha=\frac{1}{\sqrt{(1+\omega^2)(4+\omega^2)}},\quad \tan\omega\delta=\frac{3\omega}{2-\omega^2},\quad \omega=0.$$

2. $$\frac{e^{-\frac{1}{2}t}\left((\omega^2-1)\cos\frac{\sqrt{3}}{2}t-\frac{1}{\sqrt{3}}(\omega^2+1)\sin\frac{\sqrt{3}}{2}t\right)}{1-\omega^2+\omega^4}$$

$$+\frac{(1-\omega^2)\cos\omega t+\omega\sin\omega t}{1-\omega^2+\omega^4};$$

$$\alpha=\frac{1}{\sqrt{(1-\omega^2+\omega^4)}},\quad \tan\omega\delta=\frac{\omega}{1-\omega^2},\quad \omega=\frac{1}{\sqrt{2}}.$$

3. $$\frac{e^{-\frac{1}{2}t}\left(\omega\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\omega(2\omega^2-1)\sin\frac{\sqrt{3}}{2}t\right)}{1-\omega^2+\omega^4}$$

$$+\frac{(1-\omega^2)\sin\omega t-\omega\cos\omega t}{1-\omega^2+\omega^4};$$

$\alpha$, $\tan\omega\delta$, $\omega$ as in Ex. 2.

4. $$\frac{-e^{-\frac{1}{2}t}((1-2\omega^2)\cos\frac{1}{2}t+(1+2\omega^2)\sin\frac{1}{2}t)}{1+4\omega^4}$$

$$+\frac{(1-2\omega^2)\cos\omega t+2\omega\sin\omega t}{1+4\omega^4};$$

$$\alpha=\frac{1}{\sqrt{(1+4\omega^4)}},\ \tan\omega\delta=\frac{2\omega}{1-2\omega^2},\ \omega=0.$$

5. $$e^{-2t}\left(\frac{\omega^2-4}{(\omega^2+4)^2}-\frac{2t}{\omega^2+4}\right)+\frac{(4-\omega^2)\cos\omega t+4\omega\sin\omega t}{(\omega^2+4)^2}.$$

Exercises 11.3:

1. $\log(1 + y^2)(y + \sqrt{(y^2 + 1)}) + 2(1 + x)^{-1/2} = c.$

2. $(y^3 - 2x^3)/y = c.$

3. $\log y - \int^{x/y} \frac{v\,dv}{\log v - v^2} = c.$

4. $1/y = \log x + 1 + cx.$

5. $x = \arctan y - 1 + ce^{-\arctan y}.$

6. $y^2 = \sin x - \cos x + ce^{-x}.$

7. $cy^2 = e^{xy - 1/xy}.$

8. $y^2 = x - 2 + ce^{-x}.$

9. $\cos x \,.\, \cos y = c.$

10. $ye^{x/y} + x = c.$

11. $x = c_1e^t + c_2te^t + c_3t^2e^t.$

12. $x = c_1e^{3t} + c_2te^{3t} + c_3.$

13. $y = c_1 \cos x + c_2 \sin x + c_3x \cos x + c_4x \sin x.$

14. $y = c_1 + c_2e^{-x} \cos \frac{\sqrt{3}}{2} x + c_3e^{-x} \sin \frac{\sqrt{3}}{2} x.$

15. $y = c_1e^x + c_2xe^x + c_3e^{-x} + c_4xe^{-x}$
$\quad + c_5 \cos x + c_6 \sin x + c_7x \sin x + c_8x \cos x.$

16. $y = c_1 + c_2e^{x/a}.$

17. $e^{-y} = c_1 \sec(x + c_2).$

18. $y = c_1 + c_2x + c_3e^x + c_4e^{-x}.$

19. $y = c_1 \arctan x + c_2.$

20. $x = \int^y \frac{dy}{2\log|y - 1| + c_1} + c_2.$

21. $x = -1/(c_1t + c_2).$

22. $s = (\arcsin t)^2 + c_1 \arcsin t + c_2.$

23. $t + c_2 = \frac{1}{c_1}\sqrt{(c_1s^2 - 2ks)} - \frac{2k}{c_1} \arcsin \sqrt{\left(\frac{c_1s}{2k}\right)}.$

END

# Differential and Integral Calculus

**by R.Courant**

# Miscellaneous Exercises

# Answers and Hints

# Chapter I

**Introduction**

## 1. Use §5.7

**2.** $39 = 1\cdot 3^3 + 1\cdot 3^2 + 1\cdot 3 + 0$, hence the required answer is 1110.

**3.** (*a*) 10011100; (*b*) 2130.

**4.** (*a*) 758; (*b*) 5954; (*c*) 10,000; (*d*) $\cdot 2$; (*e*) $\cdot 023$; (*f*) $\cdot\dot{2}49\dot{7}$.

**5.** (*a*) $1\cdot 41 < \sqrt{2} < 1\cdot 42$; (*b*) $2\cdot 65$.

**6.** (*a*) $x \leqq \dfrac{-3-\sqrt{5}}{2}$, $x \geqq \dfrac{-3+\sqrt{5}}{2}$.

(*b*) All values of $x$.

(*c*) $x \leqq -3 - 2\sqrt{2}$; $-3 + 2\sqrt{2} \leqq x \leqq 3 - 2\sqrt{2}$; $x \geqq 3 + 2\sqrt{2}$.

(*d*) $x \geqq -2$.

**7.** Square both sides. Equality only if $a = b$.

**8.** Use Ex. 7. Equality only if $a = b$.

**9.** (*a*) Add the three inequalities $a^2 + b^2 \geqq 2ab$, $b^2 + c^2 \geqq 2bc$, $c^2 + a^2 \geqq 2ca$.

(*b*) Multiply together the three inequalities

$$\frac{a+b}{2} \geqq \sqrt{ab}, \quad \frac{b+c}{2} \geqq \sqrt{bc}, \quad \frac{c+a}{2} \geqq \sqrt{ca}.$$

(*c*) Add together the inequalities of the type $a^2b^2 + b^2c^2 \geqq 2b^2ac$.

**10.** Apply Schwarz's inequality to the numbers $x_1$, $x_2$, $x_3$ and 1, 1, 1.

**11.** From the relationship $(a_i - a_j)(b_i - b_j) \geqq 0$ we obtain

$$a_ib_i + a_jb_j \geqq a_ib_j + a_jb_i;$$

sum for all integral values of $i$ and $j$ from 1 to $n$.

**12.** (*a*) Expand $(1-1)^n$ by the binomial theorem.

(*e*) In the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$, expand and collect terms in $x^n$.

**14.** $n^2(n+1)^2/4$.

**15.** (*a*) Write $\dfrac{1}{\nu(\nu+1)(\nu+2)} = \dfrac{1}{2}\left(\dfrac{1}{\nu(\nu+1)} - \dfrac{1}{(\nu+1)(\nu+2)}\right)$ and sum from $\nu = 1$ to $n$; $\dfrac{1}{4} - \dfrac{1}{2n(n+1)}$;

(*b*) $\dfrac{n(3n+5)}{2(n+1)(n+2)}$; (*c*) $\dfrac{7n^2+21n+8}{36(n+1)(n+2)}$.

**16.** (*a*) $\frac{1}{2}(n^2 - n + 2)$; (*b*) $\frac{1}{6}(5n^3 - 18n^2 + n - 30)$.

**17.** (*a*) $n(n^2 + 5)/6$; (*b*) $n(n - 5)(5n^2 + 11n + 26)/24$.

**18.** Assume the theorem to be true for $n = m$, and then multiply by $(a + b)$, obtaining the theorem for $n = m + 1$. Verify the theorem for $n = 1, 2$.

**19.** (*a*) 1; (*b*) $\frac{1}{4}$; (*c*) $\infty$.

**25.** (*c*) If $m > n$, $|a_m - a_n| = \dfrac{1}{(n+1)!} + \dfrac{1}{(n+2)!} + \dots + \dfrac{1}{m!}$

$$= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots + \frac{1}{(n+2)\dots m}\right)$$

$$< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(n+1)^{m-n-1}}\right)$$

$$< \frac{1}{(n+1)!}\ \frac{1}{1 - \dfrac{1}{n+1}} < \frac{1}{n\,.\,n!}$$

(*d*) Similar to (*c*).

**26.** Let $c_n = \sum\limits_{\nu=0}^{n} \dfrac{1}{\nu!}$, $d_n = \sum\limits_{\tau=0}^{n} \dfrac{(-1)^\tau}{\tau!}$.

$c_n d_n = \sum\limits_{\nu,\tau=0}^{n} \dfrac{(-1)^\tau}{\tau!\,\nu!}$, and setting $\tau + \nu = \mu$, we have

$$c_n d_n = \sum_{\mu=n+1}^{2n} \sum_{\tau=0}^{n} \frac{(-1)^\tau}{\tau!\,(\mu-\tau)!} + \sum_{\mu=0}^{n} \sum_{\tau=0}^{\mu} \frac{(-1)^\tau}{\tau!\,(\mu-\tau)!}.$$

Now $\sum\limits_{\tau=0}^{\mu} \dfrac{(-1)^\tau}{\tau!\,(\mu-\tau)!} = 0$ if $\mu > 0$, so that

$$\left|c_n d_n - 1\right| = \left|\sum_{\mu=n+1}^{2n} \sum_{\tau=0}^{n} \frac{(-1)^\tau}{\tau!\,(\mu-\tau)!}\right| < \sum_{\mu=n+1}^{2n} \frac{2^\mu}{\mu!}.$$

$$< \frac{2^{n+1}}{(n+1)!}\left(1 + \frac{2}{n+1} + \frac{2^2}{(n+1)^2} + \dots\right)$$

$$< \frac{2^{n+1}}{(n+1)!}\ \frac{1}{1 - \dfrac{2}{n+1}} < \frac{2^{n+1}}{(n-1)\,.\,n!}.$$

Since $\dfrac{2^n}{n!} \to 0$ as $n \to \infty$, $c_n d_n \to 1$ and $\lim\limits_{n\to\infty} d_n = \dfrac{1}{e}$.

**27.** (*a*) The sequence is monotonically increasing and is bounded above by 2, since if $a_n < 2$, $a_{n+1} = \sqrt{2 + a_n} < \sqrt{4} < 2$.

(*b*) Let $\lim\limits_{n\to\infty} a_n = a$. Then use the relation $a_{n+1} = \sqrt{2 + a_n}$ to obtain $a = \sqrt{2 + a}$ or $a = 2$.

**33.** (*a*) 1; (*b*) 1; (*c*) $1/e$.

**35.** (*a*) $\dfrac{1}{11}$; (*b*) $\dfrac{1}{1001}$; (*c*) $\dfrac{\varepsilon}{1+\varepsilon}$.

**36.** (*a*) $4\varepsilon/(1+2\varepsilon)$; (*b*) $\varepsilon/7$; (*c*) $\arccos(1-\varepsilon)$.

**39.** Use the fact that if $x$ is rational, $n!x$ is an even integer for all sufficiently large values of $n$.

**40.** (*a*) Continuous; (*b*) Discontinuous at $x = 0$; (*c*) Discontinuous at $x = 0, \pm 1, \pm 2, \ldots$; (*d*) Discontinuous for all values of $x$.

**42.** Yes; consider signs at $x = 0$ and at $x = \pi/5$.

**44.** Let $\varepsilon$ be arbitrary; then $|f(x') - f(x'')| < \varepsilon$ provided only that $|x' - x''| < \delta$. In particular, $|f(x') - f(x'')| < \varepsilon$ if $|x' - a| < \delta$, $|x'' - a| < \delta$, which is Cauchy's criterion for convergence.

**45.** (*a*) $(x^2 + y^2 - bx)^2 = a^2(x^2 + y^2)$.

(*b*) $3x^2 - 4x - 4 + 4y^2 = 0$.

(*c*) $x^3 = y^2(2a - x)$.

(*d*) $x^3 + y^3 = 3axy$.

**47.** (*a*) Circle with centre at $-\frac{5}{3}i$ and radius $\frac{4}{3}$.

(*b*) If $k > 1$, circle with centre at $-\dfrac{1}{k^2-1}\alpha + \dfrac{k^2}{k^2-1}\beta$ and radius $\dfrac{k}{k^2-1}|\beta - \alpha|$; if $k < 1$, interchange $\alpha$ and $\beta$; if $k = 1$, the perpendicular bisector of the line joining $\alpha$, $\beta$.

(*c*) Consider the three possibilities $k < 1$, $= 1$, $> 1$.

**48.** The "triangle inequality": the sum of two sides of a triangle is greater than the third side.

**49.** The sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on all the sides of the parallelogram.

## Chapter II

The Fundamental Ideas of the Integral and Differential Calculus

52. $\sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$.

53. $f'(x) = (1 + 2x)\sin\frac{1}{x} - \left(1 + \frac{1}{x}\right)\cos\frac{1}{x}$, $x \neq 0$; $f'(0)$ does not exist, but the difference quotient $\frac{f(x) - f(0)}{x}$ as $x \to 0$ has the upper and lower limits 1 and $-1$ respectively.

54. $f''(x) = \left(\frac{2}{x^3} - \frac{1}{x}\right)\sin x - \frac{2}{x^2}\cos x$, $x \neq 0$; $f''(0) = -\frac{1}{3}$.

55. Use the mean value theorem.

56. Use the mean value theorem.

57. Consider $\varphi(x) = \{f(x + h) - f(x)\}/h$. Prove that for small (fixed) values of $h$ this assumes values above and below $\mu$; hence for some value of $x$, $\varphi(x) = \mu$. Then use the mean value theorem.

**58.** Find the equation $y = g(x)$ of the tangent; apply the mean value theorem to $f'(x) - g'(x)$, and use the result of Ex. 55.

**59.** Find the equation $y = g(x)$ of the chord joining $x = x_1$, $x = x_2$ of the curve; consider $h(x) = f(x) - g(x)$, $h''(x) = f''(x) \geqq 0$. If $h(x) > 0$ somewhere in the interval $x_1 \leqq x \leqq x_2$, there would be a point $\xi$ with $h'(\xi) = 0$, $h(\xi) > 0$; then use Ex. 58.

**60.** Use Ex. 59.

**61.** 0·006.

**62.** (*a*) $\frac{1}{2}x^{-1/2}$; (*b*) $\sec^2 x$.

**63.** Use Ex. 62: (*a*) 2; (*b*) 1.

**66.** Let $\mu = \frac{1}{a}\int_0^a u(t)\,dt$. Find the equation $y = g(x)$ of the tangent to the curve $y = f(x)$ at the point $x = \mu$. Then $f(x) \geqq g(x)$ for all values of $x$ (cf. Ex. 58). Put $x = u(t)$ and integrate.

**67.** Suppose that the acceleration is everywhere less than 4. Then $v < 4t$, and similarly $v < 4 - 4t$. Then the distance traversed, $s = \int_0^1 v\,dt$, is less than 1.

## Chapter III

### Differentiation and Integration of the Elementary Functions

**68.** (*a*) $\left(\dfrac{2\tan x}{\cos^2 x} + \cot x\right)e^{\tan^2 x + \log \sin x}$.

(*b*) $4(x+2)^3(x^2+1)^{5/7}\sqrt[3]{(1-x^2)} - \dfrac{2x}{3\sqrt[3]{(1-x^2)^2}}(x+2)^4(x^2+1)^{5/7}$

$$+ \tfrac{10}{7}x(x^2+1)^{-2/7}(x+2)^4\sqrt[3]{(1-x^2)}.$$

(*c*) $-x\sin x + \cos x + 3x^2\sin x + x^3\cos x - \dfrac{3x^2}{\sin x} + \dfrac{x^3\cos x}{\sin^2 x}$.

**69.** The denominator must not vanish for any real value of $x$; consider its discriminant. Also, the numerator of the derivative must not vanish. The conditions are $ac - b^2 > 0$, $a > 0$, $\alpha b - a\beta = 0$, $\alpha \neq 0$, or $a = b = 0$, $\alpha \neq 0$, $c \neq 0$.

**70.** Max. for $x = -1/e$, min. for $x = 1/e$, infl. at the point $(0, 1)$ and at the point given by $(2 + \log x^2)^2 + 2/x = 0$.

**74.** Let $T$ be the triangle of given area and least perimeter, and let $b$ be any side of it. Then, keeping $b$ fixed, $T$ must be the triangle of given base $b$ and given area having the least perimeter. Hence $T$ must be isosceles, and the two sides of $T$ other than $b$ are equal to one another. But $b$ is any side, and $T$ is therefore equilateral.

Analytically, we need consider isosceles triangles only. Let the coordinates of the vertices be $(-x, 0)$, $(x, 0)$ and $\left(0, \frac{A}{x}\right)$; then the perimeter is $2x + \frac{2}{x}\sqrt{(x^4 + A^2)}$. Equate the first derivative to zero, and find the second derivative.

**75.** In virtue of Ex. 71, consider isosceles triangles only.

**76.** In virtue of Ex. 72, consider isosceles triangles only.

**77.** (*a*) The derivative of $(1 + x)e^x$ is always positive for $x \geqq 0$; the minimum for $x \geqq 0$ is when $x = 0$, namely 1; (*b*) integrate (*a*) from 0 to $x$; (*c*) integrate (*b*) from 0 to $x$.

**78.** Let $f(\theta) = \dfrac{[\theta a^p + (1-\theta)b^p]^{1/p}}{[\theta a^q + (1-\theta)b^q]^{1/q}}$; then $f(0) = f(1) = 1$. Find $f'(\theta)$ and show that either $f'(\theta) \equiv 0$ or $f'(\theta) = 0$ for exactly one value of $\theta$ in the interval from 0 to 1. In the latter case, show that $f(\theta)$ is never equal to 1 for $0 < \theta < 1$. Then evaluate $f'(0)$; it is equal, except for a positive factor, to

$$b^q \frac{a^p - b^p}{p} - b^p \frac{a^q - b^q}{q} = \int_b^a b^p x^{p-1}\left[b^{q-p} - x^{q-p}\right] dx,$$

which is negative unless $a = b$. Therefore $f(\theta) \leqq 1$.

**79.** Equality sign holds only if $f'(0) = 0$, or $a = b$.

**82.** Make $a^{-\theta}b^{\theta-1}(\theta a + (1-\theta)b)$ a minimum.

**85.** (*a*) Higher; (*b*) the same; (*c*) lower; (*d*) higher.

**86.** Integrate the left-hand side, sum, then differentiate again.

**89.** $\dfrac{d^{n+1}}{dx^{n+1}}(e^{x^2/2}) = \dfrac{d^n}{dx^n}(xe^{x^2/2}) = x\dfrac{d^n}{dx^n}(e^{x^2/2}) + n\dfrac{d^{n-1}}{dx^{n-1}}(e^{x^2/2})$ by Leibnitz's rule.

**90.** Eliminate $u_{n+1}$ from both equations; $nu_{n-1} = u_n'$; differentiate one of the equations and use this relation.

**91.** $u_n(x) = x^n + \dfrac{n(n-1)}{2}x^{n-2} + \dfrac{n(n-1)(n-2)(n-3)}{2\,.\,4}x^{n-4} + \dots$.

**92.** Apply Leibnitz's rule to

$$(a)\quad \frac{d^{n+2}}{dx^{n+2}}(x^2-1)^{n+1} = \frac{d^{n+2}}{dx^{n+2}}\left[(x^2-1)\,.\,(x^2-1)^n\right];$$

$$(b)\quad \frac{d^{n+2}}{dx^{n+2}}(x^2-1)^{n+1} = \frac{d^{n+1}}{dx^{n+1}}\left[2(n+1)x\,.\,(x^2-1)^n\right].$$

(*c*) Equate the two expressions for $P'_{n+1}$ in (*a*) and (*b*).

**93.** $P_n(x) = \dfrac{(2n)!}{2^n(n!)^2}\left(x^n - \dfrac{n(n-1)}{2(2n-1)}x^{n-2} + \dfrac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}\dots\right)$.

**94.** Same as in Ex. 93.

**95.** By the binomial theorem, $\sum_{n=0}^{p} \lambda_{n,p}(x) = (x + 1 - x)^p = 1.$

Also, differentiating

$$(a + x)^p = \sum_{n=0}^{p} \binom{p}{n} a^{p-n} x^n$$

$k$ times, we have

$$\binom{p}{k}(a + x)^{p-k} = \sum_{n=k}^{p} \binom{p}{n} \binom{n}{k} a^{p-n} x^{n-k}.$$

Multiplying by $x^k$ and putting $a = 1 - x$, we have

$$\binom{p}{k} x^k = \sum_{n=k}^{p} \binom{n}{k} \binom{p}{n} (1 - x)^{p-n} x^n = \sum_{n=k}^{p} \binom{n}{k} \lambda_{n,p}(x).$$

## Chapter IV

### Further development of the Differential Calculus

**96.** $\frac{12}{13}x^{13/12} - \frac{6}{5}x^{5/6} + \frac{4}{3}x^{3/4} + \frac{12}{7}x^{7/12} - 2x^{1/2} - 3x^{1/3} + 4x^{1/4} + 12x^{1/12}$
$-2\log(1 + x^{1/4}) - 4\log(1 + x^{1/12}) - 4\sqrt{3}\ \mathrm{arc\,tan}\ \frac{2}{\sqrt{3}}(x^{1/12} - \frac{1}{2}).$

**97.** $\frac{4}{7}(1 + e^x)^{7/4} - \frac{4}{3}(1 + e^x)^{3/4}.$

**98.** $-6\sqrt[3]{(1 + x)^2}(\frac{1}{4} + \frac{1}{5}\sqrt[6]{(1 + x)} + \frac{1}{6}\sqrt[3]{(1 + x)} + \frac{1}{7}\sqrt{(1 + x)}$
$+ \frac{1}{8}\sqrt[3]{(1 + x)^2} + \frac{1}{9}\sqrt[6]{(1 + x)^5}).$

**99.** Put $x + \frac{1}{x} = t$: $\frac{1}{2}\log\frac{x^2 - x + 1}{x^2 + x + 1}.$

**100.** $\frac{1}{n}\ \mathrm{arc\,cos}\ \frac{1}{x^n}.$

**101.** $\frac{1}{n!}\Big[\log x - \binom{n}{1}\log(x + 1) + \binom{n}{2}\log(x + 2) - + \dots$
$\pm \binom{n}{n}\log(x + n)\Big].$

**102.** $\frac{(n - 1)(n - 3)\dots 1}{n(n - 2)\dots 2}\cdot\frac{\pi}{2}$ if $n$ is even; $\frac{(n - 1)(n - 3)\dots 2}{n(n - 2)\dots 3}$ if $n$ odd.

**103.** $2^{12}/(1\,.\,3\,.\,5\,.\,7\,.\,9\,.\,11\,.\,13).$

**104.** $\frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{\pi}{2}.$

**105.** $\frac{2^{2n}(n!)^2}{(2n + 1)!}.$

**106.** $\pi/16.$

**107.** $\pi/32.$

**108.** $\int x^a(\log x)^m\,dx = \frac{x^{a+1}(\log x)^m}{a + 1} - \frac{m}{a + 1}\int x^a(\log x)^{m-1}\,dx.$

**109.** $\int x^n e^{ax}\sin bx\,dx = \frac{x^n e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)$
$-\frac{an}{a^2 + b^2}\int x^{n-1}e^{ax}\sin bx\,dx + \frac{bn}{a^2 + b^2}\int x^{n-1}e^{ax}\cos bx\,dx.$

**110.** $\int x^n e^{ax} \cos bx\, dx = \dfrac{x^n e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

$$- \frac{an}{a^2+b^2}\int x^{n-1}e^{ax}\cos bx\, dx - \frac{bn}{a^2+b^2}\int x^{n-1}e^{ax}\sin bx\, dx.$$

**111.** $\int e^{ax} \sinh bx\, dx = \dfrac{e^{ax}}{b^2-a^2}(b \cosh bx - a \sinh bx).$

**112.** $\int e^{ax} \cosh bx\, dx = \dfrac{e^{ax}}{b^2-a^2}(b \sinh bx - a \cosh bx).$

**114, 115, 116.** Integrate by parts. **117.** $2^{n+1}(n!)^2/(2n+1)!$.

**118.** Convergent. **119.** Convergent.

**120.** Convergent if $n > -1$; divergent if $n \leqq -1$.

**121.** Convergent if $n > -1$, $m > -1$; otherwise divergent.

**122.** Convergent if $n > 0$, $m > -1$; otherwise divergent.

**123.** Convergent. **124.** Divergent.

**125.** Convergent. **126.** Convergent.

**127.** Convergent if $n > 0$; divergent if $n \leqq 0$.

**128.** Convergent if $m > n - 1$; divergent if $m \leqq n - 1$.

**129.** Convergent. Consider

$$\int_{\nu\pi}^{(\nu+1)\pi} \frac{dx}{1+x^4\sin^2 x} = \left(\int_{\nu\pi}^{(\nu+\epsilon)\pi} + \int_{(\nu+\epsilon)\pi}^{(\nu+1-\epsilon)\pi} + \int_{(\nu+1-\epsilon)\pi}^{(\nu+1)\pi}\right)\frac{dx}{1+x^4\sin^2 x}.$$

In the first and last integrals the integrand $< 1$, and in the second integral the integrand $< \dfrac{1}{\pi^4\nu^4\sin^2\varepsilon\pi}$, so that

$$\int_{\nu\pi}^{(\nu+1)\pi} \frac{dx}{1+x^4\sin^2 x} < 2\varepsilon\pi + \frac{\pi}{\pi^4\nu^4\sin^2\varepsilon\pi}.$$

Choose $\varepsilon = \dfrac{1}{\nu^{4/3}}$; then $\sin\varepsilon\pi > \frac{1}{2}\varepsilon\pi$, and

$$\int_{\nu\pi}^{(\nu+1)\pi} \frac{dx}{1+x^4\sin^2 x} < \frac{k}{\nu^{4/3}} < k\int_{\nu-1}^{\nu}\frac{dx}{x^{4/3}},$$

where $k$ is a constant. Finally,

$$\int_A^B \frac{dx}{1+x^4\sin^2 x} < \int_{n\pi}^{m\pi}\frac{dx}{1+x^4\sin^2 x} < k\int_{(n-1)\pi}^{(m-1)\pi}\frac{dx}{x^{4/3}}$$

$$= \frac{3k}{\pi^{1/3}}\left[\frac{1}{\sqrt[3]{n-1}} - \frac{1}{\sqrt[3]{m-1}}\right] < \frac{3k}{\pi^{1/3}\sqrt[3]{n-1}} \to 0 \text{ as } n \to \infty.$$

Or, $\displaystyle\int_{\nu\pi}^{(\nu+1)\pi} \frac{dx}{1+x^4\sin^2 x} < \int_{\nu\pi}^{(\nu+1)\pi}\frac{dx}{1+(\nu\pi)^4\sin^2 x} < \frac{\pi}{\sqrt{1+(\nu\pi)^4}} < \frac{k}{\nu^2}.$

**130.** $\int_0^A \frac{x\,dx}{1+x^2\sin^2 x} > \int_0^A \frac{x\,dx}{1+x^2} > \frac{1}{2}\log(1+A^2)$; divergent.

**131.** Convergent if $\beta < -2$, $\beta + 1 < \alpha < -1$ or $\beta > 0$, $-1 < \alpha < \beta/2 - 1$; otherwise divergent.

Suppose that $\beta \leqq 0$. Then $\int^\infty \frac{x^\alpha\,dx}{1+x^\beta \sin^2 x}$ converges only if $\alpha < -1$; $\int_0 \frac{x^\alpha\,dx}{1+x^\beta\sin^2 x}$ behaves like $\int_0 \frac{x^\alpha\,dx}{1+x^{\beta+2}}$, i.e. if $\beta + 2 \geqq 0$, then $\alpha > -1$, contrary to the preceding; if $\beta + 2 < 0$, $\alpha - \beta - 2 > -1$.

Suppose that $\beta > 0$. Then $\int_0 \frac{x^\alpha dx}{1+x^\beta \sin^2 x}$ converges only if $\alpha > -1$. Furthermore,

$$\frac{\nu^\alpha \pi^{\alpha+1}}{\sqrt{1+(\nu+1)^\beta \pi^\beta}} = \int_{\nu\pi}^{(\nu+1)\pi} \frac{(\nu\pi)^\alpha\,dx}{1+(\nu+1)^\beta\pi^\beta \sin^2 x}$$

$$< \int_{\nu\pi}^{(\nu+1)\pi} \frac{x^\alpha\,dx}{1+x^\beta \sin^2 x} < \int_{\nu\pi}^{(\nu+1)\pi} \frac{(\nu+1)^\alpha \pi^\alpha\,dx}{1+(\nu\pi)^\beta \sin^2 x} < \frac{(\nu+1)^\alpha \pi^{\alpha+1}}{\sqrt{1+(\nu\pi)^\beta}}$$

or
$$k_1 \nu^{\alpha-\beta/2} < \int_{\nu\pi}^{(\nu+1)\pi} \frac{x^\alpha\,dx}{1+x^\beta\sin^2 x} < k_2 \nu^{\alpha-\beta/2}.$$

Hence $\int_\pi^\infty \frac{x^\alpha\,dx}{1+x^\beta\sin^2 x}$ converges if, and only if, $\alpha - \beta/2 < -1$.

The integral may also be estimated by the method of Ex. 129.

**132.** $\int_a^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx = \int_{a\alpha}^\infty \frac{f(x)}{x}\,dx - \int_{a\beta}^\infty \frac{f(x)}{x}\,dx$

$$= \int_{a\alpha}^{a\beta} \frac{f(x)}{x}\,dx = L\log\frac{\beta}{\alpha} + \int_{a\alpha}^{a\beta} \frac{f(x)-L}{x}\,dx.$$

Show that this last integral tends to zero as $a \to 0$.

**134.** Consider $\int_a^b \frac{f(\alpha x) - f(\beta x)}{x}\,dx$, and proceed as in Ex. 132.

**135.** In the formula $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1}\,dt$ substitute $t = x^2$ and $t = \log\frac{1}{x}$ respectively.

# Chapter V

## Applications

**136.** (*a*) $x^5 + y^5 = 5ax^2y^2$; (*b*) $x = a \operatorname{arc} \cos \dfrac{a-y}{b} + \sqrt{b^2 - (a-y)^2}$.

**138.** (*a*) $x^2 + y^2 = \sqrt{(a^2x^2 + b^2y^2)}$. (*c*) $x(x^2 + y^2) + py^2 = 0$.

(*b*) $x^2 + y^2 = \sqrt{(a^2x^2 - b^2y^2)}$. (*d*) $x = 0$.

**141.** $x = t - \dfrac{l\sqrt{p}}{\sqrt{t+p}}$, $y^2 = 4pt\left(1 + \dfrac{l}{2\sqrt{p}\sqrt{t+p}}\right)^2$.

**142.** $5a^2/2$. **143.** $\pi b(2a + b)(a - b)^2/(2a^2)$.

**144.** $\dfrac{4b(a+b)}{a}\left(1-\cos\dfrac{a}{2b}t\right).$

**146.** Choose the axes so that the curve touches the $x$-axis at the origin, and express the ordinate $y$ as a function of the angle which the tangent at the point $(x, y)$ makes with the $x$-axis.

**147.** (a) $l^3/12$; (b) $l^3/3$; (c), (d) $l(l^2/12+d^2)$.

**148.** $r = ce^{\cot\alpha\,.\,\theta}$.

**149.** $(x-c)^2+y^2=k^2$.

**151.** $(x-c_1)^2+y^2=c_2^2$.

**152.** $y = a\cosh\dfrac{x-b}{a}$.

**153.** The length of a straight line joining the points $(r_\nu, \varphi_\nu)$, $(r_{\nu+1}, \varphi_{\nu+1})$ of the curve is

$$\sqrt{\{(r_{\nu+1}-r_\nu)^2+2r_\nu r_{\nu+1}(1-\cos(\varphi_{\nu+1}-\varphi_\nu))\}},$$

and the length of a polygonal line inscribed in the curve is

$$\sum_{\nu=0}^{n-1}\sqrt{\{(\Delta r_\nu)^2+r_\nu r_{\nu+1}(\Delta\varphi_\nu)^2+r_\nu r_{\nu+1}(\Delta\varphi_\nu)^4\,.\,R_\nu\}},$$

where the $|R_\nu|$'s are all bounded. Letting the maximum of $\Delta\varphi_\nu$ tend to zero, we obtain

$$\int\sqrt{\left\{\left(\frac{dr}{d\varphi}\right)^2+r^2\right\}}\,d\varphi.$$

## Chapter VI

Taylor's Theorem and the Approximate Expressions of Functions by

**157.** $x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \dots;\quad (\sin x)^2 = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - x^7R\right)^2$

$$= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + x^8R',$$

where $R$ and $R'$ remain bounded as $x \to 0$.

**158.** $x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots;\quad \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - x^7R}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - x^6S}$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + x^7T,$$

where $R$, $S$, $T$ are bounded as $x \to 0$.

**159.** $1 - \frac{x^2}{4} - \frac{x^4}{96} - \dots;\quad \sqrt{\cos x} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - x^6R\right)^{1/2}$

$$= 1 + \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - x^6R\right) - \frac{1}{8}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - x^6R\right)^2$$

$$+ \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - x^6R\right)^3S = 1 - \frac{x^2}{4} - \frac{x^4}{96} + x^6T,$$

where $R$, $S$, $T$ are bounded as $x \to 0$.

**160.** (a) $1-\frac{x^2}{3}-\frac{x^4}{45}-\frac{2x^6}{945}\cdots$. (b) $1-\frac{x^2}{12}+\frac{x^4}{1440}-\frac{x^6}{23712}\cdots$.

(c) $1+\frac{x^2}{2}+\frac{5}{24}x^4+\frac{61}{720}x^6+\cdots$. (d) $1+x+\frac{x^2}{2}-\frac{x^3}{8}\cdots$.

(e) $e+ex+ex^2+\frac{5}{6}ex^3+\cdots$. (f) $-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}\cdots$.

**161.** $x+\frac{1}{2}\,\frac{x^3}{3}+\frac{1\,.\,3}{2^2\,.\,2!}\,\frac{x^5}{5}+\frac{1\,.\,3\,.\,5}{2^3\,.\,3!}\,\frac{x^7}{7}+\cdots$.

**162.** $\sum\limits_{\tau=0}^{\infty}\left(\sum\limits_{\nu=0}^{\tau}\binom{2n}{n}\binom{2\tau-2n}{\tau-n}\frac{1}{(2n+1)(2\tau-2n+1)}\right)\frac{x^{2\tau+2}}{2^{2\tau}}$.

**163.** (a) $\sum\limits_{\nu=0}^{\infty}(-1)^{\nu}\frac{1\,.\,3\,.\,5\,\ldots\,(2\nu-1)}{2\,.\,4\,.\,6\,\ldots\,2\nu}\,\frac{x^{2\nu+1}}{2\nu+1}$;

(b) $\sum\limits_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\,\frac{x^{2\nu+1}}{2\nu+1}$; (c) $\sum\limits_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu+1)!}\,\frac{x^{2\nu+1}}{2\nu+1}$.

**164.** (a) $\frac{(2n)!\,x^{2n+1}}{2^{2n}(n!)^2(2n+1)}$; (b) $\frac{x^{2n+1}}{n!(2n+1)}$; (c) $\frac{x^{2n+1}}{(2n+1)!(2n+1)}$

**166.** $e-\frac{e}{2}\left(\frac{1}{x}\right)+\frac{11e}{24}\left(\frac{1}{x}\right)^2-\cdots$.

**167.** (a) $-\frac{e}{2}$; (b) $\frac{11e}{24}$; (c) 0; (d) $e^{-1/6}$; (e) 1.

**169.** (a) Minimum at $x=0$; (b) maxima and minima at points where $\tan\frac{1}{x}=\frac{1}{x}$, which occur once in each interval $\frac{1}{(n+\frac{1}{2})\pi}<x<\frac{1}{(n-\frac{1}{2})\pi}$, $n=\pm1,\pm2,\ldots$; maxima and minima alternately.

## Chapter VII

**Numerical Methods**

**170.** $5{\cdot}881a$.

**171.** 11.

**172.** $0{\cdot}82247$.

**173.** $\cdot175$, $\cdot302$, $3{\cdot}490$.

**174.** Since $\log(\alpha + x)$ is convex downward, and $\alpha > 0$,

$$\log(\alpha + 1) + \ldots + \log(\alpha + n) > \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log(\alpha + x)\,dx$$

$$= (n + \tfrac{1}{2} + \alpha)\log(n + \tfrac{1}{2} + \alpha) - (\alpha + \tfrac{1}{2})\log(\alpha + \tfrac{1}{2}) - n,$$

or $$\alpha(\alpha + 1)\ldots(\alpha + n) > \alpha\frac{(n + \frac{1}{2} + \alpha)^{n+\frac{1}{2}+\alpha}}{(\alpha + \frac{1}{2})^{\alpha+\frac{1}{2}}}e^{-n} > k(\alpha)\,n!\,n^{\alpha},$$

where $k(\alpha)$ is a positive number depending on $\alpha$. Furthermore,

$$\frac{a_n}{a_{n-1}} = \left(1 + \frac{\alpha}{n}\right)\left(1 - \frac{1}{n}\right)^{\alpha} = 1 - \frac{\alpha(\alpha + 1)}{2}\,\frac{1}{n^2} + \frac{R}{n^3},$$

where $R$ remains bounded as $n \to \infty$. Therefore, for sufficiently large values of $n$, $a_n < a_{n-1}$ and the sequence is monotonically decreasing.

**175.** $c + (n + \frac{1}{2})\log n - \sum_{\nu=1}^{l} (n_\nu + \frac{1}{2})\log n_\nu$.

## Chapter VIII

**Infinite Series and Other Limiting Processes**

**178.** If $\lim a_n \leqq 1$, the terms do not tend to zero. If $\lim a_n > k > 1$, compare the series with $\Sigma \dfrac{1}{k^n}$.

**179.** For any $\varepsilon$, $\sum\limits_{\nu=n}^{m} a_\nu < \varepsilon$ for every $n$, $m$ sufficiently large. But $\sum\limits_{\nu=n}^{m} a_\nu > (m-n)a_m$, or $ma_m < \varepsilon + na_m$. Keeping $n$ fixed, choose $m$ so large that $na_m < \varepsilon$; for every such $m$, $ma_m < 2\varepsilon$.

**180.** Apply Ex. 179.

**181.** Let $s_n$ denote the partial sums of $\sum\limits_{\nu=1}^{\infty} a_\nu$, $s$ the sum, and let $\sigma_n = s_n - s$. Then

$$\sum_{\nu=n}^{m} a_\nu b_\nu = \sum_{\nu=n}^{m} (\sigma_\nu - \sigma_{\nu-1})b_\nu = \sum_{\nu=n}^{m} \sigma_\nu(b_\nu - b_{\nu+1}) - \sigma_{n-1}b_n + \sigma_m b_{m+1}.$$

For every sufficiently large $\nu$, $|\sigma_\nu| < \varepsilon$, and

$$\left|\sum_{\nu=n}^{m} a_\nu b_\nu\right| < \varepsilon \sum_{\nu=n}^{m} |b_\nu - b_{\nu+1}| + \varepsilon|b_n| + \varepsilon|b_{m+1}|$$
$$< \varepsilon|b_n - b_{m+1}| + \varepsilon|b_n| + \varepsilon|b_{m+1}|.$$

This is in turn less than $4B\varepsilon$, where $B$ is a bound for $|b_\nu|$, and the series $\sum\limits_{\nu=1}^{\infty} a_\nu b_\nu$ converges.

**182.** Proceed as in Ex. 181:

$$\sum_{\nu=n}^{m} a_\nu b_\nu = \sum_{\nu=n}^{m} (s_\nu - s_{\nu-1})b_\nu = \sum_{\nu=n}^{m} s_\nu(b_\nu - b_{\nu+1}) - s_{n-1}b_n + s_m b_{m+1}$$

and use the monotonic character of $b_n$, the fact that $b_n \to 0$, and that $|s_\nu| < s$ for every $\nu$.

**183.** (*a*), (*b*), (*d*), (*f*) Convergent; (*c*) convergent if $\theta \neq 2n\pi$; (*e*) convergent if $\theta \neq (2n+1)\pi$.

**184.** (*a*) $\frac{1}{2}\log 2$; (*b*) $\log 2$.

**185.** (*a*) $\alpha = 1$; (*b*) $\alpha \geqq 1$.

**186.** (*a*) Diverges; (*b*) converges.

**188.** If $|a_n| < \frac{1}{n^{1+\varepsilon}}$ for every sufficiently large $n$, then

$$\log\frac{1}{|a_n|} > (1+\varepsilon)\log n \quad \text{or} \quad \frac{\log 1/|a_n|}{\log n} > 1+\varepsilon.$$

Reverse the argument: $\frac{\log 1/|a_n|}{\log n} > 1+\varepsilon$ implies $|a_n| < \frac{1}{n^{1+\varepsilon}}$. Similarly for divergence.

**189.** Apply Ex. 188.

**190.** Proceed as in Ex. 188.

**191.** The $n$-th root test may be written as follows: if $\frac{\log 1/|a_n|}{n} > \varepsilon$, the series converges; if $< -\varepsilon$, the series diverges. Write

$$\frac{\log 1/|a_n|}{\log n} = \frac{n}{\log n}\,\frac{\log 1/|a_n|}{n}.$$

**192.** If $\left|\frac{a_{n+1}}{a_n}\right| < \frac{b_{n+1}}{b_n}$ for every $n \geqq N$, then

$$|a_{n+1}| < \frac{b_{n+1}}{b_n}|a_n| < \frac{b_{n+1}}{b_n}\,\frac{b_n}{b_{n-1}}\,|a_{n-1}| < \ldots < \frac{|a_N|}{b_N}\,b_{n+1};$$

therefore $\Sigma\,|a_\nu|$ converges if $\Sigma b_\nu$ does. Similarly for divergence.

**194.** Use Ex. 192, comparing with $\sum\limits_{\nu=1}^{\infty}\frac{1}{\nu^\alpha}$. The series $\Sigma\,|a_\nu|$ converges if

$$\frac{|a_n|}{|a_{n+1}|} > \left(1+\frac{1}{n}\right)^\alpha > 1+\frac{\alpha}{n}+\frac{R}{n^2},$$

where $\alpha > 1$. Then

$$n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) > \alpha + \frac{R}{n} > 1+\varepsilon.$$

Reverse the argument:

$$n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) > 1+\varepsilon$$

implies the convergence of $\Sigma\,|a_\nu|$. Similarly for divergence.

**195.** $\Sigma\,|a_\nu|$ converges if

$$\frac{|a_n|}{|a_{n+1}|} > \left(1+\frac{1}{n}\right)\left(1+\frac{\log\left(1+\frac{1}{n}\right)}{\log n}\right)^\alpha > 1+\frac{1}{n}+\frac{\alpha}{n\log n}+\frac{R}{n^2\log n},$$

where $\alpha > 1$. Then

$$n\log n\left(\frac{|a_n|}{|a_{n+1}|} - 1 - \frac{1}{n}\right) > \alpha + \frac{R}{n} > 1+\varepsilon.$$

Reversal of this argument gives the convergence test; similarly for divergence.

**197.** (*a*) Converges if $\beta - \alpha > 1$, diverges if $\beta - \alpha \leqq 1$.

(*b*) Converges if $\gamma > \alpha + \beta$, diverges if $\gamma \leqq \alpha + \beta$.

**198.** (*a*) If $x \geqq 1 + \varepsilon$, $\sum_{\nu=1}^{\infty} \frac{1}{\nu^x} \leqq \sum_{\nu=1}^{\infty} \frac{1}{\nu^{1+\varepsilon}}$. Similarly for (*b*).

**199.** The partial sums of $\Sigma \cos \nu x$ are uniformly bounded for every $x$ in $\varepsilon \leqq x \leqq 2\pi - \varepsilon$. (Write $\cos \nu x = \frac{e^{i\nu x} + e^{-i\nu x}}{2}$ and $\sum_{\nu=0}^{n} \cos \nu x = \frac{1}{2} \sum_{\nu=-n}^{n} e^{i\nu x}$). Then prove the theorem analogous to Ex. 182 for uniform convergence.

**200.** If $x$ lies in the interval $\varepsilon \leqq x \leqq N$, then $y = \frac{x-1}{x+1}$ lies in the interval $-1 + \frac{2\varepsilon}{1+\varepsilon} \leqq y \leqq 1 - \frac{2}{N+1}$.

**201.** (*a*) $-1 < x < 1$; (*b*) $-4 < x < 4$; (*c*) $x > 1$; (*d*) $x > 0$; (*e*) any $x$; (*f*) no $x$; (*g*) $x > 1$; (*h*) $-1 < x < 1$.

**202.** If $\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^{x_0}}$ converges, write $\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^x} = \sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^{x_0}} \cdot \frac{1}{\nu^{x-x_0}}$, and use Ex. 181 or 182. If $\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^{x_0}}$ diverges, $\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^x}$ cannot converge for $x < x_0$, by what has just been proved.

**203.** Write $\Sigma \frac{a_\nu \log \nu}{\nu^x} = \Sigma \frac{a_\nu}{\nu^{x_0}} \cdot \frac{\log \nu}{\nu^{x-x_0}}$.

**204.** Clearly $\sum_{\nu=0}^{\infty} a_\nu x^\nu < \sum_{\nu=0}^{\infty} a_\nu$ for $x < 1$. On the other hand,

$$\lim_{x \to 1} \sum_{\nu=0}^{\infty} a_\nu x^\nu > \lim_{x \to 1} \sum_{\nu=0}^{N} a_\nu x^\nu = \sum_{\nu=0}^{N} a_\nu; \quad \text{or} \quad \lim_{x \to 1} \sum_{\nu=0}^{\infty} a_\nu x^\nu \geqq \sum_{\nu=0}^{\infty} a_\nu.$$

**205.** As in Ex. 204, $\lim_{x \to 1} \sum_{\nu=0}^{\infty} a_\nu x^\nu \geqq \sum_{\nu=0}^{\infty} a_\nu$ and hence is $\infty$.

**206.** Write $\sum_{\nu=0}^{\infty} a_\nu x^\nu = \sum_{\nu=0}^{\infty} a_\nu X^\nu \left(\frac{x}{X}\right)^\nu$. Then prove the theorem analogous to Ex. 181 for uniform convergence: if $\sum_{\nu=0}^{\infty} a_\nu$ converges, and if the sequence $b_0(x), b_1(x), \ldots, b_n(x), \ldots$ is monotonic for every $x$ and uniformly bounded for every $x$ in a certain interval, then $\sum_{\nu=0}^{\infty} a_\nu b_\nu(x)$ converges uniformly in that interval.

**207.** This follows from the uniform convergence of the series $\sum_{\nu=0}^{\infty} a_\nu x^\nu$ in the interval $0 \leqq x \leqq X$. For then $\sum_{\nu=0}^{\infty} a_\nu x^\nu$ is continuous in that interval.

**208.** (*a*) $x(1+x)/(1+x^2)$; (*b*) $(1-x^2)/(1-x+x^2)^2$.

**209.** (*a*) The series is equal to $\left.\frac{d}{dx}\left(\frac{e^x - 1}{x}\right)\right|_{x=1}$;

(*b*) The series is equal to $\left.\frac{\sqrt{(1+x)} - \sqrt{(1-x)}}{2}\right|_{x=1}$.

# Chapter IX

## Fourier Series

**211.** $$\pi x \cot \pi x = 1 - 2x^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu^2 - x^2} = 1 - 2x^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left( \sum_{m=0}^{\infty} \frac{x^{2m}}{\nu^{2m}} \right)$$
$$= 1 - 2 \sum_{m=1}^{\infty} \left( \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2m}} \right) x^{2m}.$$

**214.** (*a*) $\int_0^1 \frac{\log x}{1-x}\,dx = -\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}$; (*b*) $\int_0^1 \frac{\log x}{1+x}\,dx = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu^2}$.

**216.** (*a*) $\sqrt{2}$; (*b*) $\sqrt{3}$.

**217.** $$\coth \pi x - \frac{1}{\pi x} = \frac{2x}{\pi} \left( \frac{1}{1^2 + x^2} + \frac{1}{2^2 + x^2} + \frac{1}{3^2 + x^2} + \dots \right).$$

# Chapter XI

## The Differential Equations for the Simplest Types of Vibrations

**218.** $x + c = \sqrt{(a^2 - y^2)} - a \log \dfrac{a + \sqrt{(a^2 - y^2)}}{y}$.

**219.** $\frac{1}{2}ky^2 + x = c$.

**221.** $I = \dfrac{E_0}{\sqrt{(\rho^2 + \omega^2\mu^2)}} \sin(\omega t - \varphi) + \dfrac{A}{\sqrt{(\rho^2 + \omega^2\mu^2)}} e^{-\rho t/\mu}$, where $\tan\varphi = \dfrac{\omega\mu}{\rho}$.

**222.** $x^2 = a^2 - \dfrac{kt^2}{a^2}$; time of descent is $a^2/\sqrt{k}$.

**223.** Differentiate with respect to $x$ and solve the resulting differential equation for $p$ in terms of $x$:

$$y = -\frac{1}{4x^2} \quad \text{and} \quad y = \frac{c}{x} + c^2.$$

**224.** $x = -1 + \sqrt{(2y + c)} + \log(-1 + \sqrt{(2y + c)})$.

END

# Differential and Integral Calculus Vol. I

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